Chapter 1:
Coordinate Geometry

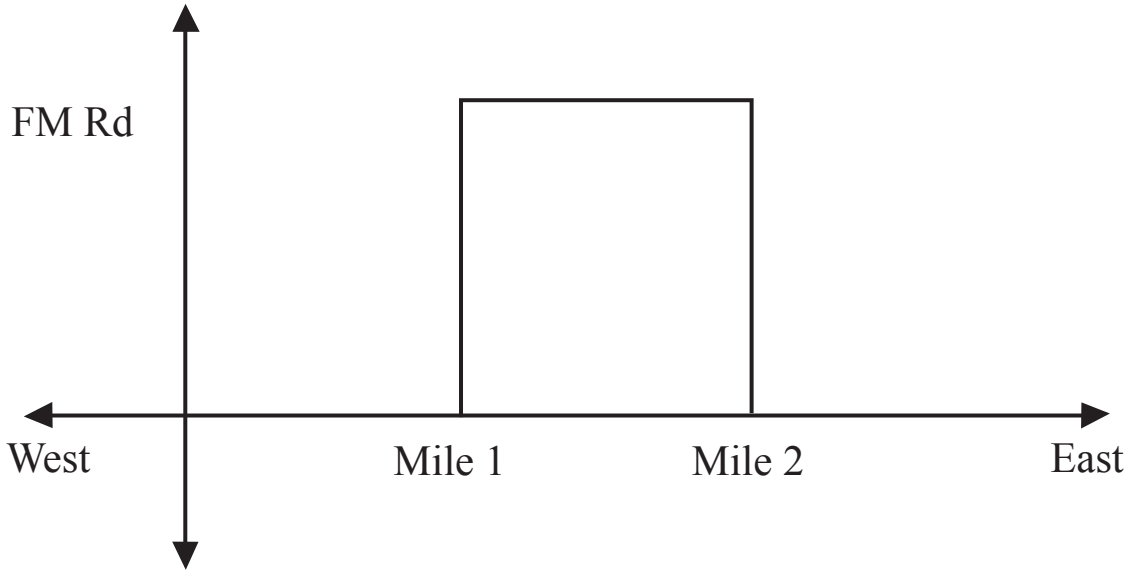
Introduction

The assessments in this chapter use the coordinate system as a convenient and efficient way of representing geometric figures and investigating geometric relationships. These problems provide the connections between what students learned in Algebra I and the geometric concepts. For example, Whitebeard's Treasure may be used as a problem to assess the student's knowledge of algebraic concepts and concepts learned about geometric figures in earlier mathematics courses.

> Geometry can be used to model and present many mathematical and real-world situations. Students perceive connections between geometry and the real and mathematical worlds and use geometric ideas, relationships, and properties to solve real problems. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999.)

Cross Country Cable

Jose owns a square parcel of land on the north side of West–East Road. The property is on the edge of the road between the one- and two-mile markers. The Farm-to-Market (FM) road runs perpendicular to the West–East Road and one mile west of the one-mile marker.



The cable company wants to run a cable through Jose's property from the intersection of the Farm-to-Market (FM) Road and West–East Road. The cable must divide Jose's parcel of land into two parts that are equal in area.

1. Determine the equation of the line that will represent the path of the buried cable.
2. Determine the length of the portion of the cable that runs across Jose's property.

Teacher Notes

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(d.2) Dimensionality and the geometry of location.
The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

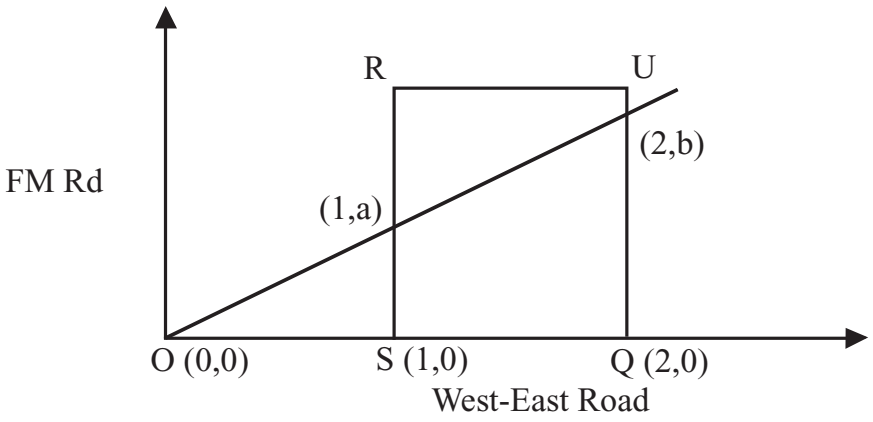
Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Scaffolding Questions:

