Chapter 3:

Interacting Linear Functions, Linear Systems

Bears' Band Booster Club

The Bears' Band Booster Club has decided to sell calendars to the band members and their parents. The cost of the calendars will be $8 per calendar, but they must also pay an initial fee of $65 for designing the cover for the calendar. They decide to sell the calendars for $12 each. Investigate the situation and determine how many calendars they must sell to make a profit.

1. Write a function that describes the relationship between cost and number of calendars.

2. Write a function that describes the relationship between revenue and the number of calendars sold.

3. How many calendars must they sell to make a profit? Describe your process for answering the question.

4. If they want to make at least $400, how many calendars must they sell?

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

Scaffolding Questions:

- Describe how to compute the cost of 10 calendars.
- Explain how to compute the cost of 15 calendars.
- What are the constants in this situation?
- What are the variables in this situation?
- Explain how to compute the revenue from the sale of 10 calendars.
- Consider the sale of 15 calendars. What is the cost of the 15 calendars? What is the revenue from the sale of 15 calendars?
- Do you make a profit when you sell 15 calendars? Explain how you know.
- How can you tell when you start making a profit?

Sample Solution:

1. The cost is 65 dollars plus 8 dollars per calendar.

   The equation is $y = 65 + 8x$ where y is the cost in dollars and x is the number of calendars.

2. The revenue is 12 dollars times the number of calendars.

   $y = 12x$ where y is the revenue in dollars and x is the number of calendars.

3. They will break even when the revenue equals the cost.

   Cost: $y = 8x + 65$.

   Revenue: $y = 12x$

   $$12x = 8x + 65$$
   $$4x = 65$$
   $$x = 16.25$$

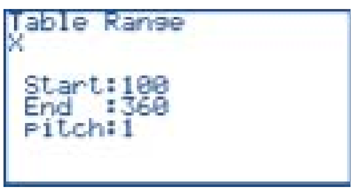
   The only reasonable values in this situation are whole numbers. You cannot sell a fraction of a calendar or a negative number of calendars.

   The first time the cost is less than the revenue is when 17 calendars are sold. They must sell at least 17 calendars to make a profit.

4. You may examine the table or the graph to see when the difference in the cost and revenue is at least $400.

   You may start with a guess of 100 calendars and examine the table.

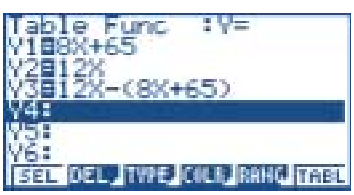
   At 110 the difference is less than 400. At 120 the difference is more than 400. The amount is between these two values. Reset the table at 110 with an increment of one.



   They must sell 117 calendars to make a profit of at least $400.

   Another approach is to write a general rule for profit.

   Profit = Revenue minus Cost
   Profit = 12*x* – (65 + 8*x*)



   Examine the table to find the first value when the profit is at least 400. This occurs when *x* is 117.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(c.4) Linear functions.
The student formulates systems of linear equations from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student:
(A) analyzes situations and formulates systems of linear equations to solve problems;

(B) solves systems of linear equations using concrete models, graphs, tables, and algebraic methods; and

(C) for given contexts, interprets and determines the reasonableness of solutions to systems of linear equations.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 4:
The student will formulate and use linear equations and inequalities.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
1 Developing Mathematical Models
   1.2 Valentine's Day Idea

II. Linear Functions
1 Linear Functions
   1.2 The Y-Intercept
3 Linear Equations and Inequalities
   3.4 Systems of Linear Equations and Inequalities

Connections to Algebra End-of-Course Exam:

Objective 4:
The student will formulate or solve linear equations/inequalities and systems of linear equations that describe real-world and mathematical situations.

Extension Questions:

- If the situation had been different and the equation for cost was written $y = 80 + 15x$, how would the situation have been described?

The set-up charge was $80, and the cost per calendar was $15.

- How could an equation be used to solve for the number of calendars when the profit is 400 dollars?

Profit = Revenue minus Cost

$$400 = 12x - (65 + 8x)$$
$$400 = 12x - 65 - 8x$$
$$465 = 4x$$
$$x = 116.25$$

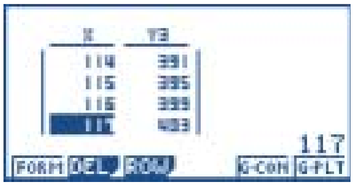
They may not sell a fraction of a calendar. They must sell 117 calendars.

- Describe how the graph could be used to answer the question.

Graph the function y = 12x–(65+8x) and the function y = 400, and find the point of intersection.

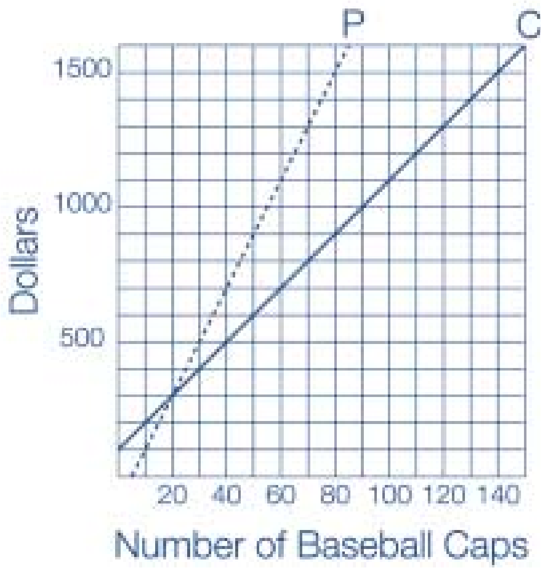
Another approach is to graph the function y = 12x–(65+8x) and the look for the value of x that gives a y-value close to 400.

