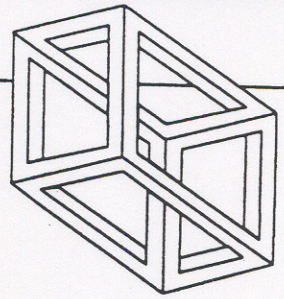
Restless Rectangles



Institute Notes

Concept: Investigate the constants of proportionality within similar shapes as "shape ratios" and the scale factors between pairs of similar shapes as "size ratios" for a set of similar rectangles and use these ratios to solve problems.

TEKS Focus: **6.3**—The student solves problems involving proportional relationships.
7.3—The student solves problems involving proportional relationships.
8.3—The student identifies proportional relationships in problem situations.

Overview: Participants will sort a group of rectangles cut from cardstock into two groups: those that have the same shape and those that do not have the same shape as the first group. The rectangles from each group are measured and their dimensions are recorded in Tables 1 and 2. The data is graphed and observations are made to determine which graphs represent a proportional relationship. Comparisons are made between any two rectangles from Group 1 (same shape group), and participants make generalizations about different ways to describe the shape and relative sizes of these rectangles. Through these investigations, participants identify the constants of proportionality within a set of similar shapes as "shape ratios" and the scale factors between pairs of similar shapes as "size ratios" and use these ideas to determine whether two geometric figures are similar or not.

Materials:
Restless Rectangles on cardstock
Scissors
1" graph paper
Cm Grid Paper
Metric rulers
Markers and/or peel-and-stick dots
Tape
Construction paper in two colors

Also:
Grade 6
4A, 11D, 12A, 13A
Grade 7
7A, 13C, 13D, 14A, 15A
Grade 8
7D, 14C,D, 15A, 16A, 16B

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Procedure:

1. Participants are to work in groups of 2 to 4 following the procedure outlined in steps a - h in Activity 1.
2. The data collected from Activity 1 sheet is to be recorded in the tables provided in Activity 2. Have participants discuss patterns they see in the data. (See *Math Notes* for Activity 2).
3. Have participants graph the data for the rectangles in Group 1 on the grid in Activity 3 or on 1" graph paper and answer the questions at the end of the activity.
   Participants should be encouraged to study Table 1 and Graph 1 to investigate the relationship between the data in the table and the graph of the ordered pairs (W,L). This connection should reinforce the definition of a ratio as "*an ordered pair of measurements.*"
4. Have participants graph the data for the rectangles in Group 2 on the grid in Activity 4 or on 1" graph paper and answer the questions at the end of the activity.

Debriefing: Ask participants to use the data from their tables and graphs to make generalizations about the rectangles in Group 1.

Extensions: a. Provide participants with a set of triangles or other

Math Notes:
Participants should observe that these rectangles have the same shape because of the constant ratio $\frac{L}{W}$, or "shape ratio," obtained from Table 1, and that the corresponding sides are proportional. They should also conclude that the scale factor or "size ratio" between each pair of rectangles describes the relative sizes of any two rectangles in the set. It would be appropriate at this time to label the rectangles in Group 1 as *similar figures* and discuss the attributes of similar figures.

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geometric figures that have the same shape and some that do not. Have them sort the geometric figures and repeat this activity.

b. Ask participants to make a conjecture about the ratio of the lengths of the diagonals for any two similar rectangles from Group 1. Then, have them measure the diagonals of two rectangles from Group 1 and compare their ratio to the ratio formed by lengths of corresponding sides.

c. Suppose the rectangle with sides 4 cm and 6 cm is cut in half to form two smaller rectangles. Would the smaller rectangle be similar to the original? Why or why not?

Assessment:

1. Have participants make a set of geometric figures that have the same shape and explain how they determined this.
2. Have participants apply their knowledge of scale factor between similar figures to answer the questions in Activity 5.

Notes:

b. Participants should recognize that the ratio of the diagonals for any two similar rectangles is the same as the ratio of their corresponding sides. Example: The diagonal of the rectangle with dimensions 4 cm by 6 cm has a length of $\sqrt{52}$ cm and the rectangle with dimensions 8 cm by 12 cm has a diagonal of length $\sqrt{208}$, obtained by using the Pythagorean Theorem. The ratio of the diagonals can be expressed as $\sqrt{52}:\sqrt{208}$. When this ratio is simplified to $2\sqrt{13}:4\sqrt{13}$ or 1:2, a comparison can more readily be observed. This is the same as the ratio of two corresponding sides of the two rectangles expressed as 4:8 or 6:12. It is important to introduce now this concept of the constant ratio of corresponding parts of similar figures, since the next activity, *One Size Fits All*, will explore this idea in depth and extend to the ratio of the perimeters of similar figures (equal to the ratio of their corresponding parts) and the ratio of areas of similar figures (equal to the square of the ratio of their corresponding parts).

