SUPPLEMENTAL

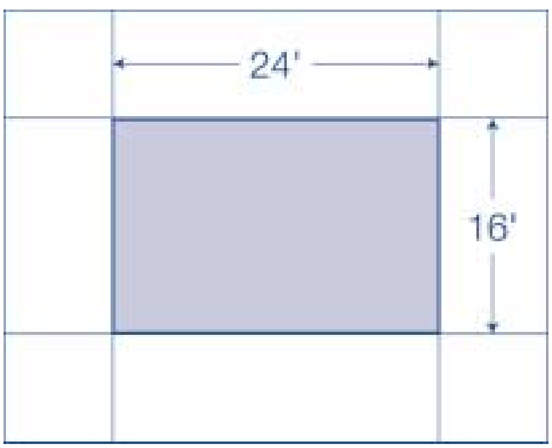
Algebra Assessments

Chapter 9:

Quadratic Functions

A Ring Around the Posies

The Lush Landscaping Company is involved in a project for the city. The garden in front of the city hall is going to be expanded by planting a border of flowers all the way around it. The current dimensions of the garden are 24 feet long by 16 feet wide. The border will have the same width around the entire garden. The flowers that will be planted in the border will fill an area of 276 $feet^2$. Find the width of the border surrounding the garden using symbolic manipulation. Explain the meaning of each number and symbol that you use.



Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

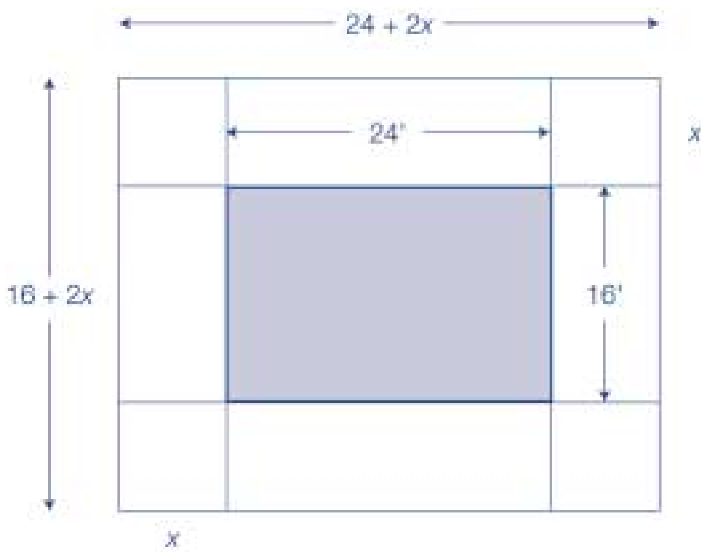
The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

Scaffolding Questions:

- How can you find the area of the existing garden?
- What are the variables in this situation?
- Name a possible length and width of a larger garden.
- Name a second possible length and width of a larger garden.
- Represent the dimensions of the larger garden algebraically.
- How can you find the area of the border?
- How does the area of 276 feet^2 relate to this situation and your equation?

Sample Solution:



The area of the original garden, having dimensions 24 feet by 16 feet, was calculated using the formula for area of rectangle = length • width.

$A = L \cdot W$
$A = 24 \cdot 16$
$A = 384$ feet^2

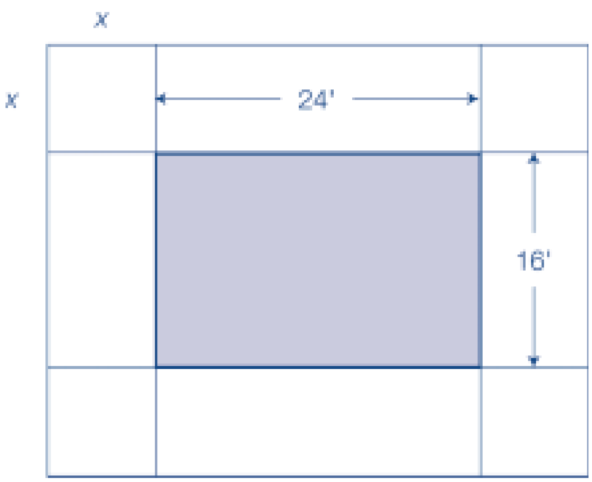
The area of the new garden with the border was calculated with the same area formula as listed above with x representing the width of the border.

$A = (24 + 2x)(16 + 2x)$
$A = 384 + 48x + 32x + 4x^2$
$A = 384 + 80x + 4x^2$

The flowers are going to be planted in the border that surrounds the 24 by 16 rectangle. To find that area, subtract the area of the small inside rectangle from the large rectangle.

Border area = large garden area – small original area
Border area = $(384 + 80x + 4x^2) - (384)$
Border area = $80x + 4x^2$

Another way to look at the border is to think of it as the sum of the rectangles that make up the border.



There are two rectangles at the top and bottom with measurements 24 and x. The two rectangles on the sides have measurements 16 and x. There are four corner squares with side length x. The border area would be

$A = 2(24x) + 2(16x) + 4x^2$
$A = 48x + 32x + 4x^2$
$A = 80x + 4x^2$

The problem stated that the flowers would fill the border that had an area of 276 feet^2. Substitute this value for the border area.

$276 = 80x + 4x^2$

Factor to solve this equation.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations; and

(B) uses the commutative, associative, and distributive properties to simplify algebraic expressions.

(d.2) Quadratic and other nonlinear functions.
The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.

The student:
(A) solves quadratic equations using concrete models, tables, graphs, and algebraic methods.

Texas Assessment of Knowledge and Skills:

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
1 Quadratic Functions
1.1 Quadratic Relationships

Connections to Algebra End-of-Course Exam:

Objective 5:
The student will formulate or solve quadratic equations that describe real-world and mathematical situations.

$0 = 4x^2 + 80x - 276$
$0 = 4(x^2 + 20x - 69)$
$0 = 4(x + 23)(x - 3)$
$0 = x + 23$ and $0 = x - 3$
$x = -23, 3$

The solutions for x are -23 and 3. Since x represents the width of the border, the value cannot be negative. The only solution that makes sense is 3. The width of the border is 3 ft.

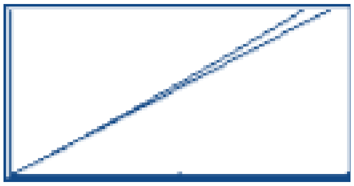
Extension Questions:

- Find the border area A as a function of border width x.

The border area is the same as illustrated in the sample solution $A = 80x + 4x^2$.

- Graph this function, and illustrate the solution to the original problem on the graph.

The top graph of this quadratic function is almost linear in the region that is relevant to the solution of this problem. The lower graph of 80x is shown for comparison.



Block That Kick

The punter on a special team unit kicks a football upward from the ground with an initial velocity of 63 feet per second. The height of the football stadium is 70 feet. The height of an object with respect to time is modeled by the equation

$$h = \frac{1}{2}gt^2 + vt + s$$

where g is -32 ft/sec ^{2}, v is the initial velocity, and s is the initial height.

1. Write a function that models this situation.
2. Sketch and describe the graph of this function.
3. At what times will the football be the same height as the top of the stadium? Explain your answer.
4. Suppose the punter's initial velocity is 68 feet per second. At what times will the football be the same height as the top of the stadium? Justify your answer.

Teacher Notes

Materials:
Graphing calculator.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

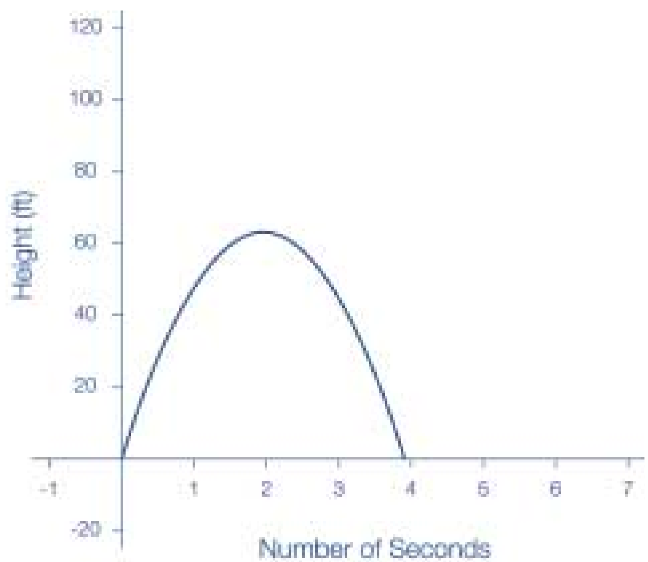
Scaffolding Questions:

- What is value of *v* in the function for this situation?
- What is the initial height?
- What do you expect the graph to look like?
- How can you tell from the graph how many solutions there will be? Explain your answer.
- How can the discriminant be used to determine the number of solutions to this problem situation?

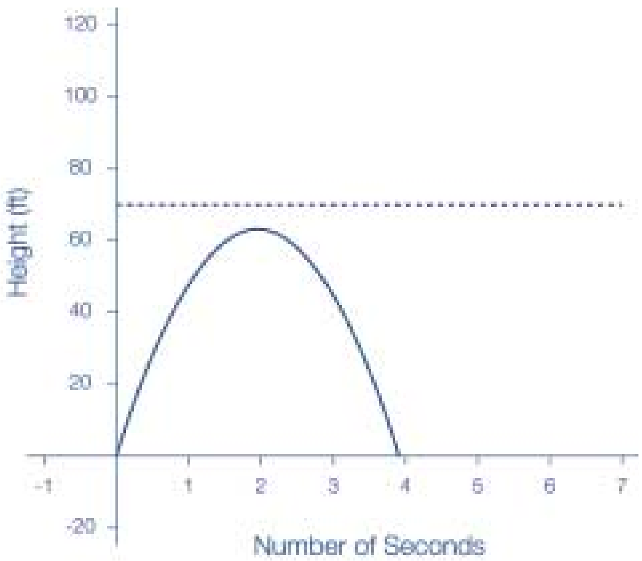
Sample Solution:

1. The initial velocity is 63 feet per second and the initial height is 0 because the ball was kicked from ground level. The height of an object after *t* seconds when projected upward is modeled by the function $f(t) = -16t^2 + 63t$.

2. The graph is a parabola that opens downward.



3. To ask when the object will be at the same height as the stadium is to ask when will the height be 70 feet. From the graph it can be seen that the height will never be 70 feet.



There is no time when the ball is at the height of 70 feet. Another way to look at this problem is to solve the equation $70 = -16t^2 + 63t$, or

$0 = -16t^2 + 63t - 70$

The value of the discriminant $b^2 - 4ac$ for a quadratic $ax^2 + bx + c$ verifies there is no solution because it is less than 0.

$$\begin{aligned} b^2 - 4ac &= (63)^2 - 4(-16)(-70) \\ &= 3969 - 4480 \\ &= -511 \end{aligned}$$

4. If the initial velocity is increased to 68 feet per second. The function becomes $h(t) = -16t^2 + 68t$. The equation becomes $70 = -16t^2 + 68t$, or

   $0 = -16t^2 + 68t - 70$

   The related quadratic function is $h(t) = -16t^2 + 68t - 70$

   The function will have 2 roots, because the graph crosses the x-axis twice.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

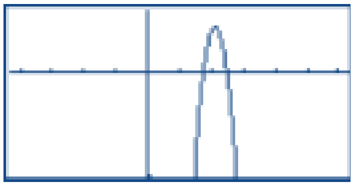
The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations;

(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

(d.2) Quadratic and other nonlinear functions.
The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.

The student:
(A) solves quadratic equations using concrete models, tables, graphs, and algebraic methods; and

(B) relates the solutions of quadratic equations to the roots of their functions.



The roots of the function are at approximately 1.75 and 2.5. This means that the football will be at 70 feet on the ascent, after 1.75 seconds, and again on the descent, at 2.5 seconds.

Extension Questions:

- How many feet off the ground would the ball have to be kicked at the given velocity to reach a height of 70 feet?

The maximum value of the parabola is about 62 feet. If the ball was kicked from 8 feet off the ground, the graph of the function would be raised vertically 8 feet and would reach the height of 70 feet.

- Suppose the football just reaches the same height of the stadium one time. Predict the initial velocity needed to have a maximum height of 70 feet to the nearest thousandth.

Look at the function $f(t) = -16t^2 + vt - 70$. *With the previous two problems the value would be between 63 and 68. If this equation has only one root, the discriminant must be equal to zero.*

$$\begin{aligned} b^2 - 4ac &= v^2 - 4(-16)(-70) \\ 0 &= v^2 - 4(-16)(-70) \\ v^2 &= 4480 \\ v &= 66.933 \end{aligned}$$

In order to have the football reach the 70-foot level one time, the initial velocity would have to be approximately 66.933 feet per second. This value comes close to producing a discriminant with a value of zero.