The profit for 116 calendars is $399, and the profit for 118 calendars from the graph is $407. The table will show the profit for 117 calendars is $403.



Cost and Profit

The Bartlett Booster Club is purchasing baseball caps to sell at school. The graph shows the line C that models the cost of making the baseball caps in terms of the number of baseball caps and the line P that models profit in terms of the number of baseball caps. The lines are represented as dashed lines because not all points on the line would represent the problem situation.



1. Write a function rule for the cost in terms of the number of baseball caps.

2. Describe how to find the cost of 60 baseball caps.

3. How many baseball caps were purchased if the cost was $340?

4. Write the function for the profit in terms of the number of baseball caps.

5. Explain how to determine the profit from the sale of 200 caps.

6. If the profit is the revenue minus the cost, what is an expression for the revenue? Describe how to determine the revenue from the sale of 54 caps?

7. What does this function rule tell you about how they sold the baseball caps?

Teacher Notes

Scaffolding Questions:

- What points on the lines really could represent the situation?
- What is the *y*-intercept of the C line? Explain what it means in this situation.
- Describe the rate at which the cost of the baseball caps is increasing? Discuss how you found the rate.
- What is the *x*-intercept of the P line, and what does it mean in this problem situation? What is the rate of change for the P line, and what does it mean in this situation?
- Describe how to determine the *y*-intercept of the P line. Explain what it means.

Sample Solution:

1. The initial cost from the graph is $100, and the rate of change is $100 for every 10 caps or $10 for 1 cap. The rule for the cost is $100 plus $10 times the number of caps.

   $C(x) = 100 + 10x$, where C is the cost and x is the number of caps.

2. To determine the cost of 60 caps, evaluate the function at $x = 60$.

   $C(60) = 100 + 10(60)$ or $700

   The cost of 60 caps is $700.

3. To determine the number of caps that will cost $340, solve the equation.

   $$\begin{aligned} 340 &= 100 + 10x \\ 240 &= 10x \\ x &= 24 \end{aligned}$$

   24 caps would cost $340.

4. There are two points on the profit line that may be used to determine the equation of the line, (20,300) (30,500). The rate of change is $200 for 10 caps or $20 for every 1 cap.

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(C) interprets situations in terms of given graphs or creates situations that fit given graphs.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables.

(c.1) Linear functions.
The student understands that linear functions can be represented in different ways and translates among their various representations.

The student:
(C) translates among and uses algebraic, tabular, graphical, or verbal descriptions of linear functions.

(c.2) Linear functions.
The student understands the meaning of the slope and intercepts of linear functions and interprets and describes the effects of changes in parameters of linear functions in real-world and mathematical situations.

The student:
(A) develops the concept of slope as rate of change and determines slopes from graphs, tables, and algebraic representations;

(B) interprets the meaning of slope and intercepts in situations using data, symbolic representations, or graphs.

(c.3) Linear functions.
The student formulates equations and inequalities based on linear functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student:
(A) analyzes situations involving linear functions and formulates linear equations or inequalities to solve problems.

To determine the y-intercept of the P line, imagine extending the graph in the negative directions from the point (20,300). Reduce y by 200 for every reduction of 10 in x until x is 0.

(20–10, 300–200) or (10,100) is a point on the graph.
(10–10, 100–200) or (0,-100) is a point on the graph.
The y-intercept is -100.

The profit is -$100 plus 20 times the number of caps sold.

$P(x) = -100 + 20x$, where x is the number of caps and $P(x)$ is the profit.

5. Let $x = 200$
   $-100 + 20(200) = 3900$

   The profit from the sale of 200 caps is $3900.

6. If profit is revenue minus cost, then revenue is profit plus cost.

   $P = R – C$
   $R = P + C$

   The function rule for the revenue is the sum of the two rules, profit and cost.

   $R = (100 + 10x) + (-100 + 20x)$
   $R = 30x$

   The revenue from the sale of 54 caps is 30 times 54 or $1620.

7. The revenue is $30 times the number of caps, so they must have sold the caps for $30 per cap.

Extension Questions:

- Explain the significance of the point of intersection of the two lines that represent cost and profit.

The point of intersection of the two lines on the graph is the point (20,300). The cost and profit are both equal to $300 at this point.
$C(20) = 100 + 10(20) = 300$
$P(20) = -100 + 20(20) = 300$

Another way to look at it is cost equals profit.

$P = R – C$
$R = P + C$
If $P = C$, then $R = C + C = 2C$.
When 20 caps are sold, the revenue is twice the cost of 20 caps.

- Suppose that they were able to find someone who would sell them caps at the same price, but with an initial cost of $80. How would the graph of the cost line be affected?