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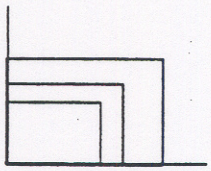
Activity 1

a. Cut out each rectangle.

b. Sort the rectangles into 2 groups:
Group 1: Rectangles that have the same shape.
Group 2: Rectangles that do not have the same shape as those in Group 1.

c. Arrange the rectangles in Group 1 in order from largest to smallest and place the largest on the Restless Rectangles Guide first followed by the next largest on top, etc. as shown below.



d. Select the largest rectangle and place the edge of a ruler along its diagonal. What do you observe?

e. Are there any rectangles that you put in Group 1 that you now think belong to Group 2? If so, put them in Group 2.

f. Repeat steps C and D above for the rectangles in Group 2. What do you observe about their diagonals?

g. How could you use the information about the diagonals of the rectangles in Group 1 to show that they have the "same shape?"

h. Write a statement about your observations concerning rectangles that have the same shape.

Answers and Math Notes:

a. Dimensions of rectangles:
A: 3 cm x 2 cm; G: 6 cm x 4 cm; I: 6 cm x 9 cm; D: 8 cm x 12 cm; E: 12 cm x 6 cm; F: 5 cm x 3 cm; C: 3cm x 4 cm; B: 10 cm x 4 cm; H: 7 cm x 5 cm

b. Participants will sort the rectangles according to "same shape" without measuring. Ask participants to explain the criteria used to sort the rectangles into two distinct groups.

c. Ask participants to observe the arrangement of the rectangles from Group 1 on the Guide. Ask them if they would like to change any of the rectangles placed in Group 1 or Group 2 and why.

d. Participants should observe that the diagonals of all rectangles in Group 1 lie on the same line through the origin. If there is any rectangle whose diagonal is not aligned with the others, its placement into this set should be reviewed.

e. These diagonals do not lie on the same straight line through the origin.

g. Arrange these rectangles on centimeter grid paper with the width and length aligned along the x- and y-axes, respectively. Observe that the diagonals lie along a line through the origin. This means that the widths and lengths of the rectangles in this set form a proportional relationship.

h. Rectangles that have the same shape will have their diagonals all on the same line that passes through the origin.

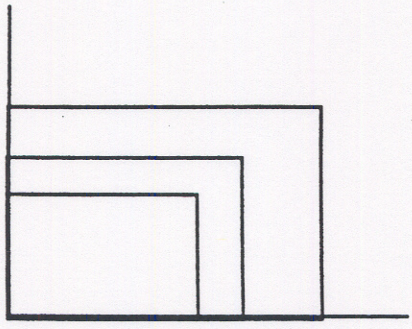
Restless Rectangles

Activity 1

a. Cut out each rectangle.

b. Sort the rectangles into 2 groups:
Group 1: Rectangles that have the same shape.
Group 2: Rectangles that do not have the same shape as those in Group 1.

c. Arrange the rectangles in Group 1 in order from largest to smallest and place the largest on the Restless Rectangles Guide first followed by the next largest on top, etc. as shown below.



d. Select the largest rectangle and place the edge of a ruler along its diagonal. What do you observe?

e. Are there any rectangles that you put in Group 1 that you now think belong to Group 2? If so, put them in Group 2.

f. Repeat steps C and D above for the rectangles in Group 2. What do you observe about their diagonals?

g. How could you use the information about the diagonals of the rectangles in Group 1 to show that they have the "same shape?"

h. Write a statement about your observations concerning rectangles that have the same shape.

