Chapter 10:

Inverse Variations, Exponential Functions, and Other Functions

College Tuition

In 1980 the average annual cost for tuition and fees at two-year colleges were $350. Since then, the cost of tuition has increased an average of 9% annually.

1. Create a function rule that models the annual growth in tuition costs since 1980. Identify the variables, and describe the dependency relationship.
2. Determine the average annual cost of tuition for 2001. Justify your answer using tables and graphs.
3. Predict the cost of tuition for the year you plan to graduate from high school.
4. When will the average cost be double the 1980 cost?
5. When will the average cost reach $1000?

Teacher Notes

Scaffolding Questions:

- Identify the variables in this situation.
- Represent the annual growth factor as a decimal.
- What is the starting value?
- Describe how you might create a table to help you determine the rule for this situation.
- Describe how tuition amounts change with each additional year.
- Create a scatterplot of the data in your table and describe the graph.
- Determine the function rule that shows the relationship between the cost of tuition and the number of years since 1980.

Sample Solution:

1. The cost of tuition depends on the number of years since 1980. Each year the tuition is the previous year's tuition plus nine percent of the previous year's tuition. This may be thought of as 100% + 9% of the previous year's tuition. This is equivalent to multiplying by 1 + .09 or 1.09. To find the cost after one year, 1.09 is multiplied with the starting amount ($350). To find the cost after 2 years, multiply 1.09 by itself and then multiply that result by the starting amount ($350).

Number of Years Since 1980	Pattern	Tuition Cost
0	350	350
1	$350 \cdot 1.09^1$	381.50
2	$350 \cdot 1.09^2$	415.84
3	$350 \cdot 1.09^3$	453.26
4	$350 \cdot 1.09^4$	494.05
5	$350 \cdot 1.09^5$	538.52
n	$350 \cdot 1.09^n$	$350 \cdot 1.09^n$

The function created by this situation is $T = 350 \cdot 1.09^n$.

The n values represent the number of years since 1980. The T values represent the cost of tuition. The minimum for the range is $350.

There was no constant rate of change in the table. Therefore, the relationship is nonlinear.

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

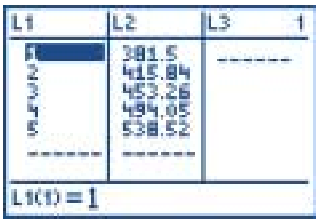
(E) interprets and makes inferences from functional relationships.

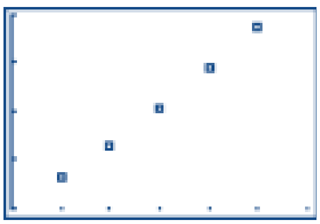
(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

2. The situation may be represented graphically and tabularly using a graphing calculator. The data may be plotted using the statistic feature on the calculator. Let *y* represent tuition cost and *x* represent the number of years.





The graph is an exponential curve showing growth. The starting amount is $350, and the growth factor is 1.09. The growth factor shows 100% of the initial cost plus 9% of the cost. To graph the function, let *y* represent the tuition cost and let *x* represent the number of years.

Use the table feature on the calculator to explore cost increases since 1980.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(d.3) Quadratic and other nonlinear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(A) uses patterns to generate the laws of exponents and applies them in problem-solving situations.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
3 Exponential Functions and Equations
3.1 Exponential Relationships
3.2 Exponential Growth and Decay
3.3 Exponential Models

Connections to Algebra End-of-Course Exam:

Objective 6:
The student will perform operations on and factor polynomials that describe real-world and mathematical situations.

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

Objective 8:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving one-variable or two-variable situations.

Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

X	Y_1
18	1651
19	1799.6
20	1961.5
21	2138.1
22	2330.5
23	2540.3
24	2768.9

X=21

The function can also be used to find the cost in the year 2001. 21 years would be used for the value of *x* minus the number of years since 1980. The cost would be $\$350 \cdot 1.09^{21} = \$2138.08$.

3. Answers will vary.

4. To determine when the average cost will be doubled, find the table value that is least 2 times 350.

X	Y_1
5	538.52
6	586.99
7	639.81
8	697.4
9	760.16
10	828.58
11	903.15

X=8

The costs will be doubled by the ninth year.

5. Look on the table for *y* values that are at least 1000.

X	Y_1
9	760.16
10	828.58
11	903.15
12	984.43
13	1073
14	1169.6
15	1274.9

X=13

The cost will reach \$1000 by the 13th year or by 1993.

Extension Questions:

- Describe the domain and range for this function.

The domain is the set of whole numbers. The domain represents the number of years since 1980. The years can only be represented in whole numbers because the increase is calculated on a yearly basis. The range represents the cost of the tuition. For this problem, the year 1980 is a starting point, and the tuition that year was $350. The cost will continue to increase at a rate of 9% per year for as long as the school is operating. The range for this problem is y greater than or equal to 350.

- How would the graph change if the annual cost increase was 12%? How would this affect costs?

The graph would be "skinnier" indicating a steeper rise per year. Cost of tuition would increase at a faster rate.

- Predict the annual cost of tuition in the year 2010 at the current growth rate.

$y = 350 \cdot 1.09^{30}$ *would equal approximately $4643.69 per year.*

Constructing Houses

Robert is part of a volunteer crew constructing houses for low-income families. It always takes 200 individual workdays to complete one house. For example, if a crew of 20 people can complete a house in 10 days, it has taken 20 times 10 individual workdays.

1. Working at the same rate, how long should it take a crew of 40 people to build the house?

2. Express the number of workdays as a function of the crew size. Define the variables, and explain how you created your function. What type of relationship is formed in the situation?

3. Write a verbal description of the effect of the crew size on the number of construction days.

4. About how long would it take a crew of 32 to complete a house?

5. If a crew can complete a house in 12.5 days, how big was the crew?

6. Express crew size as a function of the number of workdays. Compare the domain, range, and graph with your original function in question.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

Scaffolding Questions:

- How long will it take one person to complete the home by him/herself?
- Explain how a crew of 20 can complete the home in 10 days.
- Identify the variables.
- Describe how you might create a table to help you determine the rule for this situation.
- How long will it take a crew of 2 to complete the house?
- How long for a crew of 4 to complete the house?

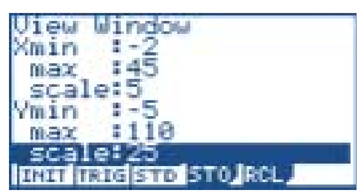
Sample Solution:

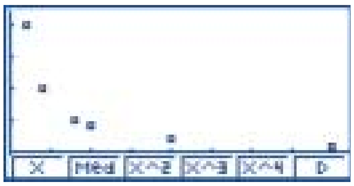
1. Use a table to determine how long it will take 40 people to complete the house.