The graph would be a line parallel to the original line, but with a y-intercept of 80. The function for cost would be $C = 10x + 80$.

- If they continue to sell the caps for $30, how is the profit affected by this new cost function? Describe the effect on the graph.

The revenue function would still be $R = 30x$.
Profit equals revenue minus cost.
$P = 30x – (10x + 80) = 20x – 80$
The previous profit function was $P = 20x – 100$. The profit would be increased by $20. The graph of the line would be raised 20 units, but would have the same slope.

(c.4) Linear functions.
The student formulates systems of linear equations from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student:
(A) analyzes situations and formulates systems of linear equations to solve problems;

(B) solves systems of linear equations using concrete models, graphs, tables, and algebraic methods; and

(C) for given contexts, interprets and determines the reasonableness of solutions to systems of linear equations.

Texas Assessment of Knowledge and Skills:

Objective 4:
The student will formulate and use linear equations and inequalities.

Connections to Algebra I: 2000 and Beyond Institute:

II. Linear Functions
1 Linear Functions
 1.3 Exploring Rates of Change
3 Linear Equations and Inequalities
 3.4 Systems of Linear Equations and Inequalities

Connections to Algebra End-of-Course Exam:

Objective 4:
The student will formulate or solve linear equations/inequalities and systems of linear equations that describe real-world and mathematical situations.

Student Work

1) C

# of T-shirts	Dollars
20	300
40	500
60	700
80	900
100	1100
120	1300
140	1500

$Y = 10x + 100$

2) $Y = 10(60) + 100$

$C = 700$

3) $20(22) - 100 = 340$

22 t-shirts

4) P

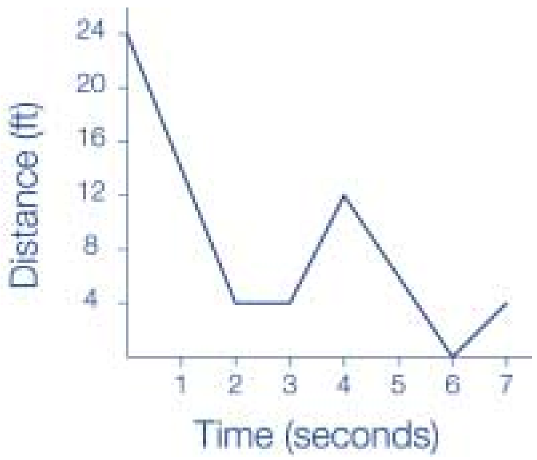
# of T-shirts	Dollars
20	300
40	700
60	1100
80	1500
100	1900
120	2300
140	2700

5) $Y = 200(20) - 100$

$Y = 3900$

Motion Detector Problem

The graph below shows how the distance between a person and a motion detector depends on the time that has elapsed since the person began walking.



1. Describe, in detail, how the person's position relative to the motion detector changes over the time interval from 0 seconds to 7 seconds. Include a description of the person's speed for each portion of the graph.

2. When is the person moving the fastest? Explain.

3. Write a function for each phase you described in problem 1.

Teacher Notes

Scaffolding Questions:

- How far from the motion detector is the person initially?
- Is he moving toward the detector or away? How fast is he moving?
- When is the person standing still? For how long?
- When is the person moving away from the motion detector?
- Does the person ever reach the sensor?

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(C) interprets situations in terms of given graphs or creates situations that fit given graphs.

Sample Solution:

1. At the beginning, the person is 24 feet away from the motion detector. He walks for two seconds and stops 4 feet from the sensor. He is walking at a rate of 20 feet per two seconds, i.e., 10 feet per second.

   He stands still 4 feet from the sensor for one second.

   He turns and walks away for one second, stopping 12 feet from the motion detector. He is walking at a rate of 8 feet per second.

   Next, he turns and walks for two seconds back toward the motion detector, going all the way up to it. He is walking at a rate of 12 feet per two seconds, i.e., 6 feet per second.

   Finally, he turns and walks away from the motion detector for one second, stopping 4 feet away from the sensor. He is walking at a rate of 4 feet per second.

2. The person is moving the fastest during the first two seconds when he is walking toward the motion detector at 10 feet per second.

3. For the first phase the y-intercept is 24 and the rate of change is -10 feet per second. The rate is negative because the distance between the person and the motion detector is decreasing as time increases. The function is $y = 24 - 10x$, where $0 \le x \le 2$.

   Between 2 seconds and 3 seconds, the distance stays constant. There are 4 feet between the person and the motion detector, so the function rule is $y = 4$, where $2 \le x \le 3$.

   Between 3 and 4 seconds, the rate of change is 8 feet per second because the distance between the person and the motion detector is increasing. Substituting either data point (3,4) or (4,12) in $y = 8x + b$ gives the y-intercept.

$$\begin{aligned} 4 &= 8(3) + b \\ 4 - 8(3) &= b \\ b &= -20 \end{aligned}$$

The function is $y = 8x - 20$ where $3 \le x \le 4$.

Between 4 seconds and 6 seconds, the rate of change is -6 feet per second. Use (4,12) or (6,0) in $y = b - 6x$ to get the y-intercept.

$$\begin{aligned} y &= b - 6x \\ 0 &= b - 6(6) \\ 0 &= b - 36 \\ b &= 36 \end{aligned}$$

The function rule is $y = 36 - 6x$, where $4 \le x \le 6$.