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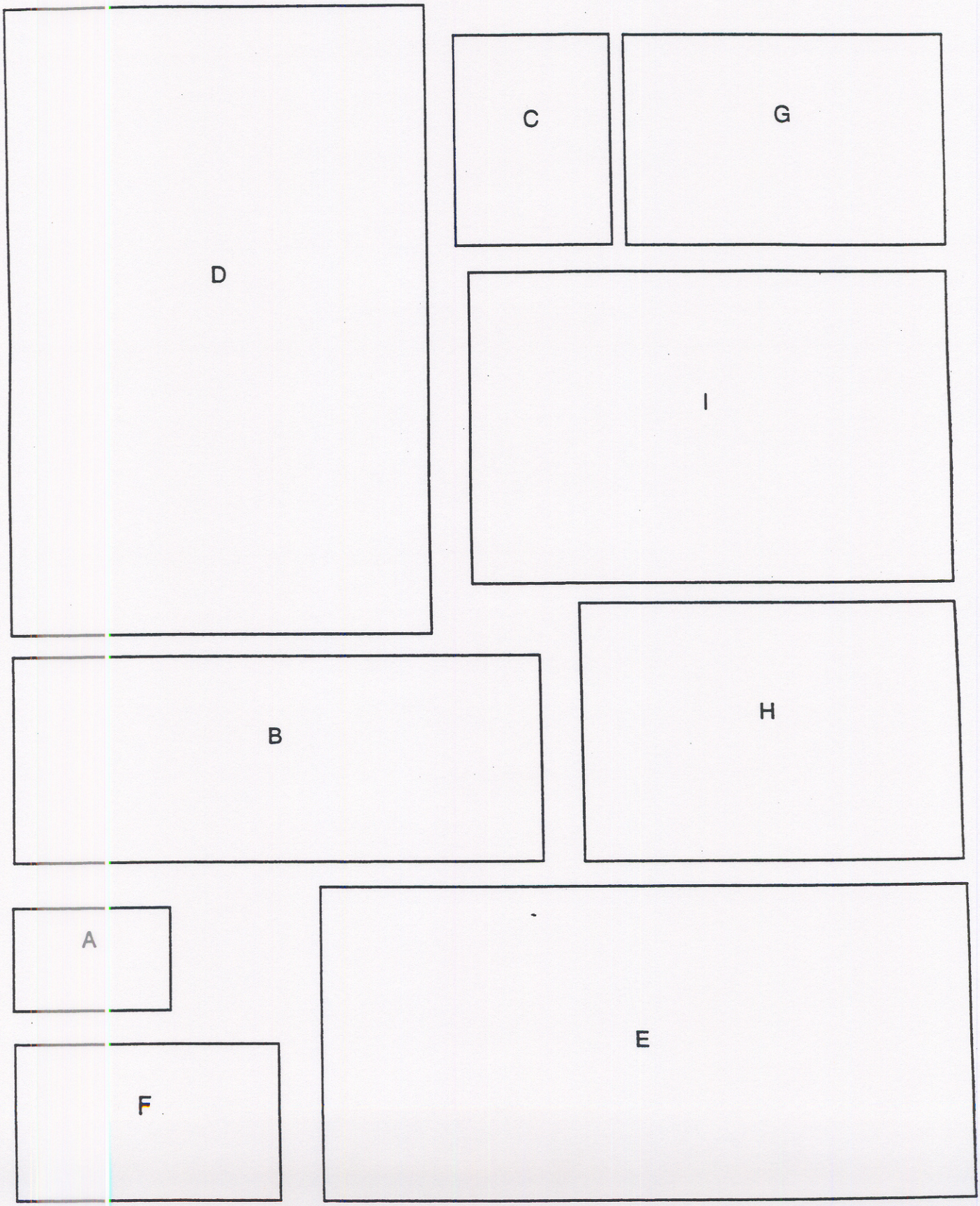
Activity 1



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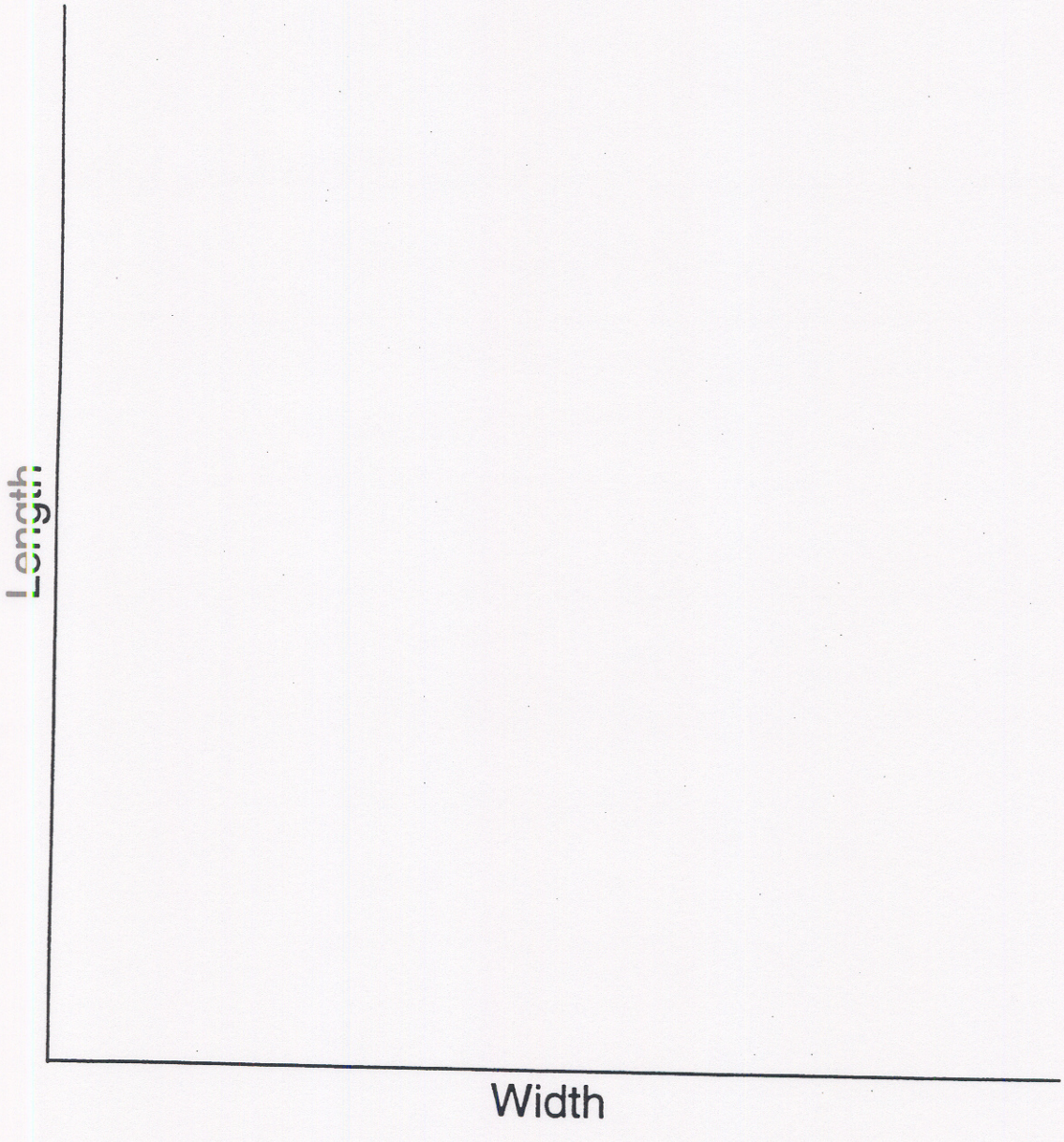
Guide for Activity 1