Texas Assessment of Knowledge and Skills:

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

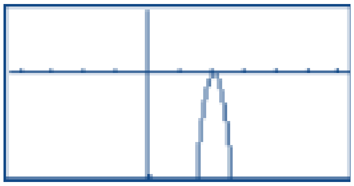
III. Nonlinear Functions
1 Quadratic Functions
  1.1 Quadratic Relationships

Connections to Algebra End-of-Course Exam:

Objective 5:
The student will formulate or solve quadratic equations that describe real-world and mathematical situations.

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

The function would be $f(t) = -16t^2 + 66.933t - 70$. The graph would have one root.



BRRR!

The windchill measures how cold the temperature feels at different wind speeds. The faster the wind carries away the warm air around your body, the colder you feel. The windchill, c, at a given temperature in Fahrenheit is a reasonably good quadratic function of the wind speed in miles per hour, s.

For example, at 0 degrees Fahrenheit, the function $c = 0.028s^2 - 2.52s + 2.7$ models the chill factor with wind speeds from 0 to 45 miles per hour.

1. Graph the function and describe how the function models the situation.

2. Use the quadratic formula to find the wind speed for a windchill of -10 degrees.

Teacher Notes

Scaffolding Questions:

- Graph the windchill function.
- Describe the graph.
- Explain why the model would not be reasonable for wind speeds above 45 miles per hour.
- Determine the roots of the function and how they relate to the given situation.

Sample Solution:

1. The graph of this function is a parabola that opens upward. According to the graph, the roots of the function are approximately at 1.0 and 88.9.

The graph doesn't make sense for wind speeds over 45 miles per hour because as the wind speed, represented by the *x* values, increases over 45 miles per hour, the y values begin to increase (warm up). Normal wind speeds don't usually go above 30 miles per hour, so the model is good for general use.

2. Windchill = -10 = *c*

$-10 = 0.028s^2 - 2.52s + 2.7$

Add 10 to each side of the equation to put it in standard form.

$0 = 0.028s^2 - 2.52s + 12.7$

Use the quadratic formula to solve, $a = 0.028$; $b = -2.52$; $c = 12.7$. Round to the nearest tenth.

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations;

(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

$$t = \frac{-(-2.52) \pm \sqrt{(-2.52)^2 - 4(0.028)(12.7)}}{2(0.028)}$$

$$= \frac{2.52 \pm \sqrt{6.4 - 1.44}}{0.056}$$

$$= \frac{2.52 \pm \sqrt{5}}{0.056}$$

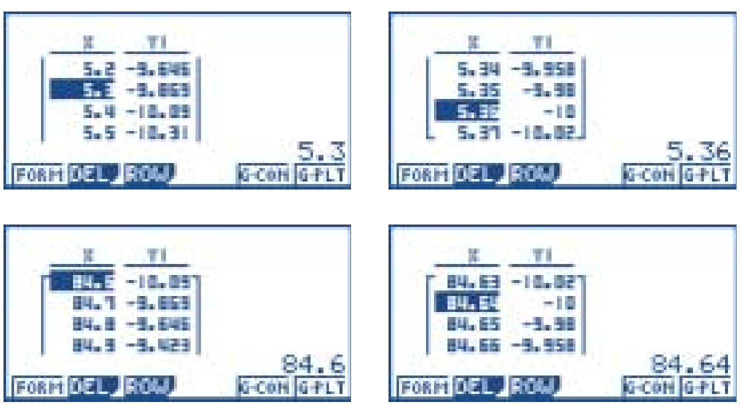
$$= 84.9, 5.4$$

Only a wind speed of 5.4 miles per hour makes sense for this situation because the value of s must be between 0 and 45 miles per hour.

Extension Questions:

- Describe how to solve the second problem using a table.

Examine the table of the function. Set the increments small enough to find a value close to -10. The values are 5.36 and 84.6 The value at 5.36 miles per hour is the reasonable value.



- Describe how to solve the second problem using a graph of a quadratic function.

The function is $c = 0.028s^2 - 2.52s + 12.7$.

The graph of the function is a parabola that opens upward and has two roots.

(d.2) Quadratic and other nonlinear functions.
The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.

The student:
(A) solves quadratic equations using concrete models, tables, graphs, and algebraic methods; and

(B) relates the solutions of quadratic equations to the roots of their functions.

Texas Assessment of Knowledge and Skills:

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

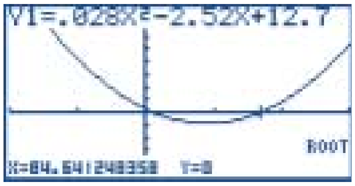
Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
2 Quadratic Equations
 2.1 Connections
 2.2 The Quadratic Formula

Connections to Algebra End-of-Course Exam:

Objective 5:
The student will formulate or solve quadratic equations that describe real-world and mathematical situations.

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.



The values are 5.36 and 84.6. The value at 5.36 miles per hour is the reasonable value.

Calculating Cost

The marketing team for Capital Computer Company is working on a project to find the most profitable selling price for its company's new laptop computer. After much research, the team decided that the function $N = -100p^2 + 300{,}000p$ represents the expected relationship between N, the net sales in dollars, and p, the retail price of the laptop computer.

1. Show how to solve the equation $-100p^2 + 300{,}000p = 0$ symbolically.
2. What information does the solution give you about the expected sales?
3. Graph the function. What does the graph tell you about the relationship between the price and the net sales?
4. What price should the marketing team propose for the laptop computer? Explain your answer.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

Scaffolding Questions:

- What does the term net sales mean?
- Describe different methods you might use to solve the equation.
- What does the variable p represent?
- What does it mean when the net sales are equal to 0?
- Describe what the graph tells you about the net sales as the retail price increases.

Sample Solution:

1. The equation $-100p^2 + 300{,}000p = 0$ is solved by factoring:
   $-100p(p - 3000) = 0$

   $-100p = 0$, therefore $p = 0$, and
   $p - 3000 = 0$, so $p = 3000$.

2. These solutions indicate that the net sales will be 0 if the price of the laptop is \$0 or \$3000.

3. The exponent of 2 in the equation indicates the graph will be a parabola. The negative sign of the coefficient indicates the graph will open downward. The graph of the function is as follows:

   The graph shows that as the price of the laptop increases from \$0 to \$1500, the net sales increase from \$0 to \$225 million. As the price continues to increase up to \$3000, the net sales decrease back to \$0.

4. The marketing team should select \$1500 as a sales price for the laptop computer because the maximum net sales occur at that point.

Extension Questions:

- If the coefficient on p is changed to 200,000, how will the change affect the solutions?

The answer will be changed to 0 and 2000.

$-100p(p - 2000) = 0$

$-100p = 0$, therefore $p = 0$, and
$p - 2000 = 0$, so $p = 2000$.

- Without graphing, tell how you expect this change to affect the graph?

The graph will have a vertex that has an x-value halfway between 0 and 2000. The maximum net sales will occur at p = 1000.

(d.2) Quadratic and other nonlinear functions.
The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.

The student:
(A) solves quadratic equations using concrete models, tables, graphs, and algebraic methods.

Texas Assessment of Knowledge and Skills:

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
2 Quadratic Equations
2.1 Connections

Connections to Algebra End-of-Course Exam:

Objective 6:
The student will perform operations on and factor polynomials that describe real-world and mathematical situations.

Investigating the Effect of *a* and *c* on the Graph of $y = ax^2 + c$

1. For each of the sets of functions in Problem Sets A – D on the following pages, complete the table to compare their graphs with the graph of the parent function $y = x^2$. Describe how changing the parameters *a* and *c* affects the shape of the graph of the parent function, its intercepts, and its orientation.

2. Each of the following describe transformations on the graph of $y = x^2$. Describe how the vertex, axis of symmetry, intercepts, and symbolic function will be changed by the transformation.

   A. Vertically stretch by a factor of 4 and translate up by 2 units.

   B. Vertically compress by a factor of $\frac{1}{2}$ and translate down by 2 units.

   C. Vertically stretch by a factor of 2 and reflect over the *x*-axis.

Activity Worksheet: Problem Set A

Complete the table to compare the graphs of the given functions with the graph of the parent function $y = x^2$. Describe how changing the parameters *a* and *c* affects the shape of the graph of the parent function, its intercepts, and its orientation.

Function	Graph Show Parent graph plain, Transformed graph bold.	Table Include vertex and images of points (±1,1) and (±2,4) in table.	Description Verbal description of transformation(s)
A1. $f(x) = 2x^2$			
A2. $g(x) = (2x)^2$			
A3. $h(x) = \frac{1}{2}x^2$			
A4. $m(x) = (0.5x)^2$			

Activity Worksheet: Problem Set B

Complete the table to compare the graphs of the given functions with the graph of the parent function $y = x^2$. Describe how changing the parameters *a* and *c* affects the shape of the graph of the parent function, its intercepts, and its orientation.

Function	Graph Show Parent graph plain, Transformed graph bold.	Table Include vertex and images of points (±1,1) and (±2,4) in table.	Description Verbal description of transformation(s)
B1. $f(x) = -x^2$			
B2. $g(x) = -3x^2$			
B3. $h(x) = (-3x)^2$			
B4. $m(x) = -(3x)^2$			

Activity Worksheet: Problem Set C

Complete the table to compare the graphs of the give functions with the graph of the parent function $y = x^2$. Describe how changing the parameters *a* and *c* affects the shape of the graph of the parent function, its intercepts, and its orientation.

Function	Graph Show Parent graph plain, Transformed graph bold.	Table Include vertex and images of points (±1,1) and (±2,4) in table.	Description Verbal description of transformation(s)
C1. $f(x) = x^2 - 1$			
C2. $g(x) = x^2 + 2$			
C3. $h(x) = (x - 1)^2$			
C4. $m(x) = (x + 2)^2$			

Activity Worksheet: Problem Set D

Complete the table to compare the graphs of the given functions with the graph of the parent function $y = x^2$. Describe how changing the parameters *a* and *c* affects the shape of the graph of the parent function, its intercepts, and its orientation.

Function	Graph Show Parent graph plain, Transformed graph bold.	Table Include vertex and images of points (±1,1) and (±2,4) in table.	Description Verbal description of transformation(s)
D1. $f(x) = 2x^2 + 1$			
D2. $g(x) = -3x^2 + 12$			
D3. $h(x) = 0.5(x - 2)^2$			
D4. $m(x) = -(x + 2)^2$			

Teacher Notes

Scaffolding Questions:

- What does the graph of the parent function $y = x^2$ look like? Is it a linear graph? What characterizes this graph?

For Problem Set A

- How are the functions different from the parent function?
- How does a table of values help you see the difference in the graph of the new function and the parent function?
- What does it mean to change a quantity by a scale factor?
- How is the point (1,1) in the original function affected by the scale factor?

For Problem Set B

- How are the functions different from the parent function?
- How are they different from the functions in Set A?
- What effect does the negative coefficient have on the graph?

For Problem Set C

- How are the functions different from the parent function?
- How does comparing a table of values for the parent function and these four functions help you see what is happening to the graph of the parent function?

For Problem Set D, you are working with combinations of transformations on the parent function.

- For each function, how would you describe the sequence of transformations performed on the parent function that give the new function?

Sample Solution:

In each set, to compare the parent function with each of the four transformed functions, we can use graphs and tables to tell us what effect each transformation has on the graph of the parent function.

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

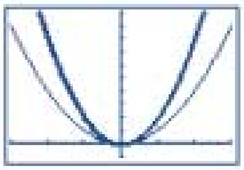
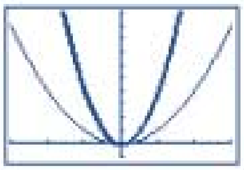
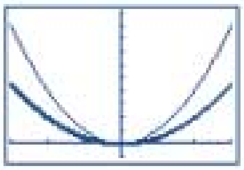
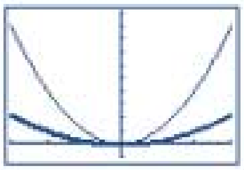
The student:
(A) identifies and sketches the parent forms of linear ($y = x$) and quadratic ($y = x^2$) functions.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(B) investigates, describes, and predicts the effects of changes in a on the graph of $y = ax^2$;

(C) investigates, describes, and predicts the effects of changes in c on the graph of $y = x^2 + c$.