Finally, between 6 seconds and 7 seconds, the rate of change is 4 feet per second. Use (6,0) or (7,4) in $y = 4x + b$ to get the y-intercept $b = -24$.

The function is $y = 4x - 24$, where $6 \le x \le 7$.

Extension Questions:

- How would the graph change for the time interval 0 to 2 seconds, if the person walked toward the motion detector with increasing speed?

The graph would be a curve opening downward. It would be shaped like part of a quadratic such as $y = x^2$, $x \le 0$.

- How would the graph change for the time interval 3 to 4 seconds if the person walked away from the motion detector with decreasing speed?

The graph would be a curve opening upward and shaped like part of a quadratic such as $y = -x^2$, $x \ge 0$.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
3 Interpreting Graphs
3.1 Interpreting Distance Versus Time Graphs

II. Linear Functions
1 Linear Functions
1.3 Exploring Rates of Change

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

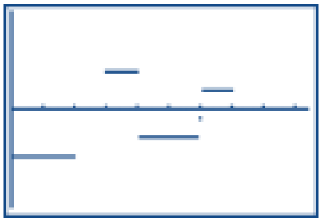
Objective 8:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving one-variable or two-variable situations.

- What would be the function rule for velocity versus time, and what would its graph look like?

The velocity is the rate of change for each section of the graph. In each section the slope is a constant. The values for y in each section would be constant. The function rules for the sections would be

If $0 \le x \le 2$, $y = -10$.
If $2 \le x \le 3$, $y = 0$.
If $3 \le x \le 4$, $y = 8$.
If $4 \le x \le 6$, $y = -6$.
If $6 \le x \le 7$, $y = 4$.

The graph would be



Student Work

Motion Detector Problem

① The person started 24 feet away at first and then within 2 minutes was within 4 feet of the motion detector. The person stayed 4 feet away for 1 minute then went 12 feet away from the motion detector in the next minute. It took 2 minutes for the person to arrive at the motion detector (0 feet away). Then, on the 7th minute, the person had travelled 4 feet.

② The person is moving the fastest between 0 and 2 minutes. We know this because in 2 minutes he moved at a steady pace and went 20 feet. This means 20 feet/2 minutes or 10 feet per minute during this period.

③

(0,24)	(1,14)	$-10/1$	$f(x)=-10x+24$
(3,4)	(4,12)	$8/1$	$f(x)=8x-20$
(4,12)	(6,0)	$-12/2=-6$	$f(x)=-6x+36$
(6,0)	(7,4).	$4/1$	$F(x)=4x-24$

Speeding Cars

Four cars start from the same city at the same time. The following data was collected on the performance of the four different cars based on miles driven in terms of hours:

Car A

Hours	Miles
0	0
2	120
3	180
5	300
6	360

Car B

Hours	Miles
0	0
1	75
2	150
3	225
4	300

Car C

Hours	Miles
0	0
5	200
10	400
15	600
20	800

Car D

Hours	Miles
0	0
1	65
2	85
3	105
4	125

1. Which car was traveling the fastest? How do you know?

2. Which car was traveling the slowest? How do you know?

3. Compare and contrast the tables.

4. Write a function rule to model each car's travel.

5. Create a scatterplot for each table. Compare and contrast the graphs. Compare the domains for the functions and the domains for the problem situation.

6. Do any of the tables represent a direct variation? Explain how you know.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(c.1) Linear functions.
The student understands that linear functions can be represented in different ways and translates among their various representations.

The student:
(A) determines whether or not given situations can be represented by linear functions;

(B) determines the domain and range values for which linear functions make sense for given situations;

(C) translates among and uses algebraic, tabular, graphical, or verbal descriptions of linear functions.

Scaffolding Questions:

- How can you use the table to determine the speed at which each car traveled?
- How can you tell if the car is traveling at a constant rate?
- What are the similarities in the table values?
- What are the differences in the table values?
- How can you tell from a table if it represents a linear function?
- How can you tell from the graph that a function is linear?
- What must be true if a set of points represents a direct variation?

Sample Solution:

1. Determine the speed or rate of change for each car.

Car A		Car B		Car C		Car D	
Hours	Miles	Hours	Miles	Hours	Miles	Hours	Miles
0	0	0	0	0	0	0	0
2	120	1	75	5	200	1	65
3	180	2	150	10	400	2	85
5	300	3	225	15	600	3	105
6	360	4	300	20	800	4	125

To find the speed at which the car is traveling find the differences in the distances and the times from the starting point (0,0).

Car A:

$\frac{120-0}{2-0}=60$ $\quad \frac{180-0}{3-0}=60$ $\quad \frac{300-0}{5-0}=60$ $\quad \frac{360-0}{6-0}=60$

Car A is traveling at 60 miles per hour.

Car B:

$\frac{75-0}{1-0}=75$ $\quad \frac{150-0}{2-0}=75$ $\quad \frac{225-0}{3-0}=75$ $\quad \frac{300-0}{4-0}=75$

Car B is traveling at 75 miles per hour.

