2.2 Compounding Interest

Overview: Participants investigate various scenarios of interest bearing bank accounts using power and exponential functions. Participants investigate continuously compounded interest accounts and the relationship to the number *e*.

Objective:

Algebra II TEKS
(f.5) The student analyzes a situation modeled by an exponential function formulates an equation or inequality, and solves the problem.

Precalculus TEKS
(c.3.A) The student is expected to use functions such as logarithmic, exponential, trigonometric, polynomial, etc. to model real-life data.

Terms: Compounding interest, *e*

Materials: Graphing calculators

Procedures:

Activity 1: Money Talk
This activity investigates the growth of a bank account balance when a constant yearly deposit is made and annual interest is earned. Participants will investigate the account numerically and develop a general formula for the growth.

1.

No. of years elapsed	Start of year balance	End of year interest	End of year deposit
0	$2,000.00	$80.00	$1,000
1	$3,080.00	$123.20	$1,000
2	$4,203.20	$168.13	$1,000
3	$5,371.33	$214.85	$1,000
4	$6,586.18	$263.45	$1,000
5	$7,849.63	$313.99	$1,000
6	$9,163.61	$366.54	$1,000

2. *Neither. It is not growing linearly because the change in balance from year to year is not constant. It is not growing exponentially because the ratio of successive balances is not constant.*

3. *Balance after one year is 2000 + 2000r + 1000 = 2000(1 + r) + 1000. After participants find the first expression above, suggest rewriting the expression using (1 + r) as shown above. Then have participants fill in the table in Exercise 4, continuing to use the (1 + r) quantity.*

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4.

In year n	Balance
0	2000
1	$2000 + 2000r + 1000 = 2000(1 + r) + 1000$
2	$\underline{[2000(1+r)+1000]} + \underline{[2000(1+r)+1000]}r + 1000$ $= \underline{[2000(1+r)+1000]}(1+r) + 1000$ $= 2000(1+r)^2 + 1000(1+r) + 1000$
3	$\underline{[2000(1+r)^2 + 1000(1+r) + 1000]} +$ $\underline{[2000(1+r)^2 + 1000(1+r) + 1000]}r + 1000$ $= \underline{[2000(1+r)^2 + 1000(1+r) + 1000]}(1+r) + 1000$ $= 2000(1+r)^3 + 1000(1+r)^2 + 1000(1+r) + 1000$
n	$2000(1+r)^n + 1000(1+r)^{n-1} + \ldots + 1000(1+r) + 1000$

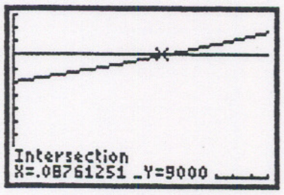
The underlined expressions above may help participants see the common factors in the expressions. You may wish to discuss the $(1+r)$ expression. This is the annual growth factor. For example, if the account earns 4% interest, $(1+r) = (1+0.04)$ or 100% + 4%. This represents 100% of the balance plus the interest, 4% of the balance. We could let $x = (1+r)$ to get a cleaner polynomial expression.

5. $p_n = 2000(1+r)^n + 1000(1+r)^{n-1} + \ldots + 1000(1+r) + 1000$
6. Graph the following functions on the graphing calculator:

$$y_1 = 2000(1+x)^5 + 1000(1+x)^4 + 1000(1+x)^3 + 1000(1+x)^2 + 1000(1+x) + 1000$$

$$y_2 = 9000$$

Find the intersection.

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WINDOW
 Xmin=0
 Xmax=.15
 Xscl=.01
 Ymin=0
 Ymax=12000
 Yscl=1000
 Xres=1
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The x-value, .088, *represents the APR. The account must earn at least* 8.8%.

Activity 2: Exploring *e*

This activity looks at accounts with various compounding time periods and the effects on the balance of the accounts. This process leads to the discovery of the number e. Have participants work in small groups on Exercises 1 – 4.

1. Assuming no other deposits or withdrawals, the 12% annual interest would result in:

   $\$1000 + \$1000(.12) = \$1000(1 + .12) = \$1000(1.12) = \$1120$.

   The account paying 1% interest every month, again assuming no other withdrawals or deposits can be explored using the chart below.

Month	Amount
1	\$1000(1.01)
2	(\$1000(1.01))(1.01)
3	\$1000(1.01)(1.01)(1.01)
12	\$1000(1.01)(1.01)(1.01)...(1.01) 12 months of 1% monthly interest

   So the result of the account paying 1% interest monthly is

   $\$1000(1.01)^{12} = \$1126.83$.

