CORE

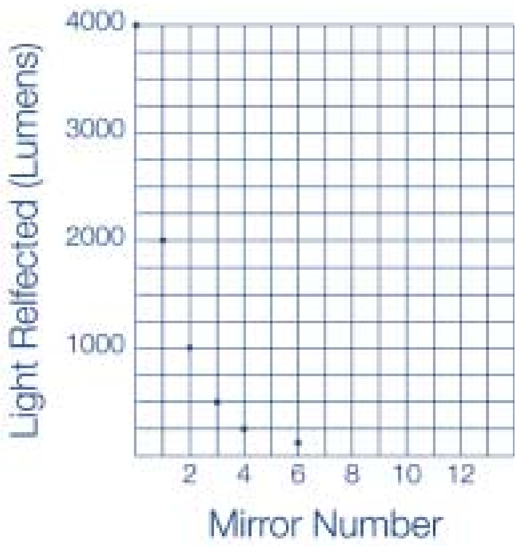
Algebra Assessments

Chapter 5:

Inverse Variations, Exponential Functions, and Other Functions

Bright Lights

The brightness of light can be described with a unit called a lumen. A light of 4000 lumens is shined on a series of mirrors. The resulting number of lumens of reflected light is recorded on this graph. The brightness of the light decreases in the same way after each reflection.



1. Describe the relationship between the mirror number and the lumens measurement. Give your description in words and symbolically. Identify the variables.

2. If this reflection continues, what would be the measurement of the sixth mirror? Explain.

3. If this reflection continues, which mirror would you expect to have a measurement of 50 lumens? Explain.

4. What mirror number might have a measurement of 3.9 lumens? Explain.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions. The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions. The student uses the properties and attributes of functions.

The student:
(C) interprets situations in terms of given graphs or creates situations that fit given graphs; and

(D) in solving problems, collects and organizes data, makes and interprets scatterplots, and models, predicts, and makes decisions and critical judgments.

Scaffolding Questions:

- Is this a linear relationship? Explain your reasoning.
- What is the relationship between the original measurement in lumens and the first measurement?
- What is the relationship between the first measurement and the second measurement?
- What fraction of light does this set of mirrors reflect?
- What would you have to do to the original measurement to get the second measurement?
- What would you have to do to the original measurement to get the third measurement?

Sample Solution:

1. The amount of light reflected each time is one-half of the previous mirror's measurement in lumens.

   The first mirror's lumens measurement is $4000\left(\frac{1}{2}\right)$.

   The second mirror's lumens measurement is $4000\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ or $4000\left(\frac{1}{2}\right)^2$.

   The third mirror's lumens measurement is $4000\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$ or $4000\left(\frac{1}{2}\right)^3$.

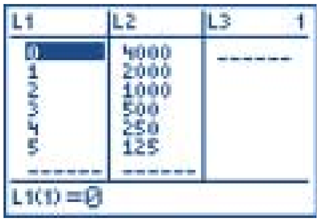
   The nth mirror's lumens measurement is $4000\left(\frac{1}{2}\right)^n$.

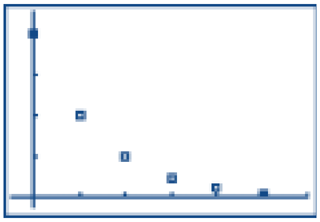
   The function that models this relationship is $L = 4000\left(\frac{1}{2}\right)^n$, where 4000 is the initial amount of light in lumens, $\left(\frac{1}{2}\right)$ is the factor by which the light decreases from mirror to mirror, and n is the mirror number.

Mirror Number	0	1	2	3	4	5
Number of Lumens	4000	2000	1000	500	250	125

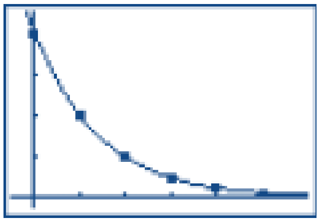
Successive ratios $\frac{2000}{4000} = \frac{1000}{2000} = \frac{500}{1000} = \frac{250}{500} = \frac{125}{250} = \frac{1}{2}$

The values for the domain (the *x*-values) are based on the number of mirrors in the table. The *y* values in the table represent the number of lumens. The ordered pairs plot a curve.





The function $y = 4000\,(0.5)^x$ models the situation. The curve went through all of the graphed points.