Car C:

$\frac{200-0}{5-0} = 40$ $\frac{400-0}{10-0} = 40$ $\frac{600-0}{15-0} = 40$ $\frac{800-0}{20-0} = 40$

Car C is traveling at 40 miles per hour.

Car D:

$\frac{65-0}{1-0} = 65$ $\frac{85-0}{2-0} = 42.5$ $\frac{105-0}{3-0} = 35$ $\frac{225-0}{4-0} = 55$

Car D is not traveling at a constant rate.

If the rates are examined for the one-hour time intervals, the differences are not the same.

$\frac{65-0}{1-0} = 65$ $\frac{85-65}{2-1} = 20$ $\frac{105-85}{3-2} = 20$ $\frac{125-105}{4-3} = 20$

Car D is traveling at a constant rate after the first hour.

Car B is traveling the fastest; for every hour it travels 75 miles.

2. Car C is traveling the slowest constant rate; for every hour it travels only 40 miles. However, Car D is traveling at a slower rate from the first to the fourth hours.

3. The tables are similar in several ways; all 4 tables start at (0,0). The tables all report miles and hours. All the tables show that as the hours increase the miles also increase. The tables are different because they don't all increase by the same hour values; the time interval is not one hour on all tables, and in table D there isn't a constant rate for the all values.

   Tables A, B, and C represent linear relationships because they indicate a constant rate of change. Table D is not a linear relationship because the first hour the car traveled 65 miles, but after the first hour it only covers 20 miles for each additional hour.

(c.2) Linear functions.
The student understands the meaning of the slope and intercepts of linear functions and interprets and describes the effects of changes in parameters of linear functions in real-world and mathematical situations.

The student:
(A) develops the concept of slope as rate of change and determines slopes from graphs, tables, and algebraic representations;

(B) interprets the meaning of slope and intercepts in situations using data, symbolic representations, or graphs;

(D) graphs and writes equations of lines given characteristics such as two points, a point and a slope, or a slope and y-intercept;

(G) relates direct variation to linear functions and solves problems involving proportional change.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

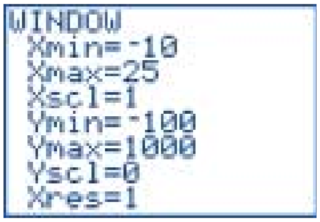
Objective 3:
The student will demonstrate an understanding of linear functions.

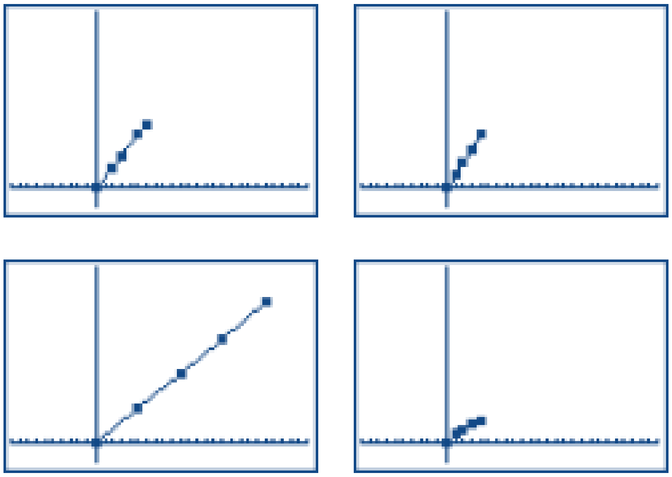
4. The first three could be represented by the equations of the form $y = mx + 0$ because the starting value is 0 where m is the slope or rate of change.

Table A: $y = 60x$
Table B: $y = 75x$
Table C: $y = 40x$
Table D: $y = 65x$ for $0 \leq x \leq 1$
$y = 20x$ for $1 \leq x \leq 4$

For values of x between 0 and 1, there is not a constant rate, and the linear model $y = 65x$ would not be appropriate.

5. Each of the graphs is a set of points. All graphs start at (0,0). The lines all have a positive slope. Graph B has the greatest slope because it is the steepest and has the greatest rate of change. Graph D could not be modeled by a line. Graph A, B, and C represent linear relationships.





Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
2 Using Patterns to Identify Relationships
2.1 Identifying Patterns
2.2 Identifying More Patterns

II. Linear Functions
1 Linear Functions
1.2 Y-Intercept
1.3 Exploring Rates of Change
1.4 Finite Differences

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

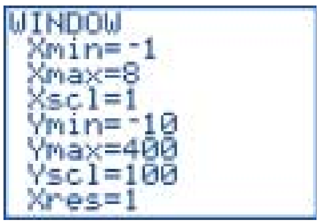
Objective 2:
The student will graph problems involving real-world and mathematical situations.

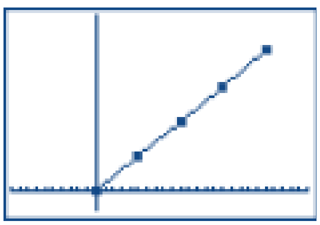
Objective 3:
The student will write linear functions (equations of lines) to model problems involving real-world and mathematical situations.

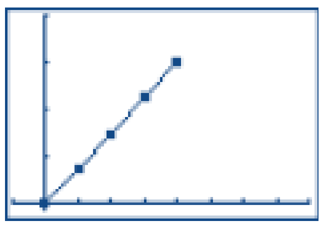
Objective 4:
The student will formulate or solve linear equations/inequalities and systems of linear equations that describe real-world and mathematical situations.