Crew Size (x)	Construction Days (y)	Individual Workdays (t)
2	100	200
4	50	200
8	25	200
10	20	200
20	10	200
40	5	200

The number of construction days, y, can be found by dividing the total number of days by the value of x, the crew size. The product of the crew size and the construction days always equals 200 individual workdays.

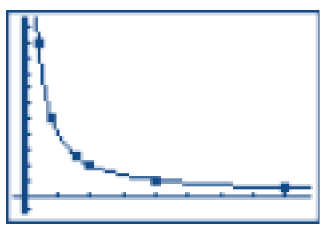
By entering the values for the crew size, x, and construction days, y, into two lists on the graphing calculator, the points can be plotted, revealing a nonlinear graph.





2. The function for this situation is $xy = 200$ or $y = \frac{200}{x}$, where x represents the crew size and y represents the number of construction days. The graph of this relationship is nonlinear.

3. As one quantity increases, the other decreases. The product of the quantities remains constant (200 total individual workdays) and forms an inverse variation. This constant product is called the constant of variation.



4. The table feature on the calculator allows exploration of the number of days it will take to build the house with various size crews.

It should take a crew of 40 people 5 days to build the house. The bigger the crew, the less time it will take to build the house. It will take 32 crew members about 6.25 days to build the house.

These values might also be verified on the home screen of the calculator by substituting into the function for each situation.

(d.3) Quadratic and other nonlinear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(B) analyzes data and represents situations involving inverse variation using concrete models, tables, graphs, or algebraic methods.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations of Functions
1 Developing Mathematical Models
1.1 Variables and Functions

Connections to Algebra End-of-Course Exam:

Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

5. If the number of days, y, is 12.5, the equation may be used to solve for x, the number of crew members.

$$x(12.5) = 200$$
$$x = \frac{200}{12.5} = 16$$

It will take 16 crew members to complete the work in 12.5 days.

6.

$$y = \frac{200}{x}$$
$$xy = 200$$
$$x = \frac{200}{y}$$

y is the independent variable, the number of workdays, and x is the dependent variable, the number of crew members.

The graph of the functions are the same as for $y = \frac{200}{x}$. However, for the problem situation, the x values must be whole numbers, since one may not use a portion of a person to build the houses. The y values will be determined by the whole number values. For example, for the original function $y = \frac{200}{x}$ the following values would be determined by the rule and the problem situation.

Crew Size	Number of Workdays
1	200
2	100
3	$200 \div 3 = 66.7$
4	50
5	40
6	$200 \div 6 = 33.3$
7	$200 \div 7 = 28.6$

Extension Questions:

- Describe how the values of *y* change as the values of *x* increase.

As the value of x increases, the value of y decreases.

- How do the values of the crew size and the construction days relate to the total number of workdays?

The product of the crew size and the number of construction days equals the total workdays, 200.

- Can your rule be written another (equivalent) way?

$xy = t$ or $y = t/x$

- Describe the graph of this relationship.

The graph is nonlinear. The graph is a firs-quadrant graph that begins at (1,200) and decreases to curve along the x-axis near (200,1).

- If the total number of construction workdays was 160, how would the equation have been written?

$xy = 160$ or $y = \frac{160}{x}$

Student Work

1. $20 \cdot 10 = 200$

$40x = 200$

$x = 5$

5 days

2. d = number of days

p = number of people

The amount of days multiplyed by the number of people always equals 200. This is a dependent function.

$dp = 200$

$d = \frac{200}{p}$

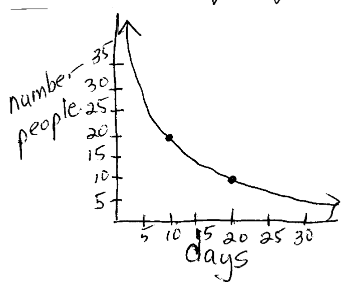
3. As the number of people increases the number of "days" decreases.

4. $32d = 200$ $\quad 32\overline{)200}$ $6\frac{1}{4}$

6.25 days

5. $12.5x = 200$

16 people



6. $p = \frac{200}{d}$

The domain, range, and graph are the same as the original function #2

Exploring Exponential Functions

A rectangular sheet of notebook paper is folded in half. The fold divides the paper into 2 rectangles. The folded paper is then folded in half again. When it is opened, there are 4 rectangles formed by the folds. Take a sheet of notebook paper and repeat this process. Record your answers in the table similar to the one shown below. Continue folding until you cannot make another fold.

Number of Folds	Number of Rectangles
0	1
1	2
2	4
3	
4	
5	

1. Identify the variables for this relationship and describe the domain and range for this situation. Describe how the number of rectangles changes as the number of folds increase.

2. Express symbolically the relationship between the variables.

3. If your paper had 128 rectangles, how many folds would you have made? Explain your answer.

4. Describe the graph of your data.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(D) in solving problems, collects and organizes data, makes and interprets scatterplots, and models, predicts, and makes decisions and critical judgments.

Scaffolding Questions:

- What is the relationship between the original number of rectangles and the number of rectangles after one fold?
- What is the relationship between the number of rectangles after one fold and the number of rectangles after two folds?
- What do you think the relationship will be between the number of rectangles after two folds and the number of rectangles after 3 folds?
- What pattern do you notice in the number of rectangles as the number of folds increase?
- What would you have to do to the original number of rectangles to get the second number of rectangles?
- How do the exponents relate to the values in your table?
- Suppose you could continue to fold the paper and extend your table to include 10 folds. How many rectangles would there be?

Sample Solution:

The paper was folded and the number of rectangles formed after each fold was recorded in the table. There was no constant change in the number of rectangles, therefore, the situation was not be linear. The values for the rectangles were all multiples of 2. The number of factors was the same as the number of folds, and a pattern with exponents emerged.

Successive ratios were constant.