2. The table feature on the calculator verifies the lumens for the sixth and seventh mirrors. The feature also tells us if any mirror would reflect exactly

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(d.3) Quadratic and other nonlinear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(A) uses patterns to generate the laws of exponents and applies them in problem-solving situations;

(C) analyzes data and represents situations involving exponential growth and decay using concrete models, tables, graphs, or algebraic methods.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

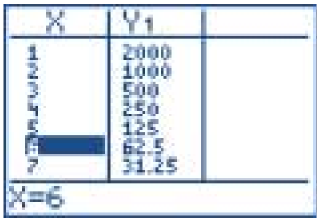
Connections to Algebra I: 2000 and Beyond Institute:

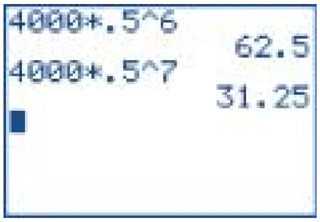
III. Nonlinear Functions
3 Exponential Functions and Equations
- 3.1 Exponential Relationships
- 3.2 Exponential Growth and Decay
- 3.3 Exponential Models

Connections to Algebra End-of-Course Exam:

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

50 lumens. The lumen value may also be verified by substituting the values into the function.





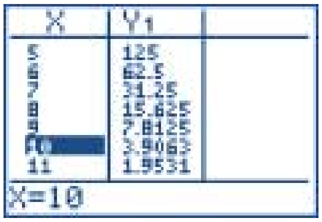
The sixth mirror would reflect $4000\left(\frac{1}{2}\right)^6$ lumens or 62.5 lumens.

The seventh mirror would reflect $4000\left(\frac{1}{2}\right)^7$ or 31.25 lumens.

3. No mirror would reflect exactly 50 lumens.

   Although the graph of the equation is a curve, not all of the points on the curve make sense for this situation. You can only consider mirrors that are whole numbers. Since 50 is between the values 62.5 and 37.5, there is no whole number that will give a function value of 50.

4. The table may be used to determine that the tenth mirror will have approximately 3.9 lumens.



Extension Questions:

- Describe the domain in this situation.

The domain as graphed is {0,1,2,3 …, n}, where n is the number of mirrors.

- Describe the range in this situation.

*The range is {4000, 2000, …, 4000 * $\left(\frac{1}{2}\right)^n$}*

- Describe the rate of change.

There is not a constant rate of change in this problem. The rates of change are:

2000 lumens per mirror after the first mirror.
1000 lumens per mirror between the second and first mirrors.
500 lumens per mirror between the second and third mirrors.
250 lumens per mirror between the third and fourth mirrors.
125 lumens per mirror between the fourth and fifth mirrors.

Because the rate of change is not constant, the equation is not linear. There is a constant ratio of successive terms: one-half.

- If the initial measurement had been 3500 lumens, how would the function have been written?

$L = 3500\left(\frac{1}{2}\right)^n$

- If the fraction of light reflected by the series of mirrors had been one fourth, how would the function be different?

The fraction would be affected and the function would become

$L = 3500\left(\frac{1}{4}\right)^n$.

- If you continued the reflection process, when will the amount of light reflected be zero or less than zero?

Theoretically, the value would never reach zero or below. It would continue to get closer to zero.

Student Work

Bright lights - Bonus

(1.) The relationship between the mirror #, and the lumens reflected, is that for every time the mirror number goes up one, the lumens reflected is half the lumens from the time before.

M= mirror number

PL: Previous lumens

$M+1 = PL/2$

(2.) $4th = 250$

$5 = \frac{250}{2} = 125$

$6 = \frac{125}{2} =$ [62.5 lumens]

I knew the 4th mirror, so using the relationship from above, I divided it in two to get the 5th mirror, and again to get the 6th, which equaled 62.5.

(3.) The measurement of 50 lumens would be in between the 6th and 7th mirror. The 6th mirror has a measure of 62.5 lumens, but the 7th mirror has a measure of 31.25 lumens, and 50 lumens is inbetween those two numbers.

(4.) 7th = 31.25/2 =

8th = 15.625/2 =

9th = 7.8125/2 =

10th = 3.90625

[The 10th mirror ≈ 3.9]

I just started with the 7th mirror and divided by two again and again untill I got to 3.9.

Music and Mathematics

Stringed instruments, like violins and guitars, produce different pitches of a musical scale depending on the length of the string and the frequency of the vibrating string. When under equal tension, the frequency of the vibrating string varies inversely with the string length.

1. Complete the table to find the string lengths for a C-major scale. Round your answers to the nearest whole number.

Pitch	C	D	E	F	G	A	B	C
Frequency (cycles/sec)	523	587	659	698	784	880	988	1046
String Length (mm)	420	___	___	___	___	___	___	___

2. Find a function that models this variation.

3. Describe how the values of the frequency change in relation to the string length.

4. Make a scatterplot of your data. Describe the graph.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(D) in solving problems, collects and organizes data, makes and interprets scatterplots, and models, predicts, and makes decisions and critical judgments.