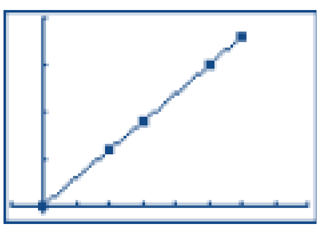
Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

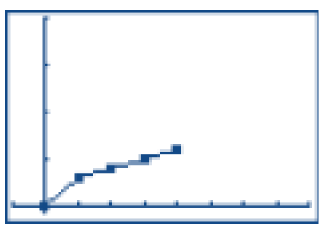
The graphs may be examined more carefully with a smaller window.











The domains for the functions are all real numbers. The domain for the problem situation are numbers greater than or equal to zero. The graphs show connected points because the functions are continuous for the values of the number of hours. However, the upper limit on the domain values depends on how long each car travels.

6. Tables A, B, and C represent direct variations because there is a constant rate of change and the relationship contains the point (0,0). Table D does not represent a proportional relationship because there is not a constant rate of change.

Extension Questions:

- If another car had traveled at a speed that was twice the speed of Car C, how would the table values have been affected?

If the car is traveling at twice the speed, the equation for the distance as a function of the number of hours would be $y = 80x$. If the x-values are the same, the y-values would have been twice the original values of Car C.

- Suppose that another car has the same values as Car A, except that 20 is added to each of the *y*-values.

Car A

Hours	Miles
0	0
2	120
3	180
5	300
6	360

New Car

Hours	Miles
0	20
2	140
3	200
5	320
6	380

Describe how this car's motion is the same or different from Car A.

The new car is traveling at the same rate as Car A , because 20 has been added to each y-value.

$$\frac{140 - 20}{2 - 0} = 60 \quad \frac{200 - 20}{3 - 0} = 60 \quad \frac{320 - 20}{5 - 0} = 60 \quad \frac{380 - 20}{6 - 0} = 60$$

One possible way to interpret the difference is to say that the new car started out 20 miles ahead of Car A.

- How is the graph of this new car's function different?

The y-intercept of this new graph would be at 20, but the graph would be parallel to the original graph of Car A.

The Walk

Two adjacent motion detectors have been set up in a room so that Pam and Abigail may walk in parallel paths in front of a motion detector. The tables below show the data that was collected for each walk. Assume the students each walked at a constant rate and started at the same time.

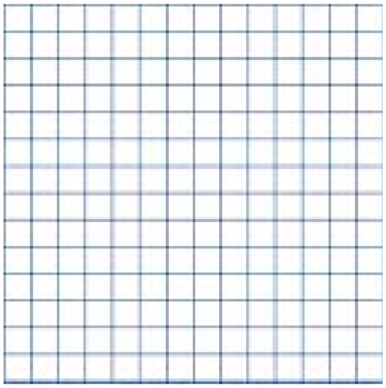
Pam's Walk

Time (seconds)	Distance (ft) from Motion Detector
1	7.9
3	5.3
6	1.4

Abigail's Walk

Time (seconds)	Distance (ft) from Motion Detector
2	3.6
4	5.2
7	7.6

1. Create a graph to model the students’ walks. Label the axes.



2. Write a function rule that models each person’s distance from the motion detector in terms of the number of seconds.

3. Determine when the two students will be next to each other when they’re walking the path.

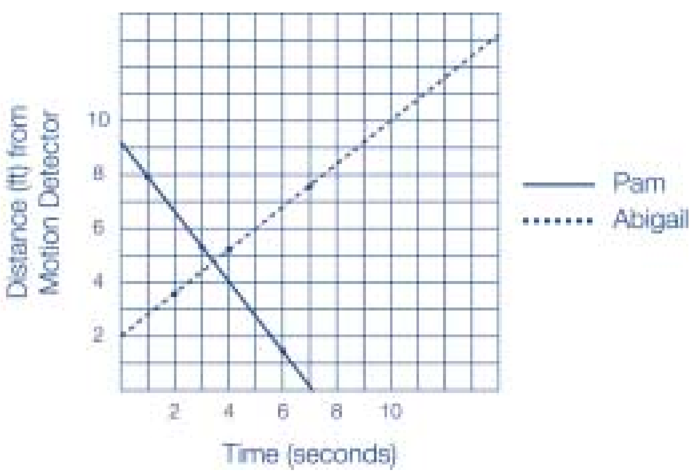
Teacher Notes

Scaffolding Questions:

- If the girls walked at a constant rate, describe the shape of the graph.
- How can you use the table to determine how fast each person was walking?
- If you plot the points, what pattern do you see?
- How is the way Pam was walking different from the way Abigail was walking?

Sample Solution:

1.



2. The differences may be used to determine how fast each person was walking.

Pam's Walk

Time (seconds)	Distance (ft) from Motion Detector
1	7.9
3	5.3
6	1.4

Time differences: 2, 3; distance differences: -2.6, -3.9

Abigail's Walk

Time (seconds)	Distance (ft) from Motion Detector
2	3.6
4	5.2
7	7.6

Time differences: 2, 3; distance differences: 1.6, 2.4

Pam was walking at a rate of -1.3 feet per second.
Abigail was walking at a rate of 0.8 feet per second.