Folds	Rectangles	Process
0	1	
1	2	$2 = 2^1$
2	4	$2 \cdot 2 = 2^2$
3	8	$2 \cdot 2 \cdot 2 = 2^3$
4	16	$2 \cdot 2 \cdot 2 \cdot 2 = 2^4$
5	32	$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$

Ratios

$2/1 = 2$
$4/2 = 2$
$8/4 = 2$
$16/8 = 2$
$32/16 = 2$

1. The n-values would represent the number of folds, and the r-values would represent the number of rectangles. The domain represents the number of folds, so n = 0, 1, 2, 3, 4, 5 …. There will be a limit to the number of

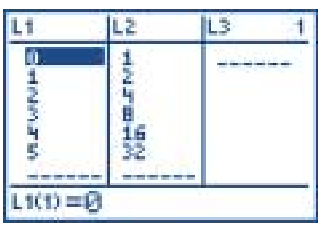
folds, depending on the type of paper. The range represents the number of rectangles formed by the folds. The range is r =1, 2, 4, 8, 16, 32….

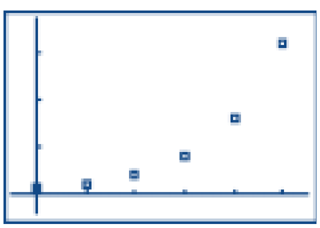
2. The pattern involves repeated multiplication by 2. You could show the pattern as powers of 2.

   $r = 2^n$

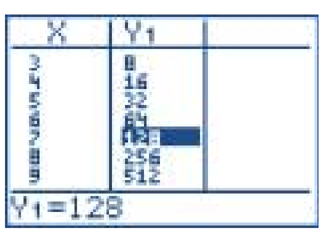
   As the value of n increases by 1, the value of r increases by a factor of 2.

3. In the graphing calculator let the *y*-values represent the number of rectangles. The *x*-values represent the number of folds.





The function rule and the table can help find the number of folds when the number of rectangles (y) was 128. It took 7 folds to get 128 rectangles.



(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(d.3) Quadratic and other nonlinear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(A) uses patterns to generate the laws of exponents and applies them in problem-solving situations;

(C) analyzes data and represents situations involving exponential growth and decay using concrete models, tables, graphs, or algebraic methods.

4. The graph is nonlinear because the rate of change is not constant. The rate of change of successive terms is a constant ratio of 2. The graph is in the first quadrant and curves upward. As the value of *x* increases, the value of *y* increases exponentially.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
2 Using Patterns to Identify Relationships
2.2 Identify More Patterns

III. Nonlinear Functions
3 Exponential Functions and Equations
3.1 Exponential Relationships

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Objective 2:
The student will graph problems involving real-world and mathematical situations.

Extension Questions:

- Describe how the situation would be different if the paper had been folded into thirds each time instead of halves.

The multiplier would be 3. The equation would be $r = 3^n$.

- Suppose that you began with a square sheet of paper that was 2 feet on a side. What is the relationship between the number of folds and the area of the rectangle after each fold?

The area of the sheet of paper is 4 square feet. Each time the paper is folded the area is multiplied by one-half.

Fold Number	Area of Rectangle (ft^2)
0	4
1	$4\left(\frac{1}{2}\right)$
2	$4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
3	$4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$
n	$4\left(\frac{1}{2}\right)^n$

The area, A, would be a function of the number of folds.

$$A = 4\left(\frac{1}{2}\right)^n$$

- Explain how the graph of this function, $A = 4\left(\frac{1}{2}\right)^n$, would be different from the graph of the original function, $A = 2^n$.

The graph of the function $A = 4\left(\frac{1}{2}\right)^n$ decreases as n gets larger. The starting value of the function is 4. The function, $A = 2^n$, increases as n increases. It has a starting value of 1.

Function Families

1. For each of the following sets of functions, compare the domains and ranges of the functions in that set. What representation best helps you see the domain and range of each function? Explain.

Set A:	Set B:	Set C:
$f(x) = 2x - 5$	$f(x) = \frac{1}{2}x^2$	$f(x) = 2^x$
$g(x) = 4$	$g(x) = -3x^2$	$g(x) = \left(\frac{1}{2}\right)^x$
$h(x) = 8 - \frac{1}{2}x$	$h(x) = 2x^2 + 5$	$h(x) = -3^x$

2. Describe the family of functions you see in each set and its parent function. Explain how each function is obtained from its parent function.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(B) for a variety of situations, identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(c.1) Linear functions.
The student understands that linear functions can be represented in different ways and translates among their various representations.

The student:
(B) determines the domain and range values for which linear functions make sense for given situations; and

(C) translates among and uses algebraic, tabular, graphical, or verbal descriptions of linear functions.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations.

Scaffolding Questions:

- What do the functions in each set have in common?
- How are the functions in each set different?
- What values, if any, make each function undefined?
- What would the graph of each function look like?
- What are the intercepts for each function?
- How are the functions from Set A to Set B to Set C different?

Sample Solution:

1. The domains and ranges of the functions in the problem sets could be seen by graphing the functions or by analyzing the expressions defining the functions.

   Set A:
   Since the functions are all linear, the domains are the set of all real numbers. There is no value of x that makes the function undefined. The ranges for functions f and h are also the set of all real numbers. If any real number is chosen as a function value, the resulting equation can be solved for the x-value that generates that function value. The function, g, however, is a constant function and has only one value in its range, 4. The function means "$g(x) = 4$ for any x-value you choose."

   Set B:
   These functions are all quadratic. Each domain is also the set of all real numbers because there is no x-value that makes the function undefined. The range for f is the set of all nonnegative real numbers, since squaring any real number and multiplying it by $\frac{1}{2}$ gives a nonnegative real number.

   x = any real number $\longrightarrow x^2 \geq 0$, making $\frac{1}{2}x^2 \geq 0$.

   The range for g is the set of all nonpositive real numbers, since squaring any real number gives a nonnegative real number. Multiplying that by -3 gives a nonpositive real number.

   x = any real number $\longrightarrow x^2 \geq 0$, making $-3x^2 \leq 0$.

The range for *h* is the set of all real numbers greater than or equal to 5, since $2x^2$ is nonnegative for any number *x*. Adding 5 to the equation gives a number greater than or equal to 5.

x = any real number $\rightarrow x^2 \geq 0$, $2x^2 \geq 0$, and $2x^2 + 5 \geq 5$.

Set C:
These functions are exponential, and, since any real number can be used as an exponent, the domain for each function is the set of all real numbers. The range for both functions *f* and *g* is the set of positive real numbers. However, the range for *h* is the set of negative real numbers, since using any real number *x* as an exponent of 3 gives a positive real number. Multiplying that by -1 gives a negative real number.

$-3^x = -1 \cdot 3^x$ and $3^x > 0 \rightarrow -1 \cdot 3^x < 0$.

The representation that best shows the domain and the range of each function is the graph of the function, because the graph gives a complete picture of input and output, intercepts, minimum or maximum points, where the function increases or decreases, and where it has minimum or maximum values.