Scaffolding Questions:

- What is true about the product of the frequency and string length for the first pitch, C?
- What does it mean to say the frequency and string length vary inversely?
- What is true about the product of the frequency and string length for the second pitch, D?
- How can you determine the value of the string length for the second pitch, D?

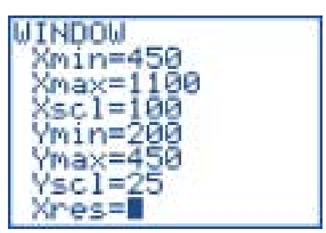
Sample Solution:

1. The problem stated the frequency of the vibrating string varies inversely with the string length. Two quantities vary inversely if their product remains constant. In this situation, the product of the frequency and the string length must remain constant. The product of the frequency for the pitch C (523) with the string length for that pitch (420) resulted in 219,660. This product is used to work backwards and obtain the string length. 219, 660 is divided by each given frequency to complete the table.

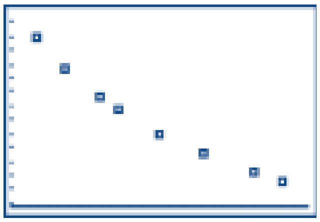
Pitch	C	D	E	F	G	A	B	C
Frequency (cycles/sec)	523	587	659	698	784	880	988	1046
String Length (mm)	420	*374*	*333*	*315*	*280*	*250*	*222*	*210*

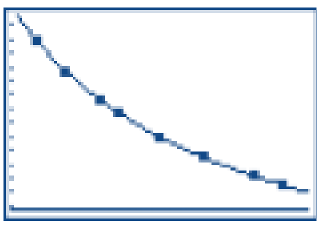
2. As the frequency increases, the string length decreases.

3. Let the *x*-values represent the frequency, and the *y*-values represent the string length.



The scatterplot would not be linear, because the rate of change was not constant. The graph is nonlinear.





4. Because the situation was described as inverse variation, the product of the quantities would remain constant, 219660. This was the pattern used to complete the table, and it is used to write the function.

$$xy = 219660$$
$$y = \frac{219660}{x}$$

where *x* represents the frequency and *y* represents the string length.

Extension Questions:

- Describe the domain and range for the function rule.

The domain and range for the function are all real numbers except zero. Zero is excluded because if one of the numbers, x or y, was zero, the product, xy, would have to be zero.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(d.3) Quadratic and other nonlinear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(B) analyzes data and represents situations involving inverse variation using concrete models, tables, graphs, or algebraic methods.

- Describe the domain and range for the problem situation.

The frequencies for the given pitches are constant as shown on the table. They may be extended in the positive direction. They may not be negative numbers. String length would approach zero but may not be less than zero.

- Compare the function $y = \frac{219660}{x}$ with the function $y = 219660x$.

The function $y = \frac{219660}{x}$ *represents an indirect variation. As the value of y increases, the value of x decreases. The domain and range are all numbers except 0. The graph of the function is not a line.*

The function y = 219660x represents a direct variation. As the value of y increases, the value of x increases. The domain and range of the function are all real numbers. The graph is a straight line.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Connections to Algebra I: 2000 and Beyond Institute:

None.

Connections to Algebra End-of-Course Exam:

Objective 2:
The student will graph problems involving real-world and mathematical situations.

The Marvel of Medicine

A doctor prescribes a dosage of 400 milligrams of medicine to treat an infection. Each hour following the initial dosage, 85% of the concentration remains in the body from the preceding hour.

1. Complete the table showing the amount of medicine remaining after each hour.

Number of Hours	Number of Milligrams Process	Number of Milligrams
0	400	400
1	400 (0.85)	340
2	400 (0.85)(0.85)	
3	400 (0.85)(0.85)(0.85)	
4		
5		

2. Using symbols and words describe the functional relationship in this situation. Discuss the domain and range of the function rule and of the problem situation.

3. Describe how to determine the amount of medicine left in the body after 10 hours.

4. When will the amount reach 60 milligrams? Explain how you know.

5. Why would it be important for the patient to repeat the dosage after a prescribed number of hours?

Teacher Notes

Scaffolding Questions:

- How much is the initial dosage?
- What percentage of the medicine is left in the body after one hour?
- Express this percentage as a decimal.
- How is the amount in the body at two hours related to the amount at one hour?
- Identify the variables in this situation.
- Describe how the variables change in relation to each other.
- Create a scatterplot of the data in the table and describe the graph.
- Explain the difference between questions 4 and 5.