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Activity 2

Use a centimeter ruler to measure the length and width of each rectangle in Group 1 and record in the table below. Do the same for the rectangles in Group 2.

Table 1—Group 1 Data

Rectangle	W	L
A	2	3
G	4	6
I	6	9
D	8	12

Table 2—Group 2 Data

Rectangle	W	L
E	6	12
F	3	5
C	3	4
B	4	10
H	5	7

a. What is the ratio of L:W for the rectangles in Group 1?

b. Is this a constant ratio?

c. How does this ratio describe the "shape" of these rectangles?

d. Compare the ratios of L:W in Group 2. Is there a constant ratio? Explain.

Reason and Communicate:
Ask participants to explain what it means to have a constant ratio in comparing L to W for the two groups.

Have participants write an equation that expresses the relationship that exists among the rectangles in Group 1. They should write one of the equations $L=kW$ or $L/W=k$. Ask them to explain what k means in each equation.

Ask participants to compare the length and width of the smallest rectangle in Group 1 to the largest. Have someone explain how to describe the relative sizes of these two rectangles. They should note that the dimensions of the largest rectangle are c times as large as the dimensions of the smallest rectangle and that c is a scale factor that describes the relative sizes of the two rectangles.

Answers:
a. L:W = 3:2
b. Yes
c. The ratio of the length to the width determines the shape of the rectangle.
d. No, the ratios vary. The rectangles do not have the same shape.

Math Notes:
Ask participants to compare the data in Tables 1 and 2 and note any likenesses or differences. They should recognize that the ratio of L:W in Table 1 is a constant ratio, while the ratio L:W in Table 2 is not constant. The constant ratio in Table 1 can be represented by k, called the constant of proportionality. This constant describes the "shape" of every rectangle in Group 1. The equation $L/W = k$ can be expressed as $L = kW$, where L is a constant multiple of W. The constant of proportionality, k, can be thought of as the "shape ratio" for this set of similar rectangles (*Dick Stanley, Dana Center of The University of California at Berkley*).

By comparing the lengths and widths of the smallest and largest rectangles from Group 1, participants should recognize that the dimensions of the largest rectangle are 4 times those of the smallest rectangle. We can write an equation that relates the length L_2 of the largest rectangle to L_1, the length of the smallest rectangle as $L_2 = 4L_1$. Ask participants to write an equation relating the widths of the largest and smallest rectangle in Group 1 ($W_2 = 4W_1$). Have them do the same for any two rectangles from Group 1. Thus, for any two rectangles in Group 1, $L_2 = cL_1$ and $W_2 = cW_1$, where c is the *scale factor*. This constant c reflects the relative size of the dimensions of one rectangle when compared to the dimensions of another rectangle in our set of rectangles that have the same shape. (For example, $L_2 = 4L_1$ and $W_2 = 4W_1$ tell us that the dimensions of the largest rectangle are four times the dimensions of the smallest rectangle). The constant **c** can be thought of as the "size ratio" between these two rectangles. (*Dick Stanley, Dana Center of The University of California at Berkley*).

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Activity 2

Use a centimeter ruler to measure the length and width of each rectangle in Group 1 and record in the table below. Do the same for the rectangles in Group 2.

Table 1—Group 1 Data

Rectangle	W	L

Table 2—Group 2 Data

Rectangle	W	L

a. What is the ratio of L:W for the rectangles in Group 1?

b. Is this a constant ratio?

c. How does this ratio describe the "shape" of these rectangles?

d. Compare the ratios of L:W in Group 2. Is there a constant ratio? Explain.