2. *Set A:*
Since all the functions in this set are linear, the parent function is $y = x$.

- To obtain $f(x) = 2x - 5$ the parent function is vertically stretched by 2 and translated down 5 units. The slope of this line is positive 2; therefore, the line goes from quadrant III to quadrant I.
- $g(x) = 4$ is a horizontal line through $y = 4$.
- $h(x) = 8 - \frac{1}{2}x$ is obtained by vertically compressing the parent function by $\frac{1}{2}$ and translating up 8 units. The slope of this line is negative $\frac{1}{2}$; therefore, the line goes from quadrant II to quadrant IV.

Texas Assessment of Knowledge and Skills:

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of the quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

II. Linear Functions
1 Linear Functions
1.1 The Linear Parent Function
1.2 The Y-Intercept

II. Nonlinear Functions
1 Quadratic Functions
1.2 Transformations

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Set B:
All of the functions in this set are quadratic, so the parent function is $y = x^2$.

- $f(x) = \frac{1}{2}x^2$ is formed by vertically compressing the parent function by $\frac{1}{2}$.

- $g(x) = -3x^2$ is obtained by vertically stretching the parent function by 3 and then reflecting it over the x-axis.

- $h(x) = 2x^2 + 5$ is made by vertically stretching the parent function by 2 and translating up 5 units.

Set C:
The functions in this set are exponential of the form $f(x) = a^x$. There is not a parent function for this set of functions. Each of these functions could be called parent functions.

Extension Questions:

- How will knowing the parent function help you determine the domains and ranges of the functions in each set?

By knowing what the parent function looks like, we can see its domain and range. We can then analyze the transformed functions in each set to see how the domain and range might need to change. In other words, we can analyze the parameter changes and the effects of those changes on the domain and/or range.

- Consider the parent function for Set A and the point (2,2) on its graph. How does $f(x) = 2x – 5$ transform this point?

The point (2,2) is transformed to the point (2,-1) because 2x "stretches" (2,2) to (2,4) and subtracting 5 translates (2,4) down to (2,-1).

- In Set A, if we relabel $f(x) = 2x - 5$ as $y = 2x - 5$, how can we show symbolically that this function generates every real number as a y-value?

We need to show that for any y-value we choose, we can find an x-value that generates it. To see what that x-value should be, we can solve $y = 2x - 5$ for x:

$$\begin{aligned} y &= 2x - 5 \\ y + 5 &= 2x \\ \frac{y+5}{2} &= x \end{aligned}$$

No matter what value we choose for y, we can find the value for x that will generate it. We can also see that the expression $\frac{y+5}{2} = x$ is always defined, no matter what y is.

- In Set B, how does the coefficient of x^2 affect the parent function?

The coefficicient of x^2 multiplies the y values of the function. The graph is stretched if the absolute value of the multiplier is greater than 1. If the absolute value of the multiplier is less than one, the graph is compressed.

- For Set C, for $f(x) = 2^x$, how would the graph of $g(x) = \frac{1}{2}2^x$ compare to $f(x)$?

The values of y would be one-half of the values of f(x).

Mathematical Domains and Ranges of Nonlinear Functions

I. For the following problems:

   A. Sketch a complete graph for the given function. Show the coordinates of any intercepts.

   B. Describe the domain and range for each mathematical situation.

Function	Graph or Table	Domain and Range
1. $f(x) = \frac{1}{2}x^2$		Domain: Range:
2. $y = x^2 + 3$		Domain: Range:
3. $y = -3x^2$		Domain: Range:
4. $y = x(5 - x)$		Domain: Range:

Function	Graph or Table	Domain and Range
5. $h(x) = 3^x$		Domain: Range:
6. $m(x) = \left(\frac{1}{3}\right)^x$		Domain: Range:
7. $g(x) = \frac{4}{x}$		Domain: Range:

II. Write a summary comparing the functions. Compare their domains and ranges and their graphs.

III. Describe a practical situation that the functions in #4, #5, #6 might represent. What restrictions will the situation make on the mathematical domain and range of the function? How will the situation affect the graph of the mathematical function?

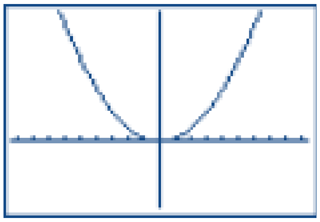
Teacher Notes

Scaffolding Questions:

- What type of function relates the variables?
- What is the dependent variable?
- What is the independent variable?
- What are the constants in the function? What do they mean?
- What restrictions does the function place on the independent variable?
- What is a reasonable domain for the function?
- What is a reasonable range for the function?

Sample Solution:

I. 1.



The *y*-intercept and the *x*-intercept for this function are the same, (0,0).

The domain for this function is the set of all real numbers, since any value for *x* can be squared and multiplied by $\frac{1}{2}$.

The range for this function is the set of all nonnegative numbers, which is the result of squaring any number and multiplying by $\frac{1}{2}$.

$x^2 \geq 0$

$\frac{1}{2}x^2 \geq 0$

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

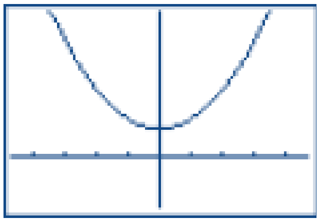
The student:
(B) for a variety of situations, identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations;

(C) interprets situations in terms of given graphs or creates situations that fit given graphs.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations.

I. 2.



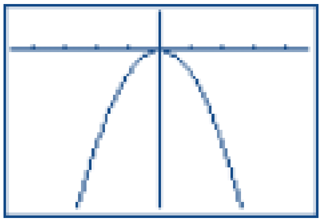
The y-intercept for this function is (0,3). There is no x-intercept because the graph does not intersect the x-axis.

The domain for this function is the set of all real numbers, since any value for x can be squared and increased by 3.

The range for this function is the set of all numbers greater than or equal to 3, since

$x^2 \geq 0$
$x^2 + 3 \geq 3 \geq 0$

I. 3.



The y-intercept and the x-intercept for this function are the same, (0,0).

The domain for this function is the set of all real numbers, since any value for x can be squared and multiplied by -3.

The range for this function is the set of all nonpositive numbers, which is the result of squaring any number and multiplying by -3.