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(B) for a variety of situations, identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations;

(C) interprets situations in terms of given graphs or creates situations that fit given graphs.

The distance Pam was from the motion detector is decreasing, so she must be walking toward the detector. Abigail's distance was increasing, so she was walking away from the detector. The function rule for Pam's walk will be the starting point plus the rate times the number of seconds. Use the table and the rates to determine the starting points. Add an extra row for time 0.

Pam's Walk

	Time (seconds)	Distance (ft) from Motion Detector	
	0	9.2	
1	1	7.9	-1.3
2	3	5.3	-2.6
3	6	1.4	-3.9

Abigail's Walk

	Time (seconds)	Distance (ft) from Motion Detector	
	0	2	
2	2	3.6	1.6
2	4	5.2	1.6
3	7	7.6	2.4

To determine the starting point add the 1.3 feet for the one second. She started at 7.9 + 1.3 or 9.2 feet from the motion detector. The rule for Pam's walk is $y = 9.2 – 1.3x$. The function rule for Abigail's walk will be her starting point plus the rate times the number of seconds.

Her starting point is 3.6 minus the distance she travels in two minutes or 3.6 – 1.6 or 2 feet.

The function rule that describes her walk is $y = 2 + 0.8x$.

3. We want to know when the two people are the same distance away from the motion detector.

   By examining the graphs one can see that at about 3 seconds they are both 5 feet away from the detector.

   To check this solution solve the system

$$y = 9.2 - 1.3x$$
$$y = 2 + 0.8x$$

$$2 + 0.8x = 9.2 - 1.3x$$
$$2.1x = 7.2$$
$$x = 3.428571$$

$$y = 2 + 0.8(3.428571) = 4.742857$$

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables.

(c.1) Linear functions.
The student understands that linear functions can be represented in different ways and translates among their various representations.

The student:
(B) determines the domain and range values for which linear functions make sense for given situations; and

(C) translates among and uses algebraic, tabular, graphical, or verbal descriptions of linear functions.

(c.2) Linear functions.
The student understands the meaning of the slope and intercepts of linear functions and interprets and describes the effects of changes in parameters of linear functions in real-world and mathematical situations.

The student:
(A) develops the concept of slope as rate of change and determines slopes from graphs, tables, and algebraic representations;

(B) interprets the meaning of slope and intercepts in situations using data, symbolic representations, or graphs;

(D) graphs and writes equations of lines given characteristics such as two points, a point and a slope, or a slope and y-intercept.

(c.3) Linear functions.
The student formulates equations and inequalities based on linear functions, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

They will be next to each other when they are about 4.74 feet from the motion detector.

The domain values must be any number greater than or equal to zero, but in an actual situation there is a limit to the number of seconds the person can walk with the motion detector. This limit is set by the calculator operator. The range values represent distance and must be positive numbers.

Extension Questions:

- What are reasonable domain and range values for this problem situation?

The domain values must be any number greater than or equal to zero, but in an actual situation there is a limit to the number of seconds the person can walk with the motion detector. This limit is set by the calculator operator. The range values represent distance and must be positive numbers.

- Suppose the motion detectors had been set up on opposite sides of the room 10 feet apart and the girls had walked on parallel paths. If the same data is used, how would your answers have been different?

If the motion detectors had been 10 feet apart on opposite sides of the room, the two walkers would have been walking in the same direction. The equations describing their motion relative to each person's motion detector would still have been

Pam: $y = 9.2 - 1.3x$.

Abigail: $y = 2 + 0.8x$.

However to consider when they would be in the same horizontal position, one must write the equations in terms of distance from one of the motion detectors. Suppose the equations are written as distance from Abigail's motion detector with respect to time.

The student:
(A) analyzes situations involving linear functions and formulates linear equations or inequalities to solve problems.

(c.4) Linear functions.
The student formulates systems of linear equations from problem situations, uses a variety of methods to solve them, and analyzes the solutions in terms of the situation.

The student:
(A) analyzes situations and formulates systems of linear equations to solve problems;

(B) solves systems of linear equations using concrete models, graphs, tables, and algebraic methods; and

(C) for given contexts, interprets and determines the reasonableness of solutions to systems of linear equations.

Texas Assessment of Knowledge and Skills:

Objective 4:
The student will formulate and use linear equations and inequalities.

Connections to Algebra I: 2000 and Beyond Institute:

II. Linear Functions
1 Linear Functions
1.3 Exploring Rates of Change
3.0 Linear Equations and Inequalities
3.4 Systems of Linear Equations and Inequalites

Connections to Algebra End-of-Course Exam:

Objective 4:
The student will formulate or solve linear equations/inequalities and systems of linear equations that describe real-world and mathematical situations.

Objective 8:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving one-variable or two-variable situations.

Abigail started 2 feet from her motion detector. Pam started 9.2 feet from her motion detector. Since the distance between the motion detectors is 10 feet, Pam would be on a horizontal distance of 10 – 9.2 or 0.8 feet from Abigail's motion detector. The equation of her movement relative to the motion detector is her starting point plus her rate times the number of minutes.