Set A:

Function	Graph	Table	Description
A1. $f(x) = 2x^2$		x | y -2 | 8 -1 | 2 0 | 0 1 | 2 2 | 8	Vertically stretch $y = x^2$ by a factor of 2. The vertex remains at (0,0).
A2. $g(x) = (2x)^2$		x | y -2 | 16 -1 | 4 0 | 0 1 | 4 2 | 16	Horizontally stretch x by a factor of 2. Then square. It is equivalent to vertical stretch by 4. The vertex remains at (0,0).
A3. $h(x) = \frac{1}{2}x^2$		x | y -2 | 2 -1 | 0.5 0 | 0 1 | 0.5 2 | 2	Vertically compress $y = x^2$ by a factor of 0.5. The vertex remains at (0,0).
A4. $m(x) = (0.5x)^2$		x | y -2 | 1 -1 | 0.25 0 | 0 1 | 0.25 2 | 1	Horizontally compress x by a factor of 0.5 and then $0.5x$ is squared. It is equivalent to a vertical compression by a factor of 0.25. The vertex remains at (0,0).

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

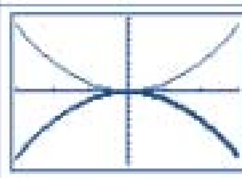
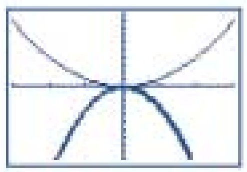
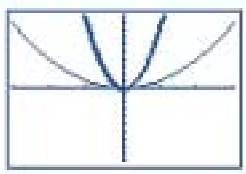
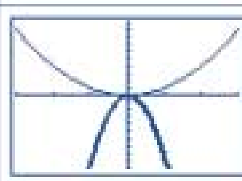
III. Nonlinear Functions
1 Quadratic Functions
1.2 Transformations

Connections to Algebra End-of-Course Exam:

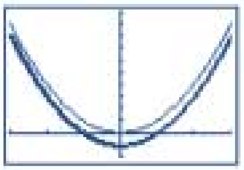
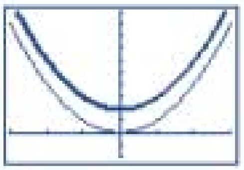
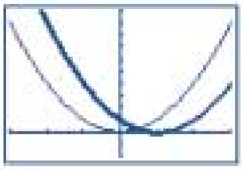
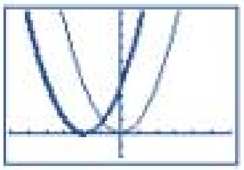
Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Objective 2:
The student will graph problems involving real-world and mathematical situations.

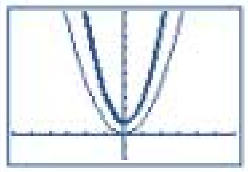
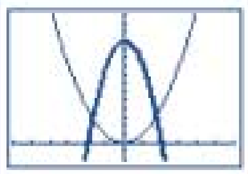
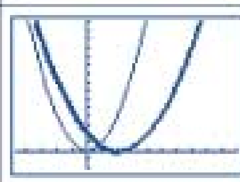
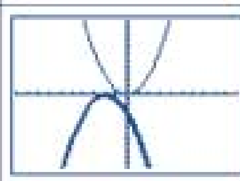
Set B:

Function	Graph	Table	Description
B1. $f(x) = -x^2$		x: -2, -1, 0, 1, 2 y: -4, -1, 0, -1, -4	Reflect graph of parent function over x-axis. The vertex remains at (0,0).
B2. $g(x) = -3x^2$		x: -2, -1, 0, 1, 2 y: -12, -3, 0, -3, -12	Vertically stretch parent graph by a factor of 3. Reflect over x-axis. The vertex remains at (0,0).
B3. $h(x) = (-3x)^2$		x: -2, -1, 0, 1, 2 y: 36, 9, 0, 9, 36	Stretch x horizontally by a factor of 3 and then the product is squared. Reflect over y-axis. It is equivalent to a vertical stretch by 9. The vertex remains at (0,0).
B4. $m(x) = -(3x)^2$		x: -2, -1, 0, 1, 2 y: -36, -9, 0, -9, -36	Stretch x horizontally by a factor of 3. Square to get y. Reflect over x-axis. The vertex remains at (0,0).

Set C:

Function	Graph	Table	Description
C1. $f(x) = x^2 - 1$		x , y -2 , 3 -1 , 0 0 , -1 1 , 0 2 , 3	Translate vertically down one unit. Vertex moves to (0,-1). y-intercept = (0,-1). x-intercepts are (-1,0), (1,0).
C2. $g(x) = x^2 + 2$		x , y -2 , 6 -1 , 3 0 , 2 1 , 3 2 , 6	Translate vertically up two units. Vertex moves to (0,2). y-intercept = (0,2). No x-intercepts.
C3. $h(x) = (x - 1)^2$		x , y -2 , 9 -1 , 4 0 , 1 1 , 0 2 , 1	Translate right one unit. Vertex moves to (1,0). y-intercept = (0,1). x-intercept = (1,0).
C4. $m(x) = (x + 2)^2$		x , y -2 , 0 -1 , 1 0 , 4 1 , 9 2 , 16	Translate left two units. Vertex moves to (-2,0). y-intercept = (0,4). x-intercept = (-2,0).

Set D:

Function	Graph	Table	Description
D1. $f(x) = 2x^2 + 1$		x: -2, -1, 0, 1, 2 y: 9, 3, 1, 3, 9	Vertically stretch parent graph by a factor of 2. Then translate up one unit. The vertex becomes (0,1). There are no x-intercepts.
D2. $g(x) = -3x^2 + 12$		x: -2, -1, 0, 1, 2 y: 0, 9, 12, 9, 0	Vertically stretch parent graph by a factor of 3. Reflect over x-axis. Translate up 12 units. The vertex becomes (0,12). The x-intercept is (±2,0).
D3. $h(x) = 0.5(x - 2)^2$		x: -2, -1, 0, 1, 2 y: 8, 4.5, 2, 0.5, 0	Translate parent graph right two units. Vertically compress by a factor of 0.5. The vertex becomes (2,0). The y-intercept is (0,2).
D4. $m(x) = -(x + 2)^2$		x: -2, -1, 0, 1, 2 y: 0, -1, -4, -9, -16	Translate parent graph left two units. Reflect over x-axis. The vertex becomes (-2,0). The y-intercept is (0,-4).

2A. If the graph of $y = x^2$ is vertically stretched by a factor of 4 and translated up 2 units, the new function is $y = 4x^2 + 2$. The original vertex (0,0) translates to (0,2). The axis of symmetry is still the y-axis. The y-intercept is the vertex. There are no x-intercepts since the equation $4x^2 + 2 = 0$ has no real solution.

2B. If the graph of $y = x^2$ is vertically compressed by a factor of 1/2 and translated down 2 units, the new function is $y = \frac{1}{2}x^2 - 2$. The original vertex (0,0) translates to (0,-2). The axis of symmetry is still the y-axis. The

y-intercept is the vertex. Solving the equation $\frac{1}{2}x^2 - 2 = 0$ implies $x = \pm 2$, so the x-intercepts are (-2,0) and (2,0).

2C. If the graph $y = x^2$ is vertically stretched by a factor of 2 and reflected over the x-axis, the new function is $y = -2x^2$. The vertex is still (0,0). The axis of symmetry is still the y-axis. The y-intercept and x-intercepts are both 0.

Extension Questions:

- What kinds of transformations on the graph of $y = x^2$ can be performed so that the resulting graph continues to be that of a function?

Since a function is a relation between x and y that generates exactly one output value, y, for each input value, x, the only transformations on $y = x^2$ we can consider are dilations, translations, and reflections over the axes.

- What parameter causes a dilation on the graph of $y = x^2$? What is another way of saying "dilation"? What else does this parameter tell you?

The parameter, a, in $y = ax^2$ causes dilation. This dilation is a vertical stretch or a vertical compression on the graph of $y = x^2$. The parameter, a, also causes a reflection over the x-axis if a is negative. As the magnitude of $|a|$ increases, the graph of $y = ax^2$ becomes steeper.

- What parameter causes a translation and what kind of translation?

The parameter, c, in $y = ax^2 + c$ causes a vertical translation. The parent graph is translated c units up if c is positive and -c units down if c is negative.

- What does the "Order of Operations" sequence tell you about the sequence of transformations performed on the parent graph to generate the graph of $y = ax^2 + c$?

If $a > 0$, vertically stretch or compress the graph of $y = x^2$ by a factor that equals the magnitude of a.

Then translate vertically c units.

If $a < 0$, vertically stretch or compress the graph of $y = x^2$ by a factor that equals the magnitude of a. Next, reflect the graph over the x-axis, and then translate vertically c units.

- What is the difference between the transformations $y = ax^2$ and $y = (ax)^2$?

The first transformation is a vertical stretch or compression of the graph of $y = x^2$. The second transformation is a horizontal stretch or compression on x before squaring. It could also be described as a vertical stretch or compression, but the dilation factor is a^2 since $(ax)^2 = a^2x^2$.

- What is the difference between the transformations $y = x^2 + c$ and $y = (x + c)^2$?

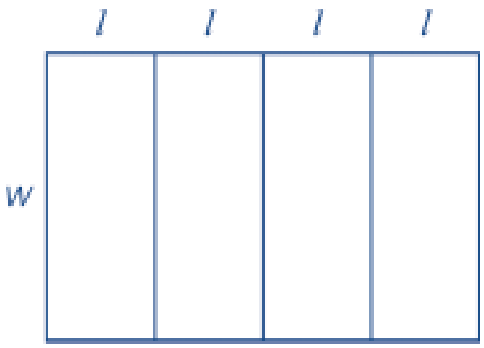
The first transformation is a vertical translation c units up if c is positive and -c units down if c is negative. The second transformation is a horizontal translation c units left if c is positive and -c units right if c is negative.

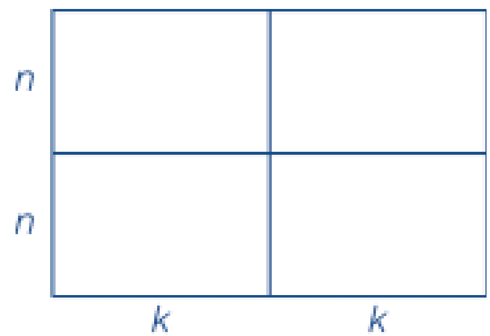
Ostrich Pen

A rancher who raises ostriches has 60 yards of fencing to enclose 4, equally sized, rectangular pens for his flock of ostriches. He is considering two options:

A. Arrange the smaller pens in a line with adjacent pens sharing one common fence.

B. Arrange the four smaller pens so that they share exactly two common fences as shown in the diagram.





1. Define the functions for the total area of each of the two options.

2. Create a graph of each function. Compare the domains and ranges for the options.

3. For each option, describe the dimensions that give maximum total area of the pens. How do the total areas of the two options compare? How do the individual pen areas in the two options compare?

4. If the total amount of fencing for the pens was doubled, how would this change your responses to Parts 1, 2, and 3?

Teacher Notes

Scaffolding Questions:

- What is an expression for the length of the large pen for possibility A?
- What is an expression for the width of the large pen for possibility A?
- What is an expression for the length of the large pen for possibility B?
- What is an expression for the width of the large pen for possibility B?
- What expression in l and w describes the total perimeter for the first set of pens?
- What expression in n and k describes the total perimeter for the second set of pens?
- Describe the restriction on the perimeter.
- What equation expresses this restriction?
- With each of the equations, how can you express one variable in terms of the other?
- Using the given variables, what area function will you write for the total area, A, for each set of pens?
- How can you express the area, A, for each set of pens just in terms of one of the variables?
- What representation(s) will best help you describe the domain and range for each set of pens? How does the situation restrict the domains and ranges?
- By looking at the graph, how can you determine the maximum area?

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(B) for a variety of situations, identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables.

Sample Solution:

1. For each set of pens the total area, A, is given by the product of the length and the width, but the total amount of fencing to be used, 60 yards, can be used to relate the two variables.

   For the first set of pens,

   $$5w + 8l = 60$$
   $$l = \frac{60 - 5w}{8}$$

   For the second set of pens,

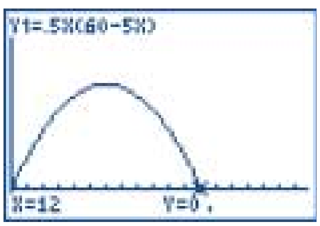
   $$6n + 6k = 60$$
   $$k = \frac{60 - 6n}{6} = \frac{6(10 - n)}{6}$$

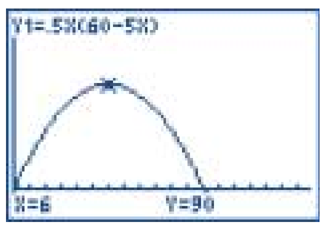
   Area equals width times length. The area functions, for the first and second sets of pens, respectively, are

$$\begin{aligned} A_1 &= w \cdot 4l \\ &= w \cdot 4\left(\frac{60 - 5w}{8}\right) \\ &= \frac{4}{8} w(60 - 5w) \\ &= 0.5w(60 - 5w) \end{aligned} \qquad \begin{aligned} A_2 &= 2n \cdot 2k \\ &= 2n \cdot 2(10 - n) \\ &= 4n \cdot (10 - n) \end{aligned}$$

2. The domains and ranges for the two area functions are easily seen by building a table or by graphing. For the first set of pens, rewrite the area function using the variables *y* and *x*.