$x^2 \geq 0$
$-3x^2 \leq 0$

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
2 Using Patterns to Identify Relationships
  2.2 Identify More Patterns

III. Nonlinear Functions
1 Quadratic Functions
  1.1 Quadratic Relationships
2 Quadratic Equations
  2.1 Connections
  2.2 The Quadratic Formula
3 Exponential Functions and Equations
  3.1 Exponential Relationships
  3.2 Exponential Growth and Decay
  3.3 Exponential Models

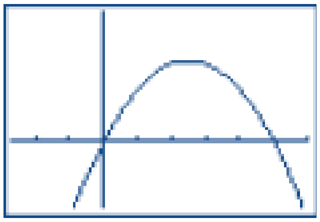
Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Objective 2:
The student will graph problems involving real-world and mathematical situations.

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

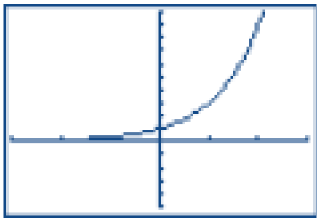
I. 4.



The *y*-intercept for this function is (0,0). The graphs cross the *x*-axis at two points so that the *x*-intercepts are (0,0) and (5,0). The domain for this function is the set of all real numbers, since the expression $x(5 - x)$ is always defined for any value of *x*.

To determine the range, trace along the graph to find the largest y value. It occurs at the point (2.5,6.25). The range for this function is the set of all numbers less than or equal to 6.25. This maximum *y*-value occurs when at $x = 2.5$, which is the average of the two *x*-intercept values, 0 and 5.

I. 5.

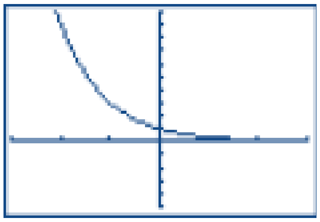


The *y*-intercept for this function is (0,1) because the value of the function at 0 is 1. There are no *x*-intercepts, since no power of 3 equals zero. The graph does not cross the *x*-axis.

The domain for this function is the set of all real numbers, since any real number can be used as an exponent on 3.

The range for this function is the set of all positive real numbers, since powers of 3 are always positive. Negative powers of 3 give *y* values between 0 and 1, and positive powers of 3 give values greater than 1.

I. 6.

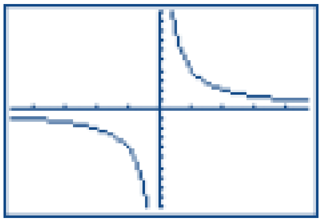


This function is the reflection of the graph in problem 5 about the *y*-axis. The *y*-intercept for this function is still (0,1). There are no *x*-intercepts, since no power of one equals zero.

The domain for this function is the set of all real numbers, since any real number can be used as an exponent.

The range for this function is the set of all positive real numbers, since powers of a positive number are always positive. Negative powers of a positive number give values more than 1, and positive powers of a negative number give values between 0 and 1.

1. 7.



The graph does not cross either axis. This function has no *y*-intercept, since division by 0 is undefined. There is no *x*-intercept, since $0 = \frac{4}{x}$ does not have a solution. The domain for this function is the set of all real numbers except 0.

The range for this function is the set of all real numbers except 0.

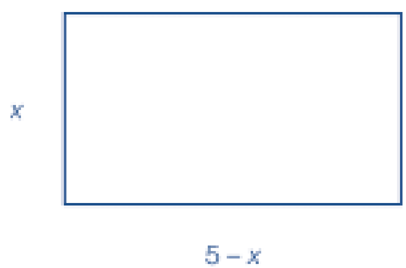
II. Comparison of the functions:

The first four of these functions are quadratic functions. They all have the same domain, the set of all real numbers. Their ranges vary because the expression ax^2 is nonnegative if $a > 0$ and is nonpositive if $a < 0$. The expression $x^2 + c$ is greater than or equal to c for any c.

The functions in #5 and #6 are exponential functions. They have the same domain and the same range.

The function in #7 is an inverse variation function with both domain and range being the set of all real numbers except zero.

III. A practical situation that $y = x(5 - x)$ could represent is the area of a rectangle.



The domain would be all numbers between 0 and 5 because the numbers less than 0 or greater than 5 result in a negative product, and the area may not be negative. The range would be the numbers between 0 and the maximum value 6.25. 6.25 would be included in this range. The graph would be that part of the parabola above the x-axis.

A practical situation that $h(x) = 3^x$ could represent is the number of rectangles formed by folding a rectangular sheet of paper repeatedly into thirds. The domain would be the number of folds made, which must be the set of whole numbers {0,1,2,3,4….}. The range would be the number of rectangles formed by the folds. The number of rectangles is a power of 3, {1,3,9,27,….}. The graph would be only those points on the original graph with nonnegative integer coordinates.

For a practical situation that might represent $m(x) = \left(\frac{1}{3}\right)^x$ consider the area of the original sheet of paper as one square unit; the function

$m(x) = \left(\frac{1}{3}\right)^x$ would represent the area of each rectangle resulting from the folds. The domain is the set of the possible number of folds {0,1,2,3,4…}, and the range is $\left\{1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots\right\}$. The graph would be the points on the original graph where *x* equals a nonnegative integer.

Extension Questions:

- Compare the range of $f(x) = \frac{1}{2}x^2$ and $f(x) = \frac{1}{2}x^2 + 3$.

The range of the first function is the set of all numbers greater than or equal to 0. If three is added to all values of the function, the range will be the set of all numbers greater than or equal to 3.

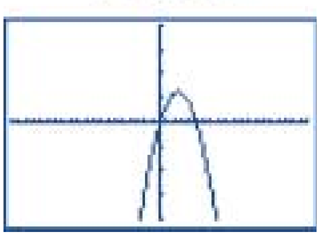
- Compare the range of $f(x) = \frac{1}{2}(x - 3)^2$ to the range of $f(x) = \frac{1}{2}x^2$.

The function will be shifted 3 units to the left. The vertex will change, but the range values will still be all the numbers greater than or equal to 0.

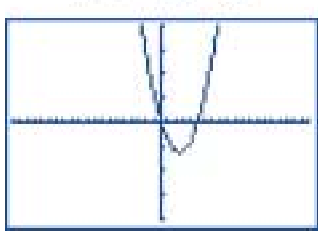
- If the function $y = x(5 - x)$ is multiplied by -1, how will the domain and range be affected?

The domain of y = x(5 – x) and y = -1x(5 – x) will both be the set of all real numbers because these products are defined for any x. However, the ranges will be different. The range of the first function is y < 6.25. The range of the second function will be y > -6.25 because all values of the function are multiplied by -1.