Sample Solution:

1. Repeated multiplication by 0.85 was used to complete the table as follows:

Number of Hours	Number of Milligrams Process	Number of Milligrams
0	400	400
1	400 (0.85)	340
2	400 (0.85)(0.85)	289
3	400 (0.85)(0.85)(0.85)	245.65
4	400 (0.85)(0.85)(0.85)(0.85) = 400 $(0.85)^4$	208.8
5	400 (0.85)(0.85)(0.85)(0.85)(0.85) = 400 $(0.85)^5$	177.48
x	400 $(0.85)^x$	

2. The amount of medicine remaining in a patient's system is 400 times the rate raised to the number of hours. The *x*-values represent the number of hours the medicine is in a patient's system. The *y*-values represent the amount of medicine (in milligrams) that is in a patient's system. A scatterplot may be used to analyze the data.

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

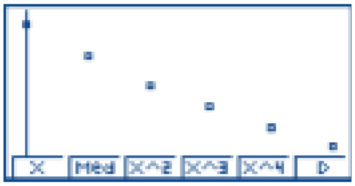
(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

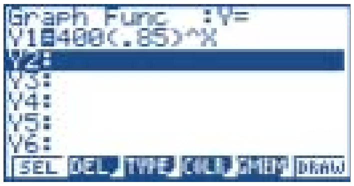
The student:
(B) for a variety of situations, identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations;

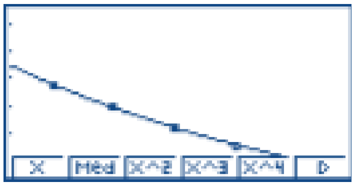
(D) in solving problems, collects and organizes data, makes and interprets scatterplots, and models, predicts, and makes decisions and critical judgments.



The graph is nonlinear. The ratio of successive terms is a constant ratio of 0.85.

A function that is used to model this situation is $y = 400(0.85)^x$. This model will apply until another dose of medicine is administered.





The domain of the function is the set of all real numbers, but the domain of the problem situation is the set of nonnegative numbers because *x* represents time in this situation. The range for the problem situation is the set of all numbers less than or equal to 400 but greater than zero. Theoretically, the amount of medicine would never reach zero.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(d.3) Quadratic and other nonlinear functions.
The student understands there are situations modeled by functions that are neither linear nor quadratic and models the situations.

The student:
(A) uses patterns to generate the laws of exponents and applies them in problem-solving situations;

(C) analyzes data and represents situations involving exponential growth and decay using concrete models, tables, graphs, or algebraic methods.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundation for Functions
2 Using Patterns to Identify Relationships
  2.2 Identify More Patterns

III. Nonlinear Functions
3 Exponential Functions and Equations
  3.1 Exponential Relationships
  3.2 Exponential Growth and Decay
  3.3 Exponential Models

Connections to Algebra End-of-Course Exam:

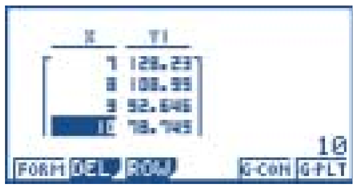
Objective 6:
The student will perform operations on and factor polynomials that describe real-world and mathematical situations.

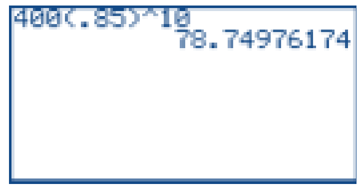
Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

Objective 8:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving one-variable or two-variable situations.

Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

3. The calculator can be used to explore the amount of medicine left in the patient's system after different numbers of hours. After 10 hours, there will be 78.75 milligrams left.





4. From the table it can be seen that a patient will have 60 milligrams left in his system between 11 and 12 hours after the initial dosage.

5. A patient would need to repeat the prescribed dosage to keep a constant amount of medicine in his system to fight the infection.

Extension Questions:

- If the rule had been $y = 500(0.85)^x$ instead of $y = 400(0.85)^x$, how would this situation be different from the given situation?

The initial amount had been changed from 400 to 500. The percent has not been changed.

- What would the equation be if the amount of medicine was reduced by 30 percent each hour?

If the amount was reduced by 30 percent each hour, then there would be 70% of the amount left. If the original amount was 400 milligrams, the equation would be $y = 400(0.70)^x$.

- If you took a new dosage of 400 milligrams at the 12th hour, how much would you expect to have in your system in the 15th hour?

From the table the amount in the system at the 12th hour is 56.897 milligrams. If 400 milligrams are added, the amount in the system is 456.897 milligrams. This amount would be multiplied by the factor .85 to obtain the amount at the 13th, 14th, and 15th hours. The amount in the system at the 15th hour is approximately 280.59 milligrams.