   $y = 0.5x(60 - 5x)$ where *y* is the area and *x* is the width.

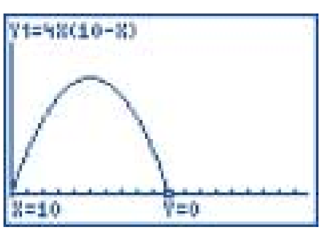


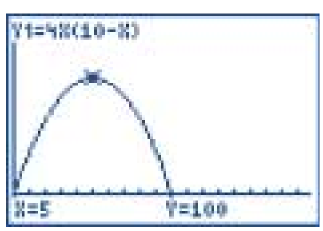


   The domain for the first set of pens is the set of all values, $0 < x < 12$. These are the only values that make sense in the situation, since the area must be positive.

   The range would be the set of all values, A, $0 < A \leq 90$, because the maximum value seen on the graph is 90.

   Similarly, the domain and range for the second set of pens may be determined by examining the graph of $y = 4x(10 - x)$.





   The domain for the first set of pens is the set of all values, $0 < x < 10$.

   The range would be the set of all values, A, $0 < A \leq 100$ because the maximum value seen on the graph is 100.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations;

(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

Texas Assessment of Knowledge and Skills:

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
1 Quadratic Functions
  1.1 Quadratic Relationships
2 Quadratic Equations
  2.1 Connections

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

3. The maximum area occurs at the vertex of the parabola, which is the graph of each area function. The maximum area for situation *A* is 90 square units when *x* is 6. *x* represented the width, *w*, in the original diagram.

$$5w + 8l = 60$$

$$l = \frac{60 - 5w}{8} = \frac{60 - 5(6)}{8} = 3.75$$

The dimensions of the large pen with a total area of 90 square yards are *w* and 4 *l* , or 6 yards and 4(3.75), or 15 yards.

For situation B the maximum area is 100 square units at *x* = 5. *x* represented the variable, *n,* in the original diagram.

The dimensions of the pen are 2*n* and 2*k*.

$$6n + 6k = 60$$

$$k = \frac{60 - 6n}{6} = \frac{6(10 - n)}{6} = 10 - n = 10 - 5 = 5$$

The dimensions of the large pen for the area of 100 square yards are 2(5) = 10 yards and 2(5) = 10 yards.

Dividing the total area by 4 gives the area of each pen. Each pen in the first set has an area of $\frac{90}{4}$ or 22.5 square yards and has a dimension of 6 yards by 3.75 yards.

Each pen in the second set has an area of $\frac{100}{4}$ or 25 square yards and has a dimension of 5 yards by 5 yards. Therefore, a square arrangement of the pens gives the maximum area.

4. If the total amount of fencing to be used were doubled or tripled, this would change the area functions and increase domains and ranges. This can be investigated easily using the table of values for the functions or the graphs.

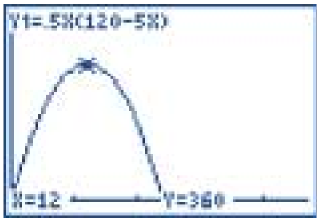
Doubling the fencing changes the 60 feet of fencing to 120 feet of fencing. The first area function changes.

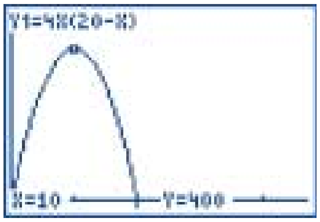
$$5w + 8l = 120$$
$$l = \frac{120 - 5w}{8}$$

$$A_1 = 4lw$$
$$= 4w\left(\frac{120 - 5w}{8}\right)$$
$$= 0.5w(120 - 5w)$$

$$6n + 6k = 120$$
$$k = \frac{120 - 6n}{6} = \frac{6(20 - n)}{6} = 20 - n$$

$$A_2 = 2n \cdot 2k$$
$$= 2n \cdot 2(20 - n)$$
$$= 4n(20 - n)$$





The parabola vertices are (12,360) and (10,400).

The square arrangement of pens still gives the greater area. Doubling the fencing doubles the dimensions of the pens and increases the area by a factor of 4.

Extension Questions:

- What quantities vary in this situation, and how do these quantities affect the total area, *A*?

The dimensions, x and y, of the pens vary, and the arrangement of the pens varies. This means that initially the area, A, is a function of the two variables, x and y. The fixed perimeter lets us relate the variables x and y, and this relation depends on the arrangement of the pens. With this relation, we can express y in terms of x, and then the total area, A, as a function of x only.

- What type of function is *A*(*x*), and how do you know this function has a maximum value? How does this help you determine the dimensions of the pens that give the maximum area?

The function A(x) is a quadratic function with a negative quadratic coefficient. Therefore, its graph is a parabola opening downward. The vertex will be the highest point on the graph, and its coordinates tell you the value of x that gives the maximum area and what that maximum area will be. Once x is known, using that value in the perimeter equation gives the corresponding value of y.

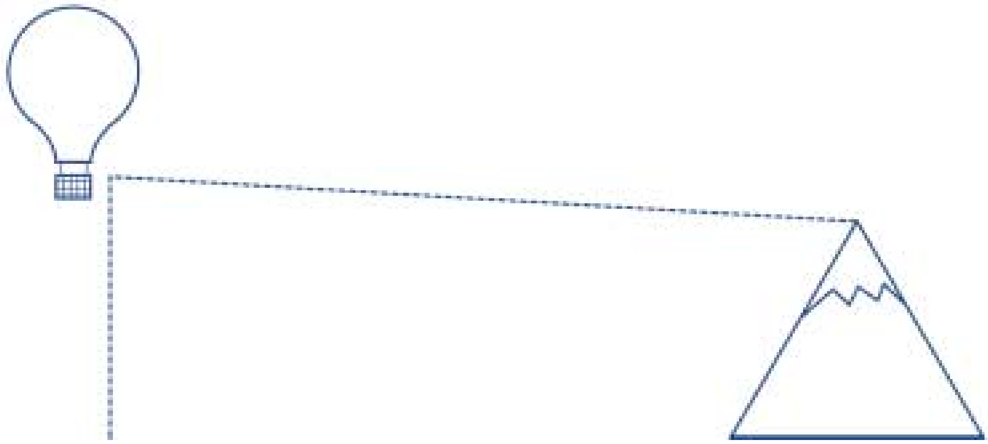
- How do you know which arrangement of pens gives the greater maximum area? Which arrangement is this, and is it realistic? What other factors might need to be considered?

By comparing the parabolas that are the graphs of the area functions, we can determine which has the higher vertex. The graph of the function for the second arrangement has the higher vertex, with a total maximum area of 100 square yards. This is realistic, but it may not be practical. The pens are square, and that may not be the best shaped pen for this animal. It depends on the space ostriches need for exercise.

Seeing the Horizon

At the county fair, you get to take a tour of the area in a hot-air balloon. As the balloon rises from the ground, you keep your eye on the top of Mount Franklyn, which is visible on the horizon. The balloon is flying away from the mountain. In order to know the balloon's position, you use a device called a GPS (Global Positioning System), to measure the distance in kilometers, *d*, from the balloon to the tower on top of Mount Franklyn, as well as the balloon's height in meters, *h*. The table below compares the two measures.

Distance (km)	10	20	30	40	50	60	70
Height (m)	8	32	72	128	200	288	392



1. How does the balloon's height vary with the distance you can see to the tower? Without actually plotting points, sketch a graph to describe this relationship. Justify your sketch.

2. Determine a function that describes the height of the balloon in terms of distance to the tower.

3. If the distance from the balloon to the tower is 80 kilometers, what is the height of the balloon?

4. When the balloon's height is 1000 meters, what is the distance from the balloon to the tower?

Teacher Notes

Scaffolding Questions:

- How is the distance changing in the table?
- How is the height changing in the table?
- Is the relationship of height to distance linear? Explain why or why not.
- How does this information help you sketch a graph without actually plotting points?
- How can you use your graphing calculator to check your thinking about the graph?
- Besides a linear function, what other types of functions have you studied in algebra that you could try as a model for this situation?

Sample Solution:

1. To sketch a graph relating the balloon's height to its distance to the tower, compare the change in the distance, *d*, with the change in the height, *h*.

Change in distance		10	10	10	10	10	10
Distance (km)	10	20	30	40	50	60	70
Height (m)	8	32	72	128	200	288	392
Change in height		24	40	56	72	88	104

The distance is increasing in equal amounts while the height is increasing more and more (faster and faster). This means the graph is not linear and must show for successive increases of 10 kilometers in distance a greater vertical change in the height. Therefore, the graph should be increasing and curving up (concave up).

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(B) for a variety of situations, identifies the mathematical domains and ranges and determines reasonable domain and range values for given situations.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

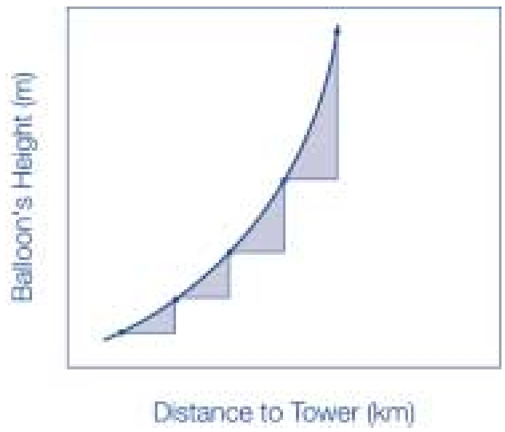
The student:
(B) given situations, looks for patterns and represents generalizations algebraically.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations;

(d.2) Quadratic and other nonlinear functions.
The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.

The student:
(A) solves quadratic equations using concrete models, tables, graphs, and algebraic methods.



2. The graph appears to be the graph of a quadratic function. Pattern analysis of the table shows that (0,0) is a correct mathematical entry in the table.

	10		10		10		10		10		10		10		
Distance, *d* (km)	0		10		20		30		40		50		60		70
Height, *h* (m)	0		8		32		72		128		200		288		392
First Order Differences		8		24		40		56		72		88		104	
Second Order Differences			16		16		16		16		16		16		

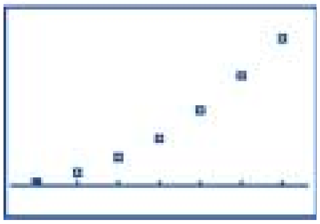
The function for this situation should have the form $y = ax^2$. Using a data point, such as (10,8), substitute for x and y and solve to get $a = 0.08$.

Check the model, $y = 0.08x^2$, against some of the other data points:

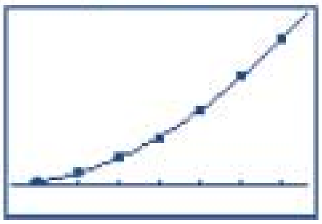
$x = 20 \rightarrow y = 0.08 \cdot 20^2 = 32$
$x = 30 \rightarrow y = 0.08 \cdot 30^2 = 72$
$x = 40 \rightarrow y = 0.08 \cdot 40^2 = 128$

Also verify the model by using the graphing calculator to make a scatterplot and fit a function to the scatterplot.

scatterplot



scatterplot with regression equation, $y = 0.08x^2$



3. If the balloon's distance from the tower is 80 kilometers, evaluate the function $y = 0.08x^2$, where $x = 80$.

   $y = 0.08(80^2) = 512$

   The height of the balloon will be 512 meters.

4. To find the balloon's distance to the tower when the balloon's height is 1000 meters, solve the equation

$$\begin{aligned} 0.08x^2 &= 1000 \\ 8x^2 &= 100000 \\ x^2 &= 12500 \\ x &= 111.803 \end{aligned}$$

   The answer is 111.803 kilometers.

Extension Questions:

- If the height, instead of the distance, increased at a constant amount, how would this change your response to Question #1?

The distance would increase by smaller and smaller amounts. To see this, build a table. Select values for the height, and solve the resulting equations for the distance.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
1 Quadratic Functions
1.2 Transformations

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

16 10 8 6

Distance, *d* (km)	11	27	37	45	51
Height, *h* (m)	10	60	110	160	210

50 50 50 50

- What are the domain and range of the function?

The domain is the set of all real numbers, and the range is the set of all nonnegative real numbers.