$y = x(5 - x)$

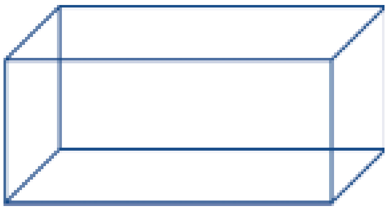


$y = -1x(5 - x)$



Paper Boxes

Marcy used a sheet of 8-inch by 11-inch paper to make an open box. She cut x by x squares out of each corner, and folded up the sides. The diagram below shows the finished box. Simplify your expressions and justify your answers to each of the questions.



1. Draw a diagram of the sheet of paper showing the fold lines needed to make the box. Label your diagram with the dimensions of the cut out pieces and the lengths of the fold lines. Use your diagram to help you find the dimensions of the open box.

2. Marcy decides to put a ribbon around the bottom edge of the box. She will need to determine the perimeter of the base. Write a polynomial to represent the perimeter of the base of the box, simplify the expression, and explain how you determined your answer.

3. Write a polynomial to represent the area of the base of the box. Explain how you found the area of the base.

4. Write a polynomial to represent the volume of the box. Justify your answer.

5. Suppose the length of the box is increased by 3 units. How will this affect the area of the base?

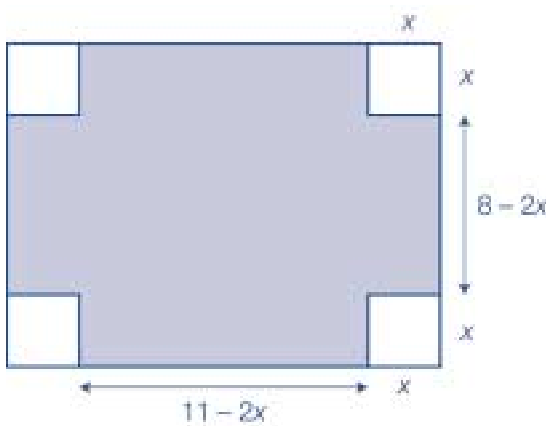
Teacher Notes

Scaffolding Questions:

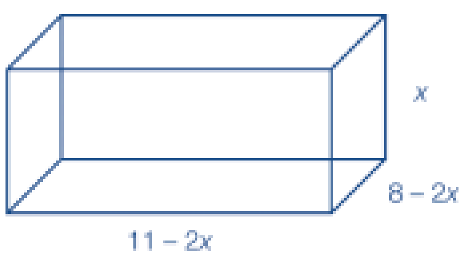
- Identify the dimensions of the base.
- How can you find the perimeter of the base?
- What would you do to find the area of the base?
- What is the formula for the volume of a rectangular prism?

Sample Solution:

1. The sheet of paper is 8 inches by 11 inches. The length of the side of the square that will be cut from each corner may vary. Let the length of the side of the square in inches be represented by x. The length of each side will be decreased by $2x$. The length of the fold lines will be $8 - 2x$ and $11 - 2x$.



The height of the open box will be the same as the length of the side of the square that was cut out.



Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities.

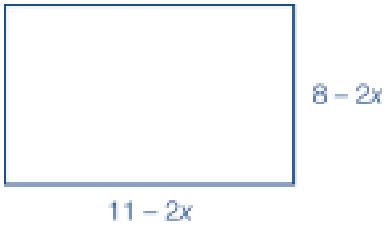
(E) interprets and makes inferences from functional relationships.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

2. The perimeter is the distance around a figure. The base of the box is a rectangle having dimensions of $11 - 2x$ and $8 - 2x$. Opposite sides of a rectangle are congruent, so there are two sides with length $11 - 2x$ and two sides with length $8 - 2x$.



Using the formula for finding the perimeter [P = 2(length) + 2(width)], the perimeter can be calculated as follows:

$P = 2(11 - 2x) + 2(8 - 2x)$
$P = 22 - 4x + 16 - 4x$
$P = 38 - 8x$

3. The base of the box is a rectangle. The area can be found using the formula for the area of a rectangle (A = length times width).

$A = (11 - 2x)(8 - 2x)$
$A = 88 - 22x - 16x + 4x^2$
$A = 88 - 38x + 4x^2$
$A = 4x^2 - 38x + 88$

4. A polynomial that represents the volume of the original box would be represented by the formula V = (Length)(Width)(Height).

The value of the length times the width was calculated when the area of the base was determined. That area was $4x^2 - 38x + 88$. The height of the box is represented by x. The volume of the original box is:

V = (Length)(Width)(Height)
$V = (4x^2 - 38x + 88)(x)$
$V = 4x^3 - 38x^2 + 88x$

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(B) uses the commutative, associative, and distributive properties to simplify algebraic expressions.

Texas Assessment of Knowledge and Skills:

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
2 Using Patterns to Identify Relationships
  2.2 Identifying Patterns
  2.3 Identifying More Patterns

Connections to Algebra End-of-Course Exam:

Objective 6:
The student will perform operations on and factor polynomials that describe real-world and mathematical situations.

5. If the length of the base is increased by 3 units, the box will have dimensions with a width of $8 - 2x$ and the length of $14 - 2x$. Using the formula A = length times width, the area of the base with an increased length of 3 units can be represented by:

$A = (14 - 2x)(8 - 2x)$
$A = 112 - 28x - 16x + 4x^2$
$A = 112 - 44x + 4x^2$
$A = 4x^2 - 44x + 112$

Extension Questions:

- Which box would hold more, a box that had an x value of 2 or a box that had an x value of 3? Justify your solution.

To find the solution, substitute the values of 2 and 3 into the volume formula to find the larger volume. If $x = 2$, the volume of the box will be:

V = (Length)(Width)(Height)
$V = (4x^2 - 38x + 88)(x)$
$V = 4(2)^3 - 38(2)^2 + 88(2)$
$V = 56 \text{ in.}^3$

If $x = 3$, the volume of the box will be:

V = (Length)(Width)(Height)
$V = (4x^2 - 38x + 88)(x)$
$V = 4(3)^3 - 38(3)^2 + 88(3)$
$V = 30 \text{ in.}^3$

Therefore, the smaller x-value of 2 yields the larger volume.