   - Why does the account that breaks the 12% interest into monthly payments end up with a larger balance at the end of a year? *The interest earned in the first month is added to the balance that the interest for the second month is calculated. This happens each month. This is what compounding means. The annual interest would also compound over the course of years, but the effect of the compounding is not seen in a single year.*

2. a. $\$1000(1.08)(1.08)(1.08) = \$1000(1.08)^3 = \$1259.71$.

   b. The annual rate of 8% is split into 12 months, so each month the interest rate is $\frac{.08}{12}$. At the end of 3 years, 36 interest payments will have been made. Therefore, the balance can be calculated as $\$1000\left(1+\frac{.08}{12}\right)^{36} = \$1270.24$.

   c. Now the annual rate is split into 52 weeks, so the interest rate per week is $\frac{.08}{52}$. After three years, $52 \bullet 3 = 156$ payments will have been made, resulting in a balance of: $\$1000\left(1+\frac{.08}{12}\right)^{156} = \$1271.01$.

   d. The balance in the account if the interest was paid daily would be: $\$1000\left(1+\frac{.08}{365}\right)^{3 \bullet 365} = \$1271.22$

- Look at the change in the balance amounts of the various compounding methods above. What do you notice? *The change from annual to monthly interest is fairly large, but from then on the change lessens. It seems to approach a limit of some sort.*
- How could you continue the list in question two in order to compound more frequently? What would (e), (f) or (g) be? *We could break the time down to hourly, by the minute or eventually to the limiting factor or continuously compounding interest.*

3. Have the participants share their formulas and discuss any differences. This formula should emerge from the work of the groups during Exercise 2: $A = P\left(1+\frac{r}{n}\right)^{nt}$. Another form of this equation is $A = P\left[\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{rt}$. Discuss with the group why these two equations are equivalent.

4. The initial deposit, P, and the annual interest rate, r, are constant and the compounding period, described by n, and the time, t change.

   As a group, consider the part of the formula above, $\left[\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]$. Let $x = \frac{n}{r}$.

   Then the expression becomes $\left[\left(1+\frac{1}{x}\right)^{x}\right]$.

5. Encourage participants to investigate the function in various manners. A starting window for the graph might be $0 \le x \le 1\ 00$, $0 \le y \le 3$. From this graph, the participants should notice that the values level off somewhere between 2 and 3. From there, the *ymin* and *ymax* of the window might be narrowed. The table feature can be used to see numerically the values approaching a limit. Allow participants to decide on a limiting value.

6. Most likely, participants will have decided that $f(x)$ approaches some limiting value of at least 2.6. The calculator will estimate $e^1 \approx 2.718281828$. Some participants may look at this value and assume that e is a repeating decimal. In fact, e does not repeat the pattern we see here. e, like π, is an irrational mathematical constant that appears in many applications, such as compounding interest.

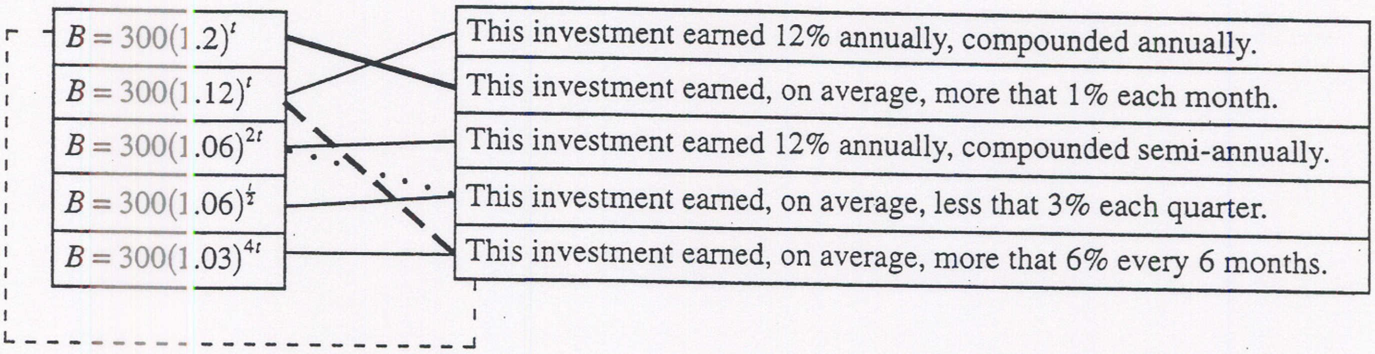
7. The table start entries below will vary from group to group. They are given here as a point of reference. The interesting point to make here is how slowly the values grow as x gets very large. Finding the x values for the last two estimates may surprise some participants.