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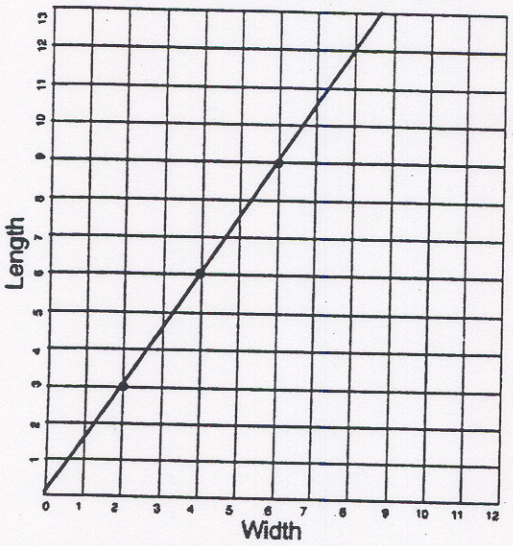
Activity 3—Group 1 Graph

Use the data from Table 1 to make a graph.



a. Use your ruler to draw a line through the points. What do you observe? Does the line pass through all the points? Does the line pass through the origin?

b. Would a rectangle with a length of 10 cm and a width of 8 cm belong in this group? Why or why not?

c. State the dimensions of another rectangle that would belong to Group 1.

Answers:

a. All the points should lie on a straight line passing through the origin.
b. No, because the coordinate (8,10) would not lie on this line.
c. Answers will vary. Example: 18 cm by 27 cm

Math Notes:

The graph of the data for the rectangles in Group 1 should lie on a straight line that passes through the origin. This result is consistent with a characteristic of a proportional relationship. (In other words, in this group of rectangles, the width is proportional to the length.) Ask participants to use this graph to give the dimensions of other rectangles that would belong to Group 1 and justify their answers. Have participants do a scatter plot of the data on a graphing calculator and a line of best fit. They could trace along the line or use a table to find dimensions of other rectangles that belong to Group 1, the group of similar rectangles that have the "same shape."

Reason and Communicate:

Ask participants to reflect on a previous activity where "groups of" objects were compared resulting in equivalent ratios (*Perfect Paint Color*). This activity builds upon the *Perfect Paint Color* activity by using "strips of three" and "strips of two" to establish a set of ordered pairs (W, L) whose ratios $\frac{L}{W}$ are equivalent.

Have participants reflect on the use of tables and graphs to represent the same data. The table helps one to make comparisons and find a constant ratio. This enables one to determine if there is a proportional relationship from a tabular form. By observing the graph of this relationship, participants should note that the vertices (W, L) from the table lie on a straight line that passes through the origin. Ask participants to state any observations about the equivalent ratios from their table and the ordered pairs graphed and write a definition of a ratio. They should observe that a ratio is an ordered pair of measurements.

While using the graph to interpret data, participants should be encouraged to consider questions like the following:
"Does a rectangle with dimensions W = 18 cm and L = 26 cm belong to Group 1? Why or why not?"
No, because the ratio of $\frac{L}{W}$ is not the same as the constant ratio $\frac{3}{2}$ for Group 1.

"What does the ordered pair (10,15) mean in the context of this problem?"
The width of the rectangle is 10 units and the length is 15 units.

"Given the ordered pair (W, 36), find the value of W so that the point will lie on the graph of the Group 1 data."
W=24