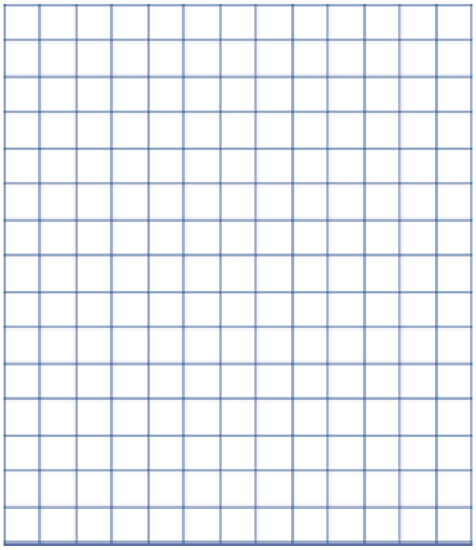
- What is the largest value of *x* that fits this situation? The smallest?

The sides of the piece of paper are 8 inches and 11 inches. If a square of side length x is cut out, the value of x must be less than 4 inches. The 8-inch side has two squares removed when the box is formed. If the sides of the squares are each 4 inches, it will cause the dimension of that side of the box to be zero. Therefore, the value for the side of the square, represented by x, must be greater than zero, and less than 4 inches.

Rebound Height

Three teams of students independently conducted experiments to relate the rebound height of a ball to the rebound number. The table below left gives the average of the three teams' results.

Rebound Number	Rebound Height (cm)
0	200
1	155
2	116
3	88
4	66
5	50
6	44



1. Construct a scatterplot of the data. Describe the functional relationship between the rebound height of the ball and the rebound number verbally and symbolically.

2. Predict the rebound height of the ball on its 10th rebound.

3. Suppose the ball stops rebounding and begins to roll across the floor when it reaches a rebound height of 3 centimeter. How many times has the ball rebounded?

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

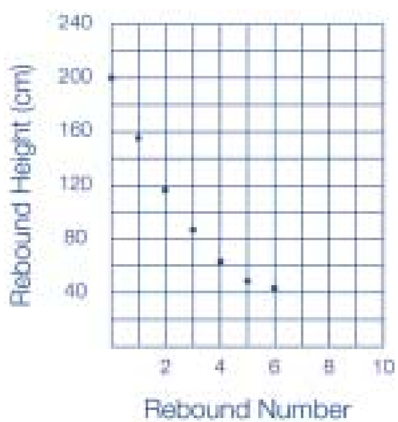
The student:
(D) in solving problems, collects and organizes data, makes and interprets scatterplots, and models, predicts, and makes decisions and critical judgments.

Scaffolding Questions:

- How will you organize the data that is collected?
- What will you need to consider to construct a scatterplot of the data?
- For the graph what will be the dependent variable? The independent variable?
- What will you need to consider to determine a reasonable interval of values and scale for each of the axes?
- What function type (linear, quadratic, exponential, inverse variation) appears to best represent your scatterplot?
- What do you need to know to determine a particular function model for your scatterplot?

Sample Solution:

Rebound Number	Rebound Height (cm)
0	200
1	155
2	116
3	88
4	66
5	50
6	44



1. The plot is clearly nonlinear and suggests that each rebound height is a fraction of the previous rebound height. The table also supports this statement, since the rate of change from one bounce to the next is not constant. The ratios of rebound height to previous rebound height are:

$$\frac{155}{200} = 0.775 \qquad \frac{66}{88} = 0.750$$

$$\frac{116}{155} = 0.748 \qquad \frac{50}{66} = 0.758$$

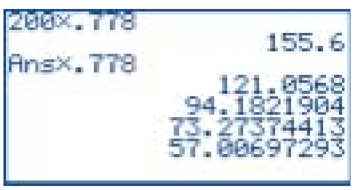
$$\frac{88}{116} = 0.759 \qquad \frac{44}{50} = 0.880$$

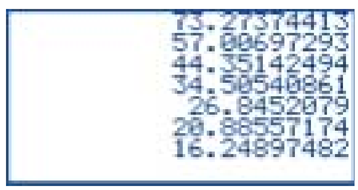
The average of these ratios is 0.778. The value at $x = 0$ is 200. A function for exponential growth is equal to (starting value) times $(\text{ratio})^x$. A function that could model this situation is $h(x) = 200(0.778)^x$, where x is the rebound number and $h(x)$ is height in centimeters.

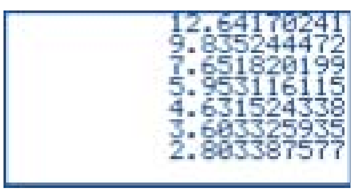
2. To predict the rebound height of the ball on its 10th rebound, substitute 10 for x in the function and evaluate: $h(10) = 200(0.778)^{10} = 16.249$.

   On the 10th rebound the ball will bounce to a height of 16.249 centimeters.

3. We need to solve the equation $200(0.778)^x = 3$. The calculator may be used to compute the values.







The value for the 16th bounce is 3.603. The value for the 17th bounce is 2.803.

This can also be done by entering the function and using the graph or a table.

(d.3) Quadratic and other non linear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(C) analyzes data and represents situations involving exponential growth and decay using concrete models, tables, graphs, or algebraic methods.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
3 Exponential Relationships
3.2 Exponential Growth and Decay
3.3 Exponential Models

Connections to Algebra End-of-Course Exam:

Objective 2:
The student will graph problems involving real-world and mathematical situations.

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

On the 17th bounce, the ball reaches a height of about 3 centimeters and rolls across the floor.

Extension Questions:

- What do the data and scatterplot suggest about the functional relationship between rebound height and rebound number?

Since the rebound numbers increase by one, the rate of change between rebounds is simply the change in rebound height. Since these are not constant, the function cannot be linear. This is also obvious by looking at the scatterplot. The scatterplot appears to be quadratic or exponentially decreasing. The ratios of consecutive rebound heights are nearly constant, suggesting the functional relationship is exponential.

- How will the scatterplot change if the original height for rebound number 0 is increased? How will this change your function?

The y-intercept will be higher, and so will each rebound height. The initial value of 200 in the function will change to the new "original height."

- Suppose the height of 200 was the height of the ball on the first rebound. How can you use your table, scatterplot, or function to predict the height before this rebound?

You could divide 200 by 0.778 to get a height of about 257. This is equivalent to evaluating the expression $h(-1) = 200(0.778)^{-1} = 257.07$.

- If the general function rule $h(x) = Ha^x$ is given to represent the rebounding ball situation, what could H, a, and x represent?

H would represent the initial height, a the fraction of the previous height by which it bounces each time, and x is the bounce number.

What is Reasonable?

For each of the following situations, determine a function that represents it. Describe the mathematical domain and range of the function and a reasonable domain and range for the situation.

A. A rectangular garden plot is to be enclosed with 40 meters of fencing. The area of the garden is a function of the dimensions of the rectangle.