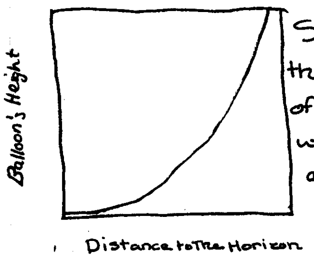
- What are the domain and range for the situation?

The domain of the situation would not include negative numbers because you cannot have a negative distance for the balloon that is moving away from the top of the mountain. The distance to the tower would be zero only when the balloon is on the top of the mountain. The balloon started at the ground, so the minimum height is 0. There is a maximum height because there is a limit to how high the balloon can rise.

The domain could be described as 0 ≤ h ≤ maximum height of the balloon. The range could be described as the numbers from 0 to the maximum distance.

Student Work

① as the balloon gets higher you can see further in the distance



Since we are looking at the distance to the horizon as the steady rate, and the height of the balloon as the dependent variable, the Graph would be a curved line due to the ever larger and faster balloon's height.

② By looking at the charts and seeing how the heights were all twice of a square root, I figured out that whatever my formula was it would be multiplied by two. Then I took the numbers which were squared and figured that my numbers would have to be squared then multiplied by two. The next part was to link the Distance to the remaining number and figured that the number that had to be squared then multiplied by two first had to be divided by five. That therefore results in this formula.

$$\left(\frac{d}{5}\right)^2 \cdot 2 = h$$

③

$\left(\frac{d}{5}\right)^2 \cdot 2 = h$	First start with the formula
$\left(\frac{80}{5}\right)^2 \cdot 2 = h$	Substitute everything in
$(16)^2 \cdot 2 = h$	Divide inside the parentheses
$356 \cdot 2 = h$	Do the squaring
512 meters high	Multiply it out to get the answer

④

$\left(\frac{d}{5}\right)^2 \cdot 2 = h$	Start with the formula
$\left(\frac{d}{5}\right)^2 \cdot 2 = 1000$	Do the substitutions
$\left(\frac{d}{5}\right)^2 = 1000$	Divide everything by two
$\left(\frac{d}{5}\right) = 22.360$	Do the square root to both sides
$d = 111.8$ km high	Multiply both sides by Five

Sky Diving

An airplane is flying at an altitude of 1000 meters. A skydiver jumps from the airplane with no upward speed.

1. Use the vertical motion formula $h = \frac{1}{2}(-9.8)t^2 + vt + s$ to write her height during free fall as a function of the time since she jumped. (h = new height; t = time in seconds; v = initial velocity; s = starting height). The initial velocity of a free fall is zero. Graph your function, and identify its roots. Relate the roots to the problem situation.

2. If the skydiver has fallen approximately 100 meters, how many seconds have passed? Explain how to use your graph to estimate the solution. Show how to find the number of seconds algebraically.

3. If the skydiver has fallen 400 meters, approximately how many seconds have passed?

Teacher Notes

Scaffolding Questions:

- If the skydiver has no upward speed, what is the initial velocity?
- Determine the domain and range for this situation.
- Which values are not reasonable for the domain and/or range? Explain.

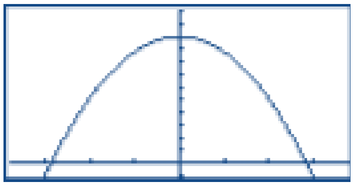
Sample Solution:

1. Using the vertical motion formula $h = \frac{1}{2}(-9.8)t^2 + vt + s$, the skydiver's height during free fall as a function of the time since she jumped, substitute values into the given formula. Given h = new height; t = time in seconds; s = starting height, v = initial velocity.

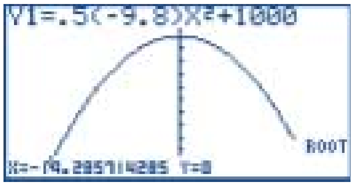
   The initial velocity for this situation is zero, therefore $v = 0$. The starting height is the altitude of 1000 meters, so $s = 1000$. The function is

   $h = \frac{1}{2}(-9.8)t^2 + 1000$ or $h = -4.9t^2 + 1000$.

   The graph of the function is a parabola that opens downward.



   The roots of the function are at approximately -14.29 and 14.29. Only the positive values make sense in the situation because this value represents the number of seconds from the time the skydiver jumps to the time she lands. The number of seconds cannot be negative.



Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations;

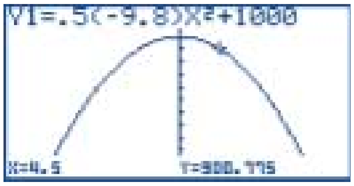
(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

(d.2) Quadratic and other nonlinear functions.
The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.

The student:
(A) solves quadratic equations using concrete models, tables, graphs, and algebraic methods; and

(B) relates the solutions of quadratic equations to the roots of their functions.

2. The graph can be used to estimate the amount of seconds that have passed when the skydiver has fallen approximately 100 meters. Using the trace function on the calculator, one can find that the value of 900 meters for the height will yield a value of about 4.5 seconds.



Another way to determine the number of seconds is to solve algebraically. Substitute the value 900 for the height into the original function and solve for *t*.

$$h = -4.9t^2 + 1000$$
$$900 = -4.9t^2 + 1000$$
$$0 = -4.9t^2 + 100$$

Use the quadratic formula to solve.
a = -4.9 b = 0 c = 100

$$t = \frac{0 \pm \sqrt{0^2 - 4(-4.9)(100)}}{2(-4.9)}$$
$$= \frac{\pm\sqrt{-4(-4.9)(100)}}{2(-4.9)}$$
$$= \frac{\pm\sqrt{1960}}{-9.8}$$
$$= \pm 4.5$$

The solutions are approximately -4.5 and 4.5. The number of seconds cannot be negative, so only the positive value is reasonable.

3. To find the elapsed time if the skydiver has fallen 400 meters, one can use the table of values. If she started at 1000 meters, her height at that time will be 600 meters from the ground. Look for a table value of approximately 600.

Texas Assessment of Knowledge and Skills:

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

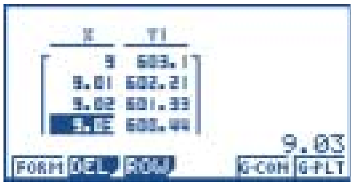
III. Nonlinear Functions
2 Quadratic Equations
2.1 Connections

Connections to Algebra End-of-Course Exam:

Objective 5:
The student will formulate or solve quadratic equations that describe real-world and mathematical situations.

Objective 7:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving exponents, quadratic situations, or right triangles.

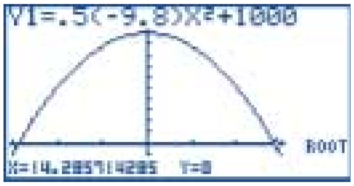
Approximately 9 seconds have passed if the skydiver has fallen 400 meters.



Extension Questions:

- Is it possible for the skydiver to wait 15 seconds before pulling the parachute cord? Justify your answer.

The skydiver cannot wait 15 seconds to open the parachute. The skydiver will land back on the ground at approximately 14.28 seconds after the jump. Waiting 15 seconds to open the parachute would be tragic.



- At the same time the skydiver jumped her friend jumped from an airplane flying at an altitude of 2500 meters. How much longer after landing must the skydiver wait for her friend to land?

First we must determine the roots of the vertical motion formula for the friend. The formula is $h = -4.9t^2 + 2500$; the only change is the starting height, which is now 2500 meters.

To find the roots we want to know when this formula equals zero.

$$
\begin{aligned}
0 &= -4.9t^2 + 2500 \\
4.9t^2 &= 2500 \\
t^2 &= \frac{2500}{4.9} \\
t^2 &\cong 510.20 \\
t &\cong \pm 22.59
\end{aligned}
$$

So the friend lands after 22.59 seconds. Disregard the negative solution because time will not be negative. The first skydiver lands after 14.29 seconds. Therefore the first skydiver must wait 22.59 – 14.29 = 8.3 seconds for her friend to land.

- Pretend that the skydiver could skydive on the moon. The moon's gravity is 1/6 that of the Earth. How long will it take the skydiver to land if she jumped from an altitude of 1000 meters on the moon?

First change the vertical motion formula to reflect the moon's gravity. Our formula for earth is $H = -4.9t^2 + 1000$. For the moon it will be:

$H = \frac{1}{6}(-4.9)t^2 + 1000$

$H = -0.817t^2 + 1000.$

To determine when the skydiver will land, determine the roots of this new formula. We set the formula equal to zero.

$$
\begin{aligned}
0 &= -0.817t^2 + 1000 \\
0.817t^2 &= 1000 \\
t^2 &= \frac{1000}{0.817} \\
t^2 &\cong 1223.99 \\
t &\cong \pm 34.99
\end{aligned}
$$

Therefore, if skydiving on the moon, the skydiver will land after 34.99 seconds.

Supply and Demand

Each year, the senior class sponsors a Star Trek Day when they will show 10 favorite Star Trek episodes. Last year, they charged $3 per ticket and sold 2500 tickets. Based on a survey of the student body, they know that for every 10¢ price increase, they would sell 50 fewer tickets. As the senior class president, you must help your classmates decide how much to charge per ticket for this year's Star Trek Day.

A. Write a function for the amount of money, A dollars, that would be collected in terms of x, the number of 10¢ price increases.

B. Obtain a reasonable graph of your function in Part A, and write a verbal description of what the graph tells you about the situation.

C. Suppose the senior class must collect at least $7800. What range of price increases would allow them to do this? How many tickets would they need to sell to meet this goal?

D. The vice president of the senior class conducts an independent survey. Based on her survey, she predicts that if the class charges $2.50 per ticket they can expect to sell 2700 tickets. Each 10¢ increase in price per ticket would result in 30 fewer tickets being purchased. If she is correct, how would the money collected under this plan compare with the previous situation?

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

Scaffolding Questions:

- What do you need to know to compute the amount of money that would be collected?
- What expression would represent the ticket price, depending on x, the number of 10¢ price increases?
- What expression would represent the number of tickets sold, depending on x, the number of 10¢ price increases?
- Now what expression would represent the amount of money collected, depending on x, the number of 10¢ price increases?
- By experimenting with your graphing calculator window, what are a reasonable domain and range for the situation?
- What does the vertex of the parabola in the graph of the situation tell you?
- What graph can you add to model the class collecting at least $7800? How does this help you answer the question asked in Part C?
- How do the graphs of the original situation and the situation in Part D compare? How do the graphs help you compare the two situations?

Sample Solution:

A. Since x is the number of 10¢ price increases and 50 fewer tickets are sold per price increase, the number of tickets sold would be given by the expression $2500 - 50x$, and the price per ticket would be given by the expression $3.00 + 0.10x$.

The amount of money collected would equal the price per ticket times the number of tickets sold, so

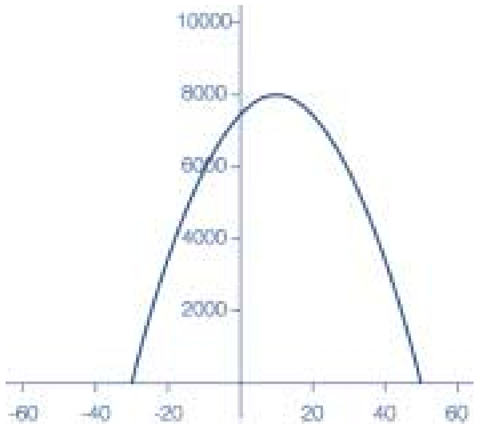
$$\begin{aligned} A &= (3.00 + 0.10x)(2500 - 50x) \\ &= 7500 + 100x - 5x^2 \end{aligned}$$

B. The domain for A is easily seen when A is in factored form and helps in setting a window for the graph. Clearly, $x \geq 0$, since x counts the number of price increases. Also, $x \leq 50$. Otherwise, $2500 - 50x$ would represent a negative number of tickets. To determine the range for A locate both intercepts, -30 and 50; find the x-coordinate of the vertex by determining the midpoint of the intercepts, $x = 10$. Evaluate the area function for $x = 10$.

$$\begin{aligned} A &= 7500 + 100(10) - 5(10)^2 \\ &= 8000 \end{aligned}$$

The range values for the function must be less than or equal to 8000. For the problem situation the range values must also be greater than or equal to 0.

The following graph represents the money collected as a function of the number of 10¢ price increases:



The intercepts that make sense for this situation are (0,7500) and (50,0). The number of 10¢ price increases can range from none ($x = 0$) to 50 ($x = 50$). The ticket prices are represented by $3.00 + 0.10x$. The ticket prices can range from $3.00 + 0.10(0)$ or \$3 to $3.00 + 0.10(50) = \$8$. When $x = 10$, the maximum amount of money, \$8000, is collected. By examining the graph it can be seen that the amount of money collected is increasing if the number of 10¢ price increases is between 0 and 10. They collect the most when the ticket price is $3.00 + 0.10(10) = \$4$, and they sell $2500 – 50(10)$ or 2000 tickets.