Estimate of e	Table Start Value	$x=$
2.7	60	74
2.71	150	163
2.718	4050	4095
2.7182	10300	10309

8. Questions 5 – 7 lead us to a definition of e as $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$. So, as n approaches infinity, the formula $A = P\left[\left(1+\frac{1}{\frac{n}{r}}\right)^{\frac{n}{r}}\right]^{rt}$ approaches $A = Pe^{rt}$. This formula gives the amount in an account over time that is compounded continuously.

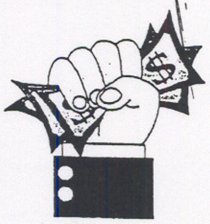
- What is the amount in the account if \$1,000 is invested at 8% compounded continuously? $1000e^{(3)(0.08)} = \$1{,}271.25$.

Answers to Reflect and Apply:



Summary: These activities give participants opportunities to explore the mathematics behind various types of banking situations. These can be modeled with modified power functions, polynomial functions, and exponential functions.

Activity 1: Money Talk



Suppose you open a bank account with $2000. Every year on the same date you deposit another $1000.

1. Suppose the account earns 4% annual interest, compounded annually. Fill in the table below.

Number of years elapsed	Start of year balance	End of year interest	End of year deposit
0	$2000.00	$80.00	$1000
1	$3080.00	$123.20	$1000
2	$4203.20		$1000
3			$1000
4			$1000
5			$1000
6			$1000

2. Describe how the balance in the account grows.

3. Suppose the account earns *R*% annual interest, compounded annually. Let *r* be the decimal form of this percent. Write an expression for the balance in the account after one year.

4. Fill in the chart below assuming the variable interest rate given in Exercise 3.

In year n	Balance
0	2000
1	$2000 + 2000r + 1000 = 2000(1 + r) + 1000$
2	
3	
n	

5. Using the chart from Exercise 4 write the equation which shows the relationship between the balance in the account and the interest rate.

6. Using the formula from Exercise 5 and a graphing calculator, determine the APR of the account in order for an investor to have a balance of \$9000 in 5 years.

Activity 2: Compounding Continuously

1. What is the difference between a bank account that pays 12% interest once per year and one that pays 1% interest every month?

2. Suppose $1000 is deposited into an account paying interest at a rate of 8% per year. Find the balance after 3 years if the interest is compounded
   a. annually
   b. monthly
   c. weekly
   d. daily.

3. Write a general formula that describes the relationship between the ending balance and the starting balance in the account.

4. In Exercise 2, which of the parameters described in Exercise 3 are constant and which are variable?

5. Investigate the following function using a graphing calculator.

$$f(x)=\left(1+\frac{1}{x}\right)^{x}$$

In this function, we can think of x as the compounding frequency described by n in our formula. If we move toward continuous compounding, the value of x would get larger and larger. Investigate the value of this function as x approaches infinity. Use the trace and table features of your calculator to help you with your investigation. Describe what happens to the value of $f(x)$ as x grows very large.

6. The limit of the function $f(x)=\left(1+\frac{1}{x}\right)^x$ as x approaches infinity defines a mathematical constant, e. Use a calculator to give an estimate of e by calculating e^1. What value do you get? How does this value compare to your conclusions in Exercise 5?

7. Using the table feature of your graphing calculator, explore what integer value of x first gives the following estimates of e.

Estimate of e	**Table Start Value**	$x =$
2.7		
2.71		
2.718		
2.7182		

8. How can our investigation of the number e be used in a formula for calculating compounding interest?

Reflect and Apply

Suppose that \$300 was deposited into one of five different bank accounts. Each of the equations on the left gives the balance of an account in dollars, as a function of the number of years elapsed, t. On the right are verbal descriptions of five different situations. Match each situation with the equation or equations it could possibly describe.

$B = 300(1.2)^t$	This investment earned 12% annually, compounded annually.
$B = 300(1.12)^t$	This investment earned, on average, more that 1% each month.
$B = 300(1.06)^{2t}$	This investment earned 12% annually, compounded semi-annually.
$B = 300(1.06)^{\frac{t}{2}}$	This investment earned, on average, less that 3% each quarter.
$B = 300(1.03)^{4t}$	This investment earned, on average, more that 6% every 6 months.