B. You fold a 12-inch square piece of paper repeatedly in half. The number of rectangles formed by the folds is a function of the number of folds you make.

C. The time it takes to proofread a certain book varies inversely with the number of people assigned to the proofreading task. Suppose 5 people proofread the book in 30 hours.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations.

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(B) for a variety of situations, identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations.

Scaffolding Questions:

For each situation,

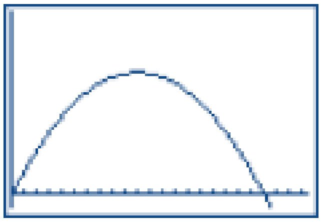
- What are the constants?
- What is the dependent variable?
- What type of function (linear, quadratic, exponential, inverse variation) relates the variables?
- What restrictions does the function place on the independent variable?
- Should you use all real numbers for the domain? Why or why not?
- What representation would best help you see the domain and range?

Sample Solution:

A. Since there are 40 meters of fencing for the perimeter of the garden, there are 20 meters of fencing for the semi-perimeter (half-way around the rectangle). The dimensions of the garden may be represented by x and $20 - x$, and the area of the garden is the product of the length and the width. The area may be expressed as a function of the width.

$A(x) = x(20 - x)$

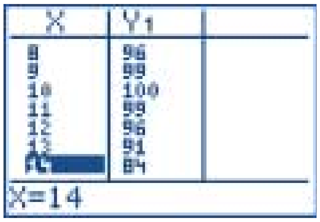
The mathematical domain for this function is all real numbers since no value for x makes the expression $x(20 - x)$ undefined. The mathematical range for the function is all real numbers less than or equal to 100.

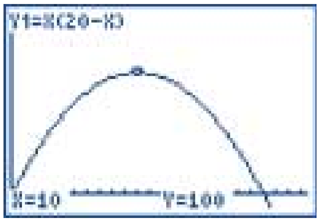


The graph shows that x can be any number (negative, zero, or positive) and that the range includes all values for y less than or equal to 100.

However, the domain of the situation is the set of all numbers between 0 and 20, because the x value must be less than 20 and greater than 0 to give a positive area. For the garden to exist, the length of a side, x, must be greater than 0 but less than half the amount of fencing, 40 meters, available for the whole garden.

The range can be determined by examining a table or a graph. The greatest value of y is 100.





The range for the function is the set of all numbers less than or equal to 100. For the problem situation the area must be positive and less than or equal to the maximum possible area, 100 square meters.

B. To determine the function, experiment with a piece of paper and record the information in a table.

Number of Folds	Number of Rectangles
0	1
1	2
2	4
3	8

Each time the paper is folded the number of rectangles is doubled or multiplied by 2. The number of rectangles is a power of 2.

Number of Folds	Number of Rectangles
0	2^0
1	2^1
2	2^2
3	2^3
4	2^4

(d.3) Quadratic and other nonlinear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(A) uses patterns to generate the laws of exponents and applies them in problem-solving situations;

(B) analyzes data and represents situations involving inverse variation using concrete models, tables, graphs, or algebraic methods.

Texas Assessment of Knowledge and Skills:

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
2 Using Patterns to Identify Relationships
2.1 Identifying Patterns

II. Nonlinear Functions
1 Quadratic Functions
1.1 Quadratic Relationships
3 Exponential Functions and Equations
3.1 Exponential Relationships
3.2 Exponential Growth and Decay
3.3 Exponential Models

The function is $r = 2^n$, where n = the number of folds made and r = the number of non-overlapping rectangles formed.

The mathematical domain for this function is the set of all real numbers since no value of n makes the function undefined. The range is the set of all positive real numbers since no power of 2 gives 0 or a negative value.

The domain of the situation is $\{0,1,2,3,...,n\}$ where n is the maximum number of folds you can make. This would depend on the dimensions of the piece of paper and its thickness.

The resulting range of the situation is $\{1,2,4,8,...,2^n\}$.

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Objective 5:
The student will formulate or solve quadratic equations that describe real-world and mathematical situations.

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

C. If two quantities vary inversely, their product is a constant. In this case the product is 5 times 30 or 150. $nh = 150$ or $h = \frac{150}{n}$, where n is the number of people proofreading the book, and h = the number of hours to proofread the book.

The mathematical domain and range are both the set of all real numbers except zero. If $n = 0$, then h would be undefined, and $h = \frac{150}{n}$ will never equal 0. The quotient of 150 and a number will never be zero.

The domain and range for the situation are best described in a table. The number of people proofreading the book must be positive integer values.

Number of People proofreading	Number of Hours to proofread
1	150
2	75
3	50
4	37.5
5	30
...	...
149	1.01
150	1

Extension Questions:

- In Situation A, describe how the domain and range would change if you change the amount of fencing you use to enclose the garden.

If you decrease the amount of fencing, both domain and range will decrease since both the dimensions and area of the garden will decrease. If you increase the amount of fencing, both the domain and range will increase since the dimensions and area of the garden will increase.

- In Situation B, determine the function that would relate the area of each of the rectangles formed in the folding process to the number of folds. Describe the domain and range of this function. Compare the area function with the "Number of Rectangles" function.

The initial area is 12^2 or 144 square inches. Each fold produces a new rectangle that is half as large as the previous rectangle. The function is

$$A = 144\left(\frac{1}{2}\right)^n$$

The domain is the set {0,1,2,...,n} where n is the maximum number of folds you can make.

The range is $\{144, 72, 36, 18, \ldots, 144\left(\frac{1}{2}\right)^n\}$.

The area function is a decreasing exponential function, while the Number of Rectangles function is an increasing exponential function. Both functions have finite domains and ranges.

- In Situation C, suppose 6 people complete the proofreading task in 30 hours and that the time to complete the task must be measured in whole hours. How will this change your function and the domain and range for the situation?

The function becomes $h = \frac{180}{n}$ *since the task now requires 6 times 30 or 180 people-hours to complete. The domain and range now consist of factor pairs of 180, since we are measuring by both whole people and whole hours. There are 18 factor pairs for the new situation.*

Number of People	Number of Hours
1	180
2	90
3	60
4	45
5	36
6	30
9	20
10	18
12	15

Number of People	Number of Hours
15	12
18	10
20	9
30	6
36	5
45	4
60	3
90	2
180	1

- Describe, in general, how changing the number of people affects the time to complete the proofreading task.

As the number of people increases, the time to complete the task decreases, and this occurs at a nonlinear rate. This can be seen by building a table where x, the number of people, increases by a constant amount, and comparing the corresponding change in the time y, to complete the task.