C. To see what range in the number of price increases the class must consider to collect at least \$7800, you could add the graph of $y = 7800$ to the money collected graph and determine the x-coordinates of the points of intersection of the two graphs:

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations;

(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

(d.2) Quadratic and other nonlinear functions.
The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.

The student:
(A) solves quadratic equations using concrete models, tables, graphs, and algebraic methods; and

(B) relates the solutions of quadratic equations to the roots of their functions.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

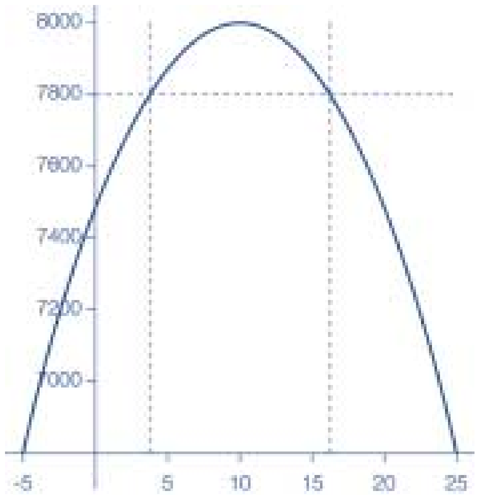
Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
1 Quadratic Functions
 1.1 Quadratic Relationships
2 Quadratic Equations
 2.1 Connections

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

Objective 5:
The student will formulate or solve quadratic equations that describe real-world and mathematical situations.



The number of 10¢ price increases must fall between 4 and 16. This means the class should keep the ticket price between 3.00 + 0.10(4) = \$3.40 and 3.00 + 0.10(16) = \$4.60 inclusive, and they need to sell between 2500 – 4(50) = 2300 and 2500 – 16(50) = 1700 tickets.

D. In this second situation the number of tickets is changed to 2700, and the price per ticket is changed from the original situation to \$2.50.

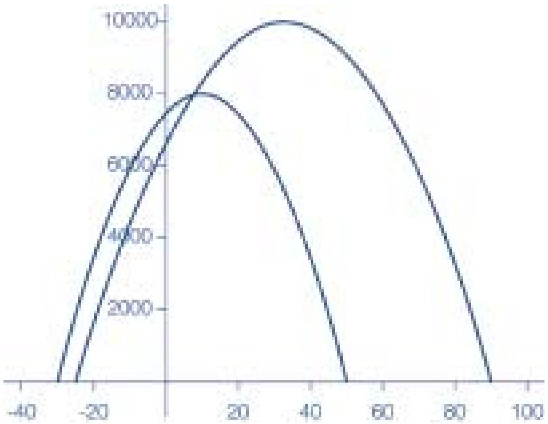
The number of tickets sold would be given by the expression $2700 - 30x$, and the price per ticket would be given by the expression $2.50 + 0.10x$.

The money collected function for this second situation would be

$$\begin{aligned} A &= (2.50 + 0.10x)(2700 - 30x) \\ &= 6750 + 195x - 3x^2 \end{aligned}$$

where the first factor in the expression for *A* is the price per ticket per 10¢ increase, and the second factor is the number of tickets sold.

The following graph compares the two situations:



For the second situation, the maximum amount of money that can be collected is when the number of 10¢ price increases per ticket is between 32 and 33. At 32 the ticket price is 2.50 + 0.10(32) = \$5.70, and at 33 the ticket price is 2.50 + 0.10(33) = \$5.80.

If the function is evaluated at both 32 and 33, the value of the function is \$9918.

$A(32) = 6750 + 195(32) - 3(32)^2 = 9918$
$A(33) = 6750 + 195(33) - 3(33)^2 = 9918$

The fact that the vertex of the parabola for the second situation is higher and further to the right than that of the first situation shows that the second situation will result in more tickets sold and more money made.

The *y*-intercept for the first graph is (0,7500), while the *y*-intercept for the second graph is (0,6750). This means with no increase in ticket price the money collected for the first situation would be \$7500, and the money collected for the second situation would be \$6750. By finding the intersection of the two graphs, we know that the first situation is better up to an increase of 69¢ per ticket. With an increase of more than 69¢, the second situation will make more money for the class.

The wider spread of the graph of the second situation also shows that more students are willing to accept the greater number of 10¢ price increases.

Extension Questions:

- In this situation, what decisions must the senior class make in order to determine the possible amount of money they can collect with their fundraiser?

They know that for each 10¢ increase in ticket price they will sell 50 fewer tickets. Therefore, the price per ticket depends on the number of 10¢ increases in price per ticket and the number of tickets sold also depends on the number of 10¢ increases they might make. The amount of money they can collect would be price per ticket times number of tickets sold. Since both of these quantities depend on the number of 10¢ increases in price per ticket, the amount of money to be collected will also be affected by the number of 10¢ price increases.

- What properties of the graph of the money collected function help you to draw conclusions about this situation?

Since the price per ticket will have a positive rate of change (10¢ increase per ticket) and the number of tickets sold (50 fewer per 10¢ increase per ticket) will have a negative rate of change, their product will produce a negative quadratic coefficient. Therefore, the graph of the quadratic function for the money collected will be a parabola opening downward. The vertex of the parabola will tell us the maximum number of 10¢ price increases per ticket to make in order to collect the most amount of money.

The Dog Run

The owner of a kennel that raises Saint Bernards needs to build a dog run for a new litter of puppies. He has 22 meters of chain link fence to enclose all four sides.

1. Construct a table of values (at least five entries) relating the area of the run in square meters, A, to the length of the pen in meters, l.

Length, l, in meters	Area, A, in square meters

2. Write a function rule relating A and l, and construct a graph.

3. What length maximizes the area of the pen? Should the kennel owner build the pen to maximize the area? Why or why not?

4. How will your function change if the perimeter of the dog run is 24 meters? 26 meters? 28 meters?

5. Describe how you can find the maximum area and corresponding length for any given perimeter.

Teacher Notes

Scaffolding Questions:

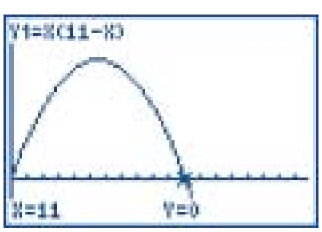
- What lengths make sense in this situation?
- If the length were 1 meter, what would the width be?
- If you are to organize the information in a table, what columns might you have in the table?
- What is the relationship between the length and the width?
- What are the corresponding areas?
- What function type describes the area in terms of the dog run's length?
- What is the parent for this function?
- What does the graph of the parent function look like?
- What will the graph of your function look like?
- How does your table and/or graph help you find the maximum area and corresponding length?

Sample Solution:

1. Since the perimeter of the dog run is to be 22 meters, it takes 11 meters to fence a width and a length. The width of the pen will be 11 minus the length. If the length is represented by l, the width may be represented by $11 - l$ meters.

Length (*m*)	Width (*m*)	Area Process	Area (m^2)
1	10	1(10)	10
3	8	3(8)	24
5	6	5(6)	30
7	4	7(4)	28
9	2	9(2)	18
l	$11 - l$	$l(11 - l)$	

2. The function is $A = l(11 - l)$ where $0 < l < 11$. The following is a graph of the situation:



Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(A) describes independent and dependent quantities in functional relationships;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(A) determines the domain and range values for which quadratic functions make sense for given situations;

(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

3. The maximum area occurs when the length equals the *x*-coordinate that is halfway between the *x*-intercepts, 0 and 11. Therefore, the length that gives the maximum area is $l = 5.5$ meters, and the maximum area is $A = 5.5(11-5.5)$ or 30.25 square meters.

   The shape of the dog run that maximizes the area is a square. Usually, a dog run is longer than it is wide so that the dog has plenty of room to run. The kennel owner may decide not to build a square run. He should research what could be the best length to width ratio for the run and use that information to decide on the dimensions of the dog run.

4. The sum of the length and the width must be one-half of the total amount of fencing. The table below gives area, *A*, as a function of length, l, for varying perimeters:

Perimeter	Area as a function of Length
22	$A = l(11 - l)$
24	$A = l(12 - l)$
26	$A = l(13 - l)$
28	$A = l(14 - l)$

   In general, for any given perimeter, *P*, the area as a function of length is

$$A = l\left(\frac{P}{2} - l\right)$$

5. To find the maximum area for any given perimeter, you can trace along the graph or look at the table. The maximum area will occur when the length equals the width. The dog run with maximum area is square in shape. The length and the width would be one-fourth of the total perimeter.

$$w = l = \frac{P}{4}$$

$$A = w \cdot l = \frac{P}{4} \cdot \frac{P}{4} = \frac{P^2}{16}$$

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
1 Quadratic Functions
1.1 Quadratic Relationships

Connections to Algebra End-of-Course Exam:

Objective 2:
The student will graph problems involving real-world and mathematical situations.

Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

Extension Questions:

- Suppose the dog owner investigates and determines that the best pen for the dog is one in which the ratio of the length to the width of the pen is 2:1. What is the area of this pen?

If the ratio of the length to width ratio is 2:1, then $l = 2w$ and $l + w = 11$.

$$\begin{aligned} 2w + w &= 11 \\ 3w &= 11 \\ w &= \frac{11}{3} = 3\frac{2}{3} \\ l &= 2\left(3\frac{2}{3}\right) = 7\frac{1}{3} \end{aligned}$$

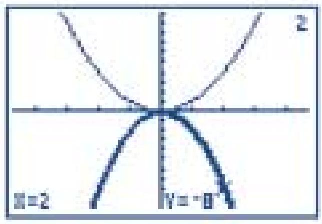
The dimensions of the dog run would be $3\frac{2}{3}$ meters and $7\frac{1}{3}$ meters.

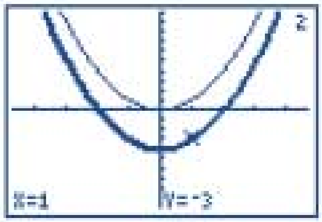
- If a given perimeter is doubled, how will this pen's maximum area and its corresponding value be related to the maximum area and its corresponding value for the given perimeter?

Let the original perimeter be represented by P. The area for the original perimeter is given by the rule $A = l\left(\frac{P}{2} - l\right)$. When the perimeter is doubled, the area becomes $A = l\left(\frac{2P}{2} - l\right) = l(P - l)$. For the original perimeter the maximum area occurs at $l = \frac{P}{4}$. The area would be $\frac{P}{4} \cdot \frac{P}{4}$ or $\frac{P^2}{16}$. For the new perimeter the maximum area occurs at $\frac{P}{2}$ which is twice the value for the original perimeter. The maximum area would be $\frac{P}{2} \cdot \frac{P}{2}$ or $\frac{P^2}{4}$. This value is four times the maximum area for the original perimeter.

Transformations of Quadratic Functions

In this activity, your task is to investigate, describe, and predict the effects of the changes in the parameters *a* and *c* on the graph of $y = ax^2 + c$ as compared to the graph of the parent function $y = x^2$. One representation of the parameter change is given, and you are to complete the others. Include in the verbal description the images of (0,0) and (1,1) under the transformation. For each problem, graph the parent function and the transformed function on the same grid. Draw the parent function in red and the transformed function in blue.