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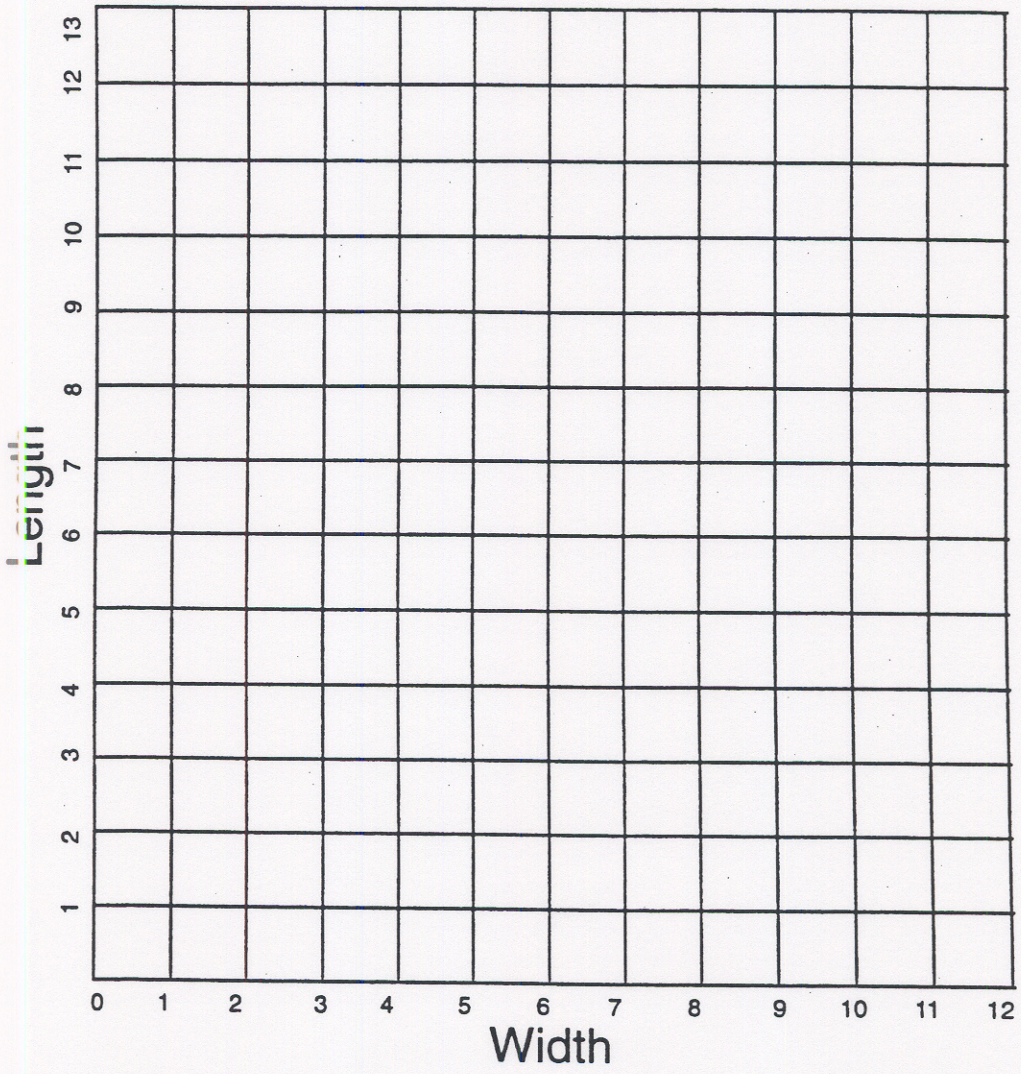
Activity 3—Group 1 Graph

Use the data from Table 1 to make a graph.



a. Use your ruler to draw a line through the points. What do you observe? Does the line pass through all the points? Does the line pass through the origin?

b. Would a rectangle with a length of 10 cm and a width of 8 cm belong in this group? Why or why not?

c. State the dimensions of another rectangle that would belong to Group 1.

Restless Rectangles

Activity 3—Group 1 Graph continued

d. Place each rectangle from Group 1 on the grid so that the length and width are aligned with the axes and trace. State your observations.
Cut out one 3 cm by 15 cm strip from one color of construction paper. Cut out another strip 2 cm by 15 cm from a different color. Use the wider strip to mark off 3 cm sections along the vertical axis. Use the other strip to mark off 2 cm sections along the horizontal axis.

e. How many groups of 3 cm are there in the length of rectangle A? rectangle G? How many groups of 2 cm are there in the width of rectangle A? rectangle G?

f. Use these strips to compare groups of 3 cm and 2 cm for the other rectangles in Group 1. Do these rectangles seem to be "growing" in the same way as you move from smallest to largest? Explain.

g. The dimensions of rectangle G are how many times the dimensions of rectangle A? This factor is called a scale factor.

h. Write an equation that describes the relationship between the length, L_2, of rectangle G and the length, L_1, of rectangle A. Write another equation that describes the relationship between the width, W_2, of rectangle G and the width, W_1, of rectangle A.

i. What seems to be the same in each equation? What is another name for this constant?

j. What does this constant describe about rectangle A and rectangle G? What would the ratio between their lengths describe about any two rectangles in Group 1?

k. Select any two rectangles from Group 1 (other than A and G) and identify the scale factor from one to the other for a "sizing up" and "sizing down" situation.

Answers:

d. The endpoints of a diagonal of each rectangle lie on the line.

e. Rectangle A: 1 group of 3 cm, 1 group of 2 cm
Rectangle G: 2 groups of 3 cm, 2 groups of 2 cm

f. Yes, the lengths and widths of any two rectangles in Group 1 "grow" by the same scale factor.

g. 2 times

h. $L_2 = 2L_1$, $W_2 = 2W_1$

i. In each equation, there is a constant factor 2 called the scale factor.

j. This constant "c" for any two rectangles in Group 1 is also referred to as the "size ratio".

k. Example: What is the scale factor from the rectangle with *w:l of 4:6* to the rectangle with *w:l of 10:15*?

$\frac{5}{2}$

What scale factor would represent a reduction of the rectangle with sides 10 cm and 15 cm to a rectangle with sides 2 cm and 3 cm?