$y = 0.8 + 1.3x.$

The rate is positive because her distance is increasing from Abigail's motion detector.

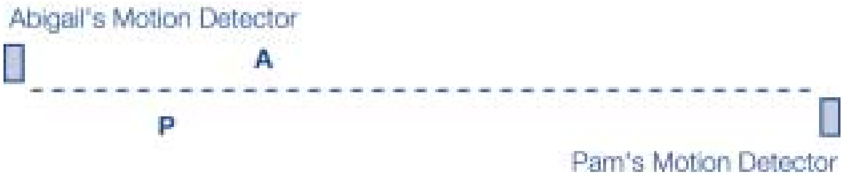
Abigails's rule is $y = 2 + 0.8x.$

Solving this system of equations results in a solution of

$$0.8 + 1.3x = 2 + 0.8x$$
$$0.5x = 1.2$$
$$x = 2.4$$

$$y = 2 + 0.8(2.4) = 3.92$$

They will both be 3.92 feet from Abigail's motion detector 2.4 seconds after they started walking.



- How would the functions that describe the motion in this last situation have been changed if the motion detectors had been positioned 12 meters apart instead of 10 meters?

Pam would be on a horizontal distance of 12 – 9.2 or 2.8 feet from Abigail's motion detector. The equation of her movement relative to Abigail's motion detector is her starting point plus her rate times the number of minutes.

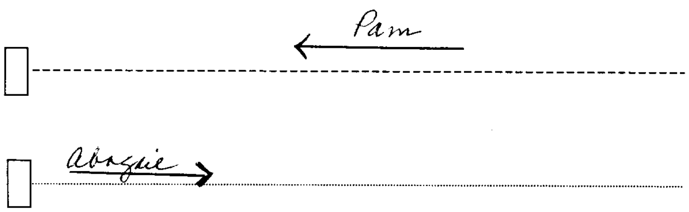
$y = 2.8 + 1.3x.$

Pam's equation would be $y = 2.8 + 1.3x.$
Abigail's rule is not changed. It would be $y = 2 + 0.8x.$

Student Work

The Walk

Two adjacent motion detectors have been set up in a room so that Pam and Abigail may walk in parallel paths in front of a motion detector. The table below shows the data that was collected for each walk. Assume the students each walked at a constant rate and started at the same time.



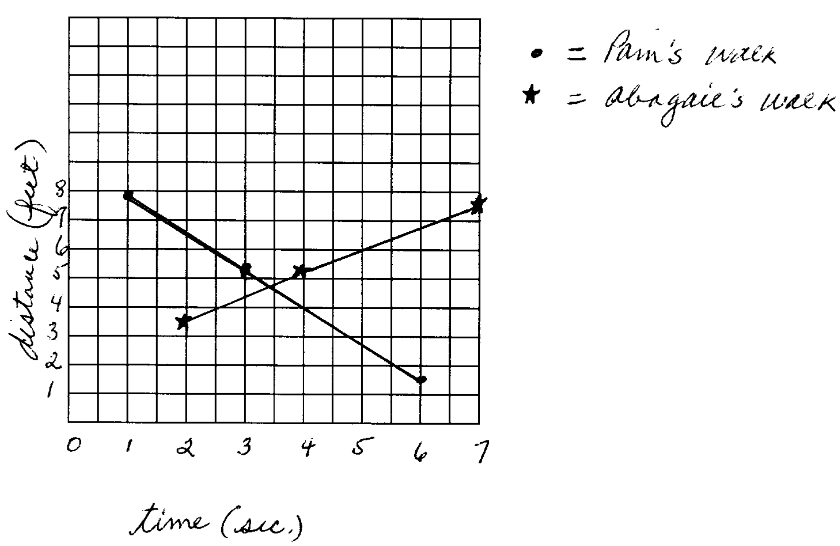
Pam's Walk

Time in seconds	Distance in feet from the motion detector
1	7.9
3	5.3
6	1.4

Abigail's Walk

Time in seconds	Distance in feet from the motion detector
2	3.6
4	5.2
7	7.6

1. Create a graph to model the students' walks. Label the axes.



2. x = time in seconds
y = distance from motion dietector in feet

Pam's walk

	x	y	
	0	9.2	
1+	1	7.9	-1.3
1+	2	6.6	-1.3
1+	3	5.3	-1.3
1+	4	4	-1.3
1+	5	2.7	-1.3
	6	1.4	-1.3.

abagaie's walk

	x	y	
	0	2	
1+	1	2.8	+.8
1+	2	3.6	+.8
1+	3	4.4	+.8
1+	4	5.2	+.8
1+	5	6	+.8
	6	6.8	
	7	7.6	

To get my tables, I divided the measurments by the time intervals so I could see the relationship when it had time intervals of one second. That showed me it was a linear relationship because x and y had a constant charge.

(Pam) $y = -1.3x + 9.2$
(abagaie) $y = .8x + 2$

3. $-1.3x + 9.2 = .8x + 2$
$-.8x \quad\quad -.8x$
$-2.1x + 9.2 = 2$

$-2.1x + 9.2 = 2$
$-9.2 \quad -9.2$
$-2.1x = -7.2$
$\overline{-2.1} \quad \overline{-2.1}$

x = 3.43 seconds