<table>
<tr><th>Function</th><th>Description
Verbal description of transformation(s)</th><th>Graph
Graph should show the vertex and x- and y-intercepts.</th><th>Table
Include vertex and images of (±1,1) and (±2,4).</th></tr>
<tr><td>1a.
$y = 3x^2$</td><td></td><td></td><td><table><tr><td>$y = x^2$</td><td>$y =$</td></tr><tr><td><table><tr><th>x</th><th>y</th></tr><tr><td>-2</td><td>4</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td><td><table><tr><th>x</th><th>y</th></tr></table></td></tr></table></td></tr>
<tr><td>1b.</td><td>Vertically compress $y = x^2$ by a factor of $\frac{1}{2}$.</td><td></td><td><table><tr><td>$y = x^2$</td><td>$y =$</td></tr><tr><td><table><tr><th>x</th><th>y</th></tr><tr><td>-2</td><td>4</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td><td><table><tr><th>x</th><th>y</th></tr><tr><td>-2</td><td>2</td></tr><tr><td>-1</td><td>0.5</td></tr></table></td></tr></table></td></tr>
<tr><td>1c.</td><td></td><td>
</td><td><table><tr><td>$y = x^2$</td><td>$y =$</td></tr><tr><td><table><tr><th>x</th><th>y</th></tr><tr><td>-2</td><td>4</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td><td><table><tr><th>x</th><th>y</th></tr></table></td></tr></table></td></tr>
<tr><td>1d.</td><td></td><td></td><td><table><tr><td>$y = x^2$</td><td>$y =$</td></tr><tr><td><table><tr><th>x</th><th>y</th></tr><tr><td>-2</td><td>4</td></tr><tr><td>-1</td><td>1</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td></tr><tr><td>2</td><td>4</td></tr></table></td><td><table><tr><th>x</th><th>y</th></tr><tr><td>-2</td><td>16</td></tr><tr><td>-1</td><td>4</td></tr><tr><td>0</td><td>0</td></tr><tr><td>1</td><td>4</td></tr><tr><td>2</td><td>16</td></tr></table></td></tr></table></td></tr>
</table>

Function	Description Verbal description of transformation(s)	Graph Graph should show the vertex and x- and y-intercepts.	Table Include vertex and images of (±1,1) and (±2,4).
2a. $y = x^2 + 1$			$y = x^2$: x = -2, -1, 0, 1, 2; y = 4, 1, 0, 1, 4 $y =$: x | y
2b.			$y = x^2$: x = -2, -1, 0, 1, 2; y = 4, 1, 0, 1, 4 $y =$: x | y
2c.	Vertically stretch $y = x^2$ by a factor of 2 and translate down 2 units. (0,0) ⟶ (0,-2) (1,1) ⟶ (1,0)		$y = x^2$: x = -2, -1, 0, 1, 2; y = 4, 1, 0, 1, 4 $y =$: x | y
2d.			$y = x^2$: x = -2, -1, 0, 1, 2; y = 4, 1, 0, 1, 4 $y =$: x = -2, -1, 0, 1, 2; y = 1, 4, 5, 4, 1

3. Write a summary of the effects of the changes in the parameters a and c on the graph of $y = ax^2 + c$. Include a description of the effects of the transformations on the vertex, axis of symmetry, and the intercepts.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(A) identifies and sketches the parent forms of linear (y = x) and quadratic (y = x^2) functions.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(B) investigates, describes, and predicts the effects of changes in a on the graph of y = ax^2;

(C) investigates, describes, and predicts the effects of changes in c on the graph of y = x^2 + c; and

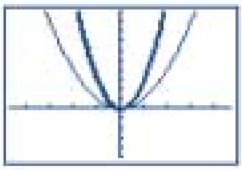
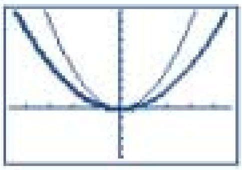
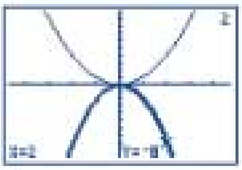
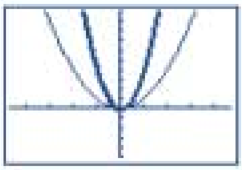
(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

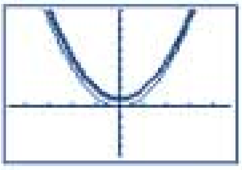
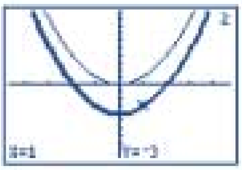
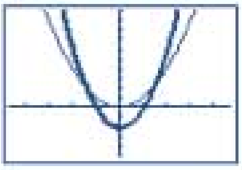
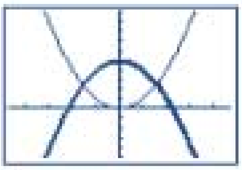
Scaffolding Questions:

- What points help you graph the parent function? What are its vertex, axis of symmetry, and intercepts?
- What does the graph show you about the effect of *a* on the shape and orientation of the graph of $y = ax^2$?
- Describe how the value of *y* varies in your tables as the value of *a* varies.
- What does the graph show you about the effect of *c* on the position of the graph of $y = ax^2 + c$?
- Describe how the value of *c* affects the value of *y* in your tables.
- What is the order of operations in the expression $ax^2 + c$?

Sample Solution:

Note: The lighter lined graph is the graph of $y = x^2$. The darker graph is the transformed graph.

Function	Description	Graph	Table
1a. $y = 3x^2$	The parent graph is vertically stretched by a factor of 3. (0,0) → (0,0) (1,1) → (1,3)		x: -2, -1, 0, 1, 2 y: 12, 3, 0, 3, 12
1b. $y = \frac{1}{2}x^2$	Vertically compress $y = x^2$ by a factor of $\frac{1}{2}$. (0,0) → (0,0) (1,1) → (1,0.5)		x: -2, -1, 0, 1, 2 y: 2, 0.5, 0, 0.5, 2
1c. $y = -2x^2$	Vertically stretch $y = x^2$ by a factor of 2 and reflect over the *x*-axis. (0,0) → (0,0) (1,1) → (1,-2)		x: -2, -1, 0, 1, 2 y: -8, -2, 0, -2, -8
1d. $y = 4x^2$	Vertically stretch $y = x^2$ by a factor of 4. (0,0) → (0,0) (1,1) → (1,4)		x: -2, -1, 0, 1, 2 y: 16, 4, 0, 4, 16

Function	Description	Graph	Table
2a. $y = x^2 + 1$	Translate $y = x^2$ up one unit. (0,0) → (0,1) (1,1) → (1,2)		x: -2, -1, 0, 1, 2; y: 5, 2, 1, 2, 5
2b. $y = x^2 - 4$	Translate $y = x^2$ down four units. (0,0) → (0,-4) (1,1) → (1,-3)		x: -2, -1, 0, 1, 2; y: 0, -3, -4, -3, 0
2c. $y = 2x^2 - 2$	Vertically stretch $y = x^2$ by a factor of 2 and vertically translate down 2 units. (0,0) → (0,-2) (1,1) → (1,0)		x: -2, -1, 0, 1, 2; y: 6, 0, -2, 0, 6
2d. $y = 5 - x^2$	Reflect $y = x^2$ over x-axis and translate up 5 units. (0,0) → (0,5) (1,1) → (1,4)		x: -2, -1, 0, 1, 2; y: 1, 4, 5, 4, 1

3. Summary: The effect of *a* on the graph of $y = ax^2$ depends on the signed value of *a* and the magnitude of *a*. If $a > 0$, the graph of the transformed function still opens up. If $a < 0$, the graph of $y = x^2$ is reflected over the *x*-axis, and the transformed function opens downward. If the magnitude of *a* is greater than one, the graph of $y = x^2$ is vertically stretched by a factor of the magnitude of *a*. If *a* is less than one in magnitude, the graph of $y = x^2$ is vertically compressed by a factor of the magnitude of *a*; in particular, $(1,1) \rightarrow (1,|a|)$. The vertex and the axis of symmetry are preserved under this transformation.

   The effect of *c* on the graph of $y = x^2 + c$ is to translate the graph of $y = x^2$ vertically *c* units. If $c > 0$, the graph of $y = x^2$ is translated up *c* units. If $c < 0$, the graph of $y = x^2$ is translated down $|c| = -c$ units. The *x*-coordinate of the vertex is still 0, and the axis of symmetry is still $x = 0$. But the vertex has been translated to the point $(0, c)$. This is the new

Texas Assessment of Knowledge and Skills:

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

III. Nonlinear Functions
1 Quadratic Functions
1.2 Transformations

Connections to Algebra End-of-Course Exam:

Objective 1:
The student will demonstrate an understanding of the characteristics of graphing in problems involving real-world and mathematical situations.

y-intercept. The new x-intercepts are $(\pm\sqrt{-c}, 0)$, since to get the x-intercepts, we solve $x^2 + c = 0$.

Extension Questions:

- What kinds of transformations on the graph of $y = x^2$ can be performed so that the resulting graph continues to be that of a function?

Since a function is a relation between x and y that generates exactly one output value, y, for each input value, x, the only transformations on $y = x^2$ we can consider are dilations, translations, and reflections over the axes.

- What parameter causes a dilation on the graph of $y = x^2$? What is another way of saying "dilation?" What else does this parameter tell you?

The parameter, a, in $y = ax^2$ causes dilation. If $|a| > 1$, the dilation is a vertical stretch. If $|a| < 1$, then there is a vertical compression on the graph of $y = x^2$. The parameter, a, also causes a reflection over the x-axis if a is negative.

- What parameter causes a translation and what kind of translation?

The parameter, c, in $y = x^2 + c$ causes a vertical translation. The parent graph is translated c units up if c is positive and $|c|$ units down if c is negative.

- What does the "Order of Operations" sequence tell you about the sequence of transformation performed on the parent graph to generate the graph of $y = ax^2 + c$?

If $a > 0$, vertically stretch or compress the graph of $y = x^2$ by a factor that equals the magnitude of a. Then translate vertically $|c|$ units. If $a < 0$, vertically stretch or compress the graph of $y = x^2$ by a factor that equals the magnitude of a. Next, reflect the graph over the x-axis, and then translate vertically $|c|$ units.

- If $a > 0$, what is the difference between the transformations $y = ax^2$ and $y = (ax)^2$?

The first transformation is a vertical stretch or compression of the graph of $y = x^2$. The second transformation is a horizontal stretch or compression on x before squaring. It could also be described as a vertical stretch or compression, but the dilation factor is a^2 since $(ax)^2 = a^2x^2$.

- What is the difference between the transformations $y = x^2 + c$ and $y = (x + c)^2$?

The first transformation is a vertical translation c units up if c is positive and -c units down if c is negative. The second transformation is a horizontal translation c units left if c is positive and -c units right if c is negative.

What is the Best Price?

Laura makes earrings to sell at craft fairs. Because of her expenses she has decided that the cheapest price at which she can sell them is $15. She has tried different selling prices at several different fairs and has recorded the data in a table.

Selling Price ($)	Number Sold
15	118
16	115
17.50	110
19	102
20	99
21.50	93
22	91
24	79
25	75
27.50	62
28.50	56
30	51
35	27

She thinks the number sold seems to depend on the selling price. The revenue, the amount of money she receives from the sales, depends on the selling price and the number sold.

1. Use a graphing calculator to create a scatter plot of the data. Determine a model for the number of earrings sold as a function of the selling price.

2. If she had set the selling price at $32, how many might she have expected to sell?

3. Revenue is the amount of money received from sales. For example, if you sold 118 items for $15, the revenue would be $1770. Make a table comparing the selling price and the revenue. Create a scatter plot of the points (revenue, selling price).

4. Use the function rule you found for the number of items to find a function for the revenue in terms of the selling price.

5. Evaluate the revenue function for the selling price of $32.

6. Explain what you think the selling price should be to have the greatest revenue. Justify your reasoning using algebraic representations and tables or graphs.

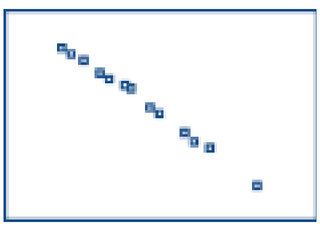
Teacher Notes

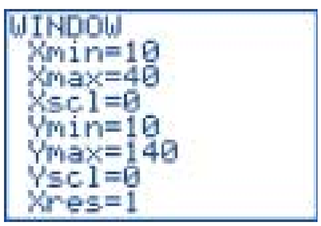
Scaffolding Questions:

- What are the variables in this situation? Which one is the dependent variable?
- Examine the table and determine the rate of change. What is the average rate of change in the scatterplot?
- How could you determine the *y*-intercept for the linear model?
- What is the function rule that shows the relationship between the number sold and the selling price?
- What do you need to know to determine the revenue?
- What is the function rule for the revenue?

Sample Solution:

1. The points determined by the table were plotted with the number sold depending on the selling price. The graph appears to be linear.





The finite differences were computed to determine the rate of change.

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.1) Foundations for functions.
The student understands that a function represents a dependence of one quantity on another and can be described in a variety of ways.

The student:
(B) gathers and records data, or uses data sets, to determine functional (systematic) relationships between quantities;

(C) describes functional relationships for given problem situations and writes equations or inequalities to answer questions arising from the situations;

(D) represents relationships among quantities using concrete models, tables, graphs, diagrams, verbal descriptions, equations, and inequalities; and

(E) interprets and makes inferences from functional relationships.

(b.2) Foundations for functions.
The student uses the properties and attributes of functions.

The student:
(D) in solving problems, collects and organizes data, makes and interprets scatterplots, and models, predicts, and makes decisions and critical judgements.

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables; and

(B) given situations, looks for patterns and represents generalizations algebraically.

(c.1) Linear functions.
The student understands that linear functions can be represented in different ways and translates among their various representations.

The student:
(A) determines whether or not given situations can be represented by linear functions.

(d.1) Quadratic and other nonlinear functions.
The student understands that the graphs of quadratic functions are affected by the parameters of the function and can interpret and describe the effects of changes in the parameters of quadratic functions.

The student:
(D) for problem situations, analyzes graphs of quadratic functions and draws conclusions.