$\frac{1}{5}$

Restless Rectangles

Activity 3—Group 1 Graph continued

d. Place each rectangle from Group 1 on the grid so that the length and width are aligned with the axes and trace. State your observations.
Cut out one 3 cm by 15 cm strip from one color of construction paper. Cut out another strip 2 cm by 15 cm from a different color. Use the wider strip to mark off 3 cm sections along the vertical axis. Use the other strip to mark off 2 cm sections along the horizontal axis.

e. How many groups of 3 cm are there in the length of rectangle A? rectangle G? How many groups of 2 cm are there in the width of rectangle A? rectangle G?

f. Use these strips to compare groups of 3 cm and 2 cm for the other rectangles in Group 1. Do these rectangles seem to be "growing" in the same way as you move from smallest to largest? Explain.

g. The dimensions of rectangle G are how many times the dimensions of rectangle A? This factor is called a scale factor.

h. Write an equation that describes the relationship between the length, L_2, of rectangle G and the length, L_1, of rectangle A. Write another equation that describes the relationship between the width, W_2, of rectangle G and the width, W_1, of rectangle A.

i. What seems to be the same in each equation? What is another name for this constant?

j. What does this constant describe about rectangle A and rectangle G? What would the ratio between their lengths describe about any two rectangles in Group 1?

k. Select any two rectangles from Group 1 (other than A and G) and identify the scale factor from one to the other for a "sizing up" and "sizing down" situation.

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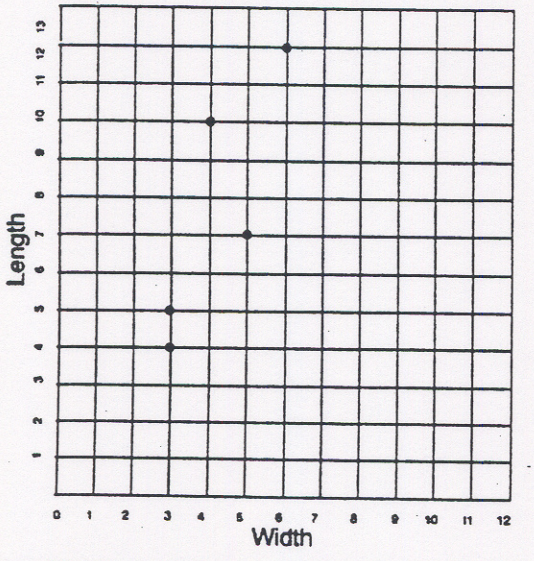
Activity 4—Group 2 Graph

Use the data from Table 2 to make a graph.



a. Is it possible to draw a line through these points like you did in Graph 1?

b. Place each rectangle from Group 2 on the grid and trace.

c. What do you observe about these rectangles? Are they "growing" in the same way? Explain.

Reason and Communicate:
Participants should observe that the rectangles in this group have diagonals that do not line up on the same line. Ask them what this means. Ask participants to explain how they could use Table 2 to help them determine if this is a proportional relationship.

Answers:
a. No
c. The dimensions of the rectangle are not growing in the same way. The diagonals do not lie on the same line.

Math Notes:
The graph of the data for Group 2 rectangles does not represent a straight line. One can conclude that the data for Group 2 does not represent a proportional relationship. This conclusion can also be verified by examining the data in Table 2. There is not a constant ratio when comparing L to W.

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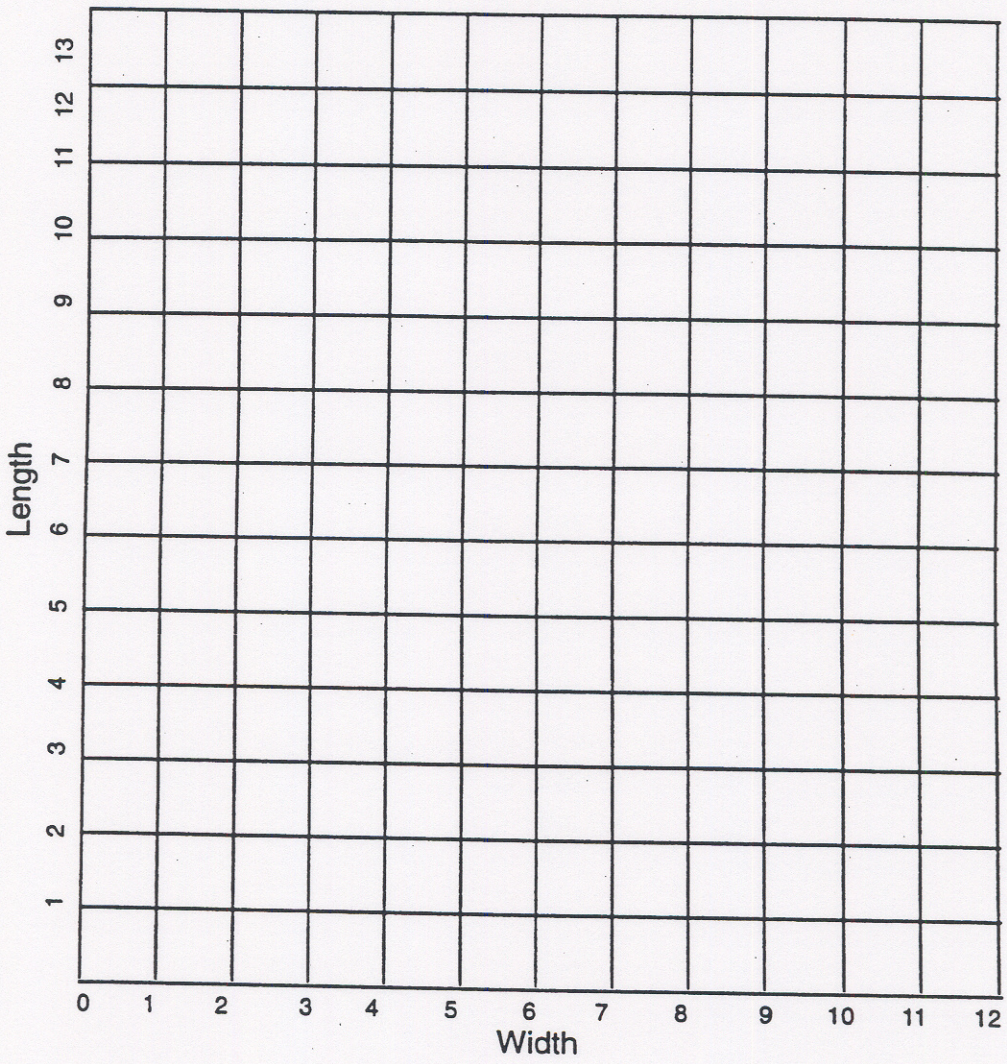
Activity 4—Group 2 Graph

Use the data from Table 2 to make a graph.



a. Is it possible to draw a line through these points like you did in Graph 1?

b. Place each rectangle from Group 2 on the grid and trace.

c. What do you observe about these rectangles? Are they "growing" in the same way? Explain.

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Activity 5

Use your Restless Rectangles to solve these problems involving a scale factor.

a. Identify the scale factor from rectangle G to rectangle C.

b. What is the scale factor from rectangle I to rectangle A? Did this scale factor create an enlargement or reduction from I to A?

c. Is there a scale factor from rectangle D to rectangle E? Explain.

d. Use centimeter grid paper to trace rectangle D. Then draw rectangle J so that the scale factor from rectangle D to rectangle J is 3. State the dimensions of rectangle J. How do these dimensions compare to those of rectangle D?

e. Trace rectangle C on centimeter grid paper. Next, draw Rectangle L so that the scale factor from rectangle C to rectangle L is 1/3. What are the dimensions of rectangle L? How do these compare to those of rectangle C?

f. If you use a scale factor (less than one) from figure R to figure S, then the result is a(n)_____. (enlargement , reduction)

g. If you use a scale factor (greater than one) from figure W to figure Z, then the result is a(n) _____. (enlargement, reduction)

h. If you use a scale factor of 1 from figure P to figure Q, then the result is a _____.

Answers:

a. There isn't one because the rectangles are not the same shape (i.e., they are not similar).

b. $\frac{2}{6} = \frac{1}{3}$, a reduction

c. No, because the two rectangles are not similar.

d. The dimensions of rectangle J are 24 units x 36 units; the dimensions of rectangle J are 3 times the corresponding dimensions of rectangle D

e. The dimensions of rectangle L are 1 unit x $\frac{4}{3}$ units;

the dimensions of rectangle L are 1/3 the length of the corresponding dimensions of rectangle C.

f. reduction

g. enlargement

h. congruent figure

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Activity 5

Use your Restless Rectangles to solve these problems involving a scale factor.

a. Identify the scale factor from rectangle G to rectangle C.

b. What is the scale factor from rectangle I to rectangle A? Did this scale factor create an enlargement or reduction from I to A?

c. Is there a scale factor from rectangle D to rectangle E? Explain.

d. Use centimeter grid paper to trace rectangle D. Then draw rectangle J so that the scale factor from rectangle D to rectangle J is 3. State the dimensions of rectangle J. How do these dimensions compare to those of rectangle D?

e. Trace rectangle C on centimeter grid paper. Next, draw Rectangle L so that the scale factor from rectangle C to rectangle L is 1/3. What are the dimensions of rectangle L? How do these compare to those of rectangle C?

f. If you use a scale factor (less than one) from figure R to figure S, then the result is a(n)_____. (enlargement , reduction)

g. If you use a scale factor (greater than one) from figure W to figure Z, then the result is a(n) _____. (enlargement, reduction)

h. If you use a scale factor of 1 from figure P to figure Q, then the result is a _____.