Price difference	Selling Price ($)	Number Sold	Number difference
	15	118	
1			-3
	16	115	
1.5			-5
	17.50	110	
1.5			-8
	19	102	
1			-3
	20	99	
1.5			-6
	21.50	93	
0.5			-2
	22	91	
2			-12
	24	79	
1			-4
	25	75	
2.5			-13
	27.50	62	
1			-6
	28.50	56	
1.5			-5
	30	51	
5			-24
	35	27	

The rates of change are found by finding the ratios of the differences.

$\frac{-3}{1} = -3$ $\quad \frac{-5}{1.5} = -3.33$ $\quad \frac{-8}{1.5} = -5.33$

$\frac{-3}{1} = -3$ $\quad \frac{-6}{1.5} = -4$ $\quad \frac{-2}{0.5} = -4$

$\frac{-12}{2} = -6$ $\quad \frac{-4}{1} = -4$ $\quad \frac{-13}{2.5} = -5.2$

$\frac{-6}{1} = -6$ $\quad \frac{-5}{1.5} = -3.33$ $\quad \frac{-24}{5} = -4.8$

The ratios are -3, -3.33, -5.33, -3, -4, -4, -6, -4, -5.2, -6, -3.33, -4.8.

The average of this set of numbers is found by adding up these rates and dividing by 12. The average is -4.33. This number may be used as the rate of change of the linear function that models the set of data. The function rule is of the form $y = -4.33x + b$. Use one of the given points, (20,99), and solve for b.

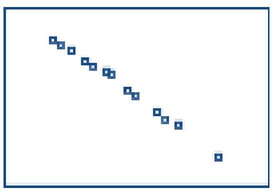
$$99 = -4.33(20) + b$$
$$b = 185.6$$

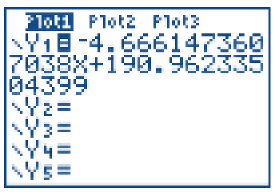
The number sold, n, as a function of the selling price, p, is

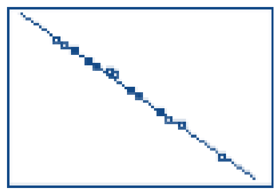
$n = -4.33p + 185.6$

Note that using a different data point would give a different y-intercept and thus a different rule. This is an approximate value. Students may find other approximations.

It is also possible to use the regression line from a graphing calculator.







$y = -4.66x + 190.96$ where y is the number sold, and x is the selling price.

2. If she had set a selling price of $32, the function must be evaluated for $p = 32$.

   $n = -4.33(32) + 185.6 = 47.04$.

   She could expect to sell about 47 items using the first model. Using the second model

   $y = -4.66x + 190.96 = -4.66(32) + 190.96 = 41.84$

   she would sell 41 items.

3. To find the revenue, multiply the selling price by the number sold.

Texas Assessment of Knowledge and Skills:

Objective 1:
The student will describe functional relationships in a variety of ways.

Objective 2:
The student will demonstrate an understanding of the properties and attributes of functions.

Objective 5:
The student will demonstrate an understanding of quadratic and other nonlinear functions.

Connections to Algebra I: 2000 and Beyond Institute:

II. Linear Functions
1 Linear Functions
  1.1 The Linear Parent Function
2 Interpreting Relationships Between Data Sets
  2.1 Out for the Stretch

III. Nonlinear Functions
1 Quadratic Functions
  1.2 Transformations

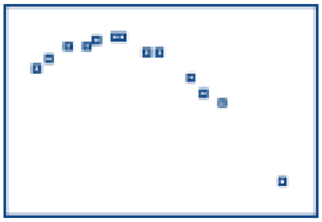
Connections to Algebra End-of-Course Exam:

Objective 2:
The student will graph problems involving real-world and mathematical situations.

Objective 9:
The student will use problem-solving strategies to analyze, solve, and/or justify solutions to real-world and mathematical problems involving probability, ratio and proportions, or graphical and tabular data.

Selling Price ($)	Number Sold	Revenue ($)
15	118	1770
16	115	1840
17.50	110	1925
19	102	1938
20	99	1980
21.50	93	1999.5
22	91	2002
24	79	1896
25	75	1875
27.50	62	1705
28.50	56	1596
30	51	1530
35	27	945

The scatter plot of selling price and the revenue.



To develop the symbolic representation, remember that revenue = number sold times selling price.

$R = np$

Method 1:
$R = (-4.33p + 185.6)p$
$R = -4.33p^2 + 185.6p$

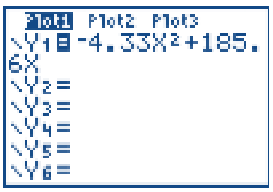
Method 2 (linear regression):
$R = (-4.66p + 190.96)p$
$R = -4.66p^2 + 190.96p$

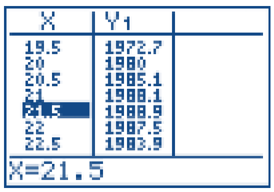
5. The value of the function at $32 is

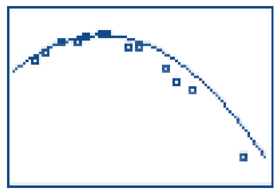
Method 1: $-4.33(32)^2 + 185.6(32)$ or $1505.28
Method 2: $-4.66(32)^2 + 190.96(32)$ or $1338.88

6. Enter the rule into the calculator and examine the table to determine the selling price that will give the highest revenue.

Method 1:

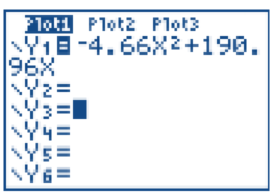


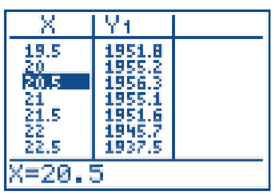


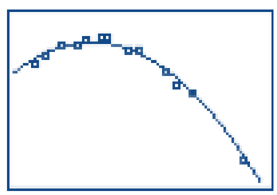


The table gives a maximum revenue of $1988.90 at the selling price of $21.50.

Method 2:







The table gives a maximum revenue of $1956.30 at the selling price of $20.50.

The answers may vary slightly depending on the model selected. However, the regression equation gives the model that more accurately matches the data.

Extension Questions:

- Describe the domain of the linear function used to model the situation. Compare the domain of the function to the domain for the problem situation.

The domain of this linear function is all real numbers, but the domain of this problem situation requires that p be a selling price in dollars and cents, thus it must be a positive rational number with at most two decimal places. Further restrictions given in the problem require that p be greater than or equal to 15. The x-intercept is between 42 and 43. For any integer greater than 42, the value of n will be negative. Since the number sold may not be negative, $15 \le x \le 42$.

- Explain how the domain of the revenue function compares to the domain of the linear model.

There are no restrictions for the function rules on x, but for the problem situation the same restrictions apply to x, $15 \le x \le 42$.

- What other conditions might be considered in this situation?

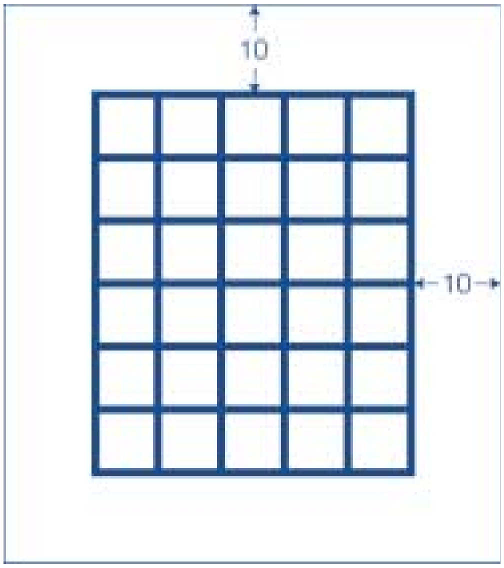
The cost of the production of the goods would affect how many earnings she would make. The amount of time required to produce the product would also affect her production.

- What is the meaning of the slope in the linear equation?

For every 4.33 dollars decrease in price, one more of the items is sold.

Window Panes

The window shown below is made up of squares: wooden strips surrounding each square, and a border that frames all of the squares. The individual white squares have dimensions of x inches by x inches. The width of the wooden strips surrounding each of the squares is y inches. The width of the border that frames all of the squares is 10 inches.



1. Write expressions in terms of x and y for the dimensions of the entire window (including the border).

2. Write an expression in terms of x and y for the perimeter of the window. Simplify the expression.

3. Write an expression in terms of x and y for the area of the window. Simplify the expression.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Algebra I TEKS and Performance Descriptions:

(b.3) Foundations for functions.
The student understands how algebra can be used to express generalizations and recognizes and uses the power of symbols to represent situations.

The student:
(A) uses symbols to represent unknowns and variables.

(b.4) Foundations for functions.
The student understands the importance of the skills required to manipulate symbols in order to solve problems and uses the necessary algebraic skills required to simplify algebraic expressions and solve equations and inequalities in problem situations.

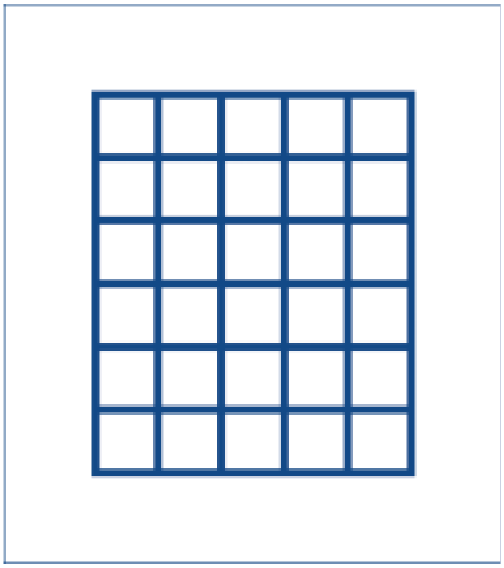
The student:
(A) finds specific function values, simplifies polynomial expressions, transforms and solves equations, and factors as necessary in problem situations; and

(B) uses the commutative, associative, and distributive properties to simplify algebraic expressions.

Scaffolding Questions:

- If the width of the border is 10 inches, each wooden strip surrounding a square is y inches wide, and the side of each square is x inches, how can you find the width of the entire window? The length?
- Describe the shape of the window.
- How can you determine the perimeter of a rectangle?
- How can you determine the area of a rectangle?

Sample Solution:



1. Along the horizontal side there are 5 widths of the window plus 6 wooden strip widths and 2 border widths. The width is represented by $5x + 6y + 20$.

   The height of the window is 6 widths of the window plus 7 wooden strip widths and 2 border widths. The height is $6x + 7y + 20$.

   The dimensions of the window are $(5x + 6y + 20)$ by $(6x + 7y + 20)$.

2. The perimeter is found by using the formula $P = 2$ (length) $+ 2$ (width).

$P = 2(5x + 6y + 20) + 2(6x + 7y + 20)$
$P = 10x + 12y + 40 + 12x + 14y + 40$
$P = 22x + 26y + 80$

3. The area of the window would be found by multiplying the length and width.

   $A = (5x + 6y + 20)(6x + 7y + 20)$
   $A = 30x^2 + 35xy + 100x + 36xy + 42y^2 + 120y + 120x + 140y + 400$
   $A = 30x^2 + 71xy + 220x + 42y^2 + 260y + 400$

Extension Question:

- Suppose you want your window to be 98 inches wide by 113 inches long. Write and solve a system of equations to find the widths of the squares (x) and the width of the wooden strips surrounding the squares (y).

The expression for the width is 5x + 6y + 20. This amount must equal 98 inches.

$5x + 6y + 20 = 98$

The expression for the length is 6x +7y + 20. The length must equal 113 inches.

$6x + 7y + 20 = 113$

Subtract 20 from each side of each equation.

$5x + 6y = 78$
$6x + 7y = 93$

Multiply the first equation by 6, the second by -5.

$6(5x + 6y + 20 = 98)$ $\quad 30x + 36y = 468$
$-5(6x + 7y + 20 = 113)$ $\quad -30x - 35y = -465$
$y = 3$

Texas Assessment of Knowledge and Skills:

Objective 2:
The student will demonstrate an understanding of properties and attributes of functions.

Connections to Algebra I: 2000 and Beyond Institute:

I. Foundations for Functions
2.1 Using Patterns to Identify Relationships

Connections to Algebra End-of-Course Exam:

Objective 6:
The student will perform operations on and factor polynomials that describe real-world and mathematical situations.

Substitute back into one of the original equations to find x:

$$\begin{aligned} 5x + 6y &= 78 \\ 5x + 6(3) &= 78 \\ 5x + 18 &= 78 \\ 5x &= 60 \\ x &= 12 \end{aligned}$$

The width of the square window (x) is 12 inches, and the width of the wooden strips (y) is 3 inches.