- What information do you know about the roads near Jose's property and the location of his parcel of land?
- What needs to be added to the given figure before the equation of the line can be found?
- Where will the cable line originate? How can you label this point?
- In how many different ways might the cable intersect his property?
- Which way must the cable be drawn to divide it into two portions that are equal in area?
- How can you label the points where the cable will cross the boundaries of Jose's property?
- How can you determine the slope of the cable line?

Sample Solution:

1. Draw a figure in the coordinate plane to represent Jose's land.



Since SQUR is a square, the coordinates at R are (1,1), and the coordinates at U are (2,1). The coordinates of the

intersection of $\overline{SR}$ and the cable line are (1,*a*), and the point of intersection of $\overline{UQ}$ and the cable line are (2,*b*).

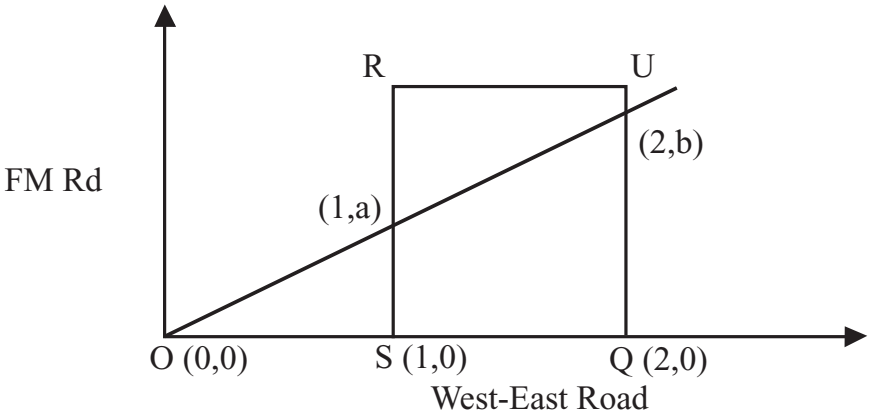
The cable originated at the intersection of the FM Road and the West–East Road. This point is the origin. The cable is going to begin at the origin, therefore one point on the line is (0,0). The *y*-intercept of the line will also be 0. The line will also contain point (1,*a*).

The slope of the line is $\frac{a-0}{1-0} = a$. The equation of this line is $y = ax$.

By using points (2,*b*) and (1,*a*), the slope of the line can also be expressed as $\frac{b-a}{2-1} = b-a$.

When the two slopes are set equal to each other, $a = b - a$, the equation can be simplified into $2a = b$ or $b = 2a$.

The cable divides the area of the square into two equal parts. Each upper and lower area is in the shape of a trapezoid.



We know that the area of the land is 1 square mile, and the area of one trapezoid is $\frac{1}{2}$ square mile. Using this information the area of one trapezoid can be calculated, and the numerical value of the slope can be found.

The lower trapezoid (below the cable line) has vertical bases with lengths *a* and *b*. The height of the trapezoid is the length of the side running along the West–East Road. The length is 1 mile. The area of the trapezoid is found by

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

III. Triangles: Pythagorean Theorem

III. Triangles: Right triangle Ratios

Teacher's Comment:

"To introduce the problem to the students, go over the criteria on the solution guide and allow students to read and decide how to solve the problem. Tell the students to read the problem silently before working as a group--it made it easier to work."

multiplying one-half by the product of the sum of the bases and the height.

$$A = \frac{1}{2}(a+b) \bullet 1 = \frac{a+b}{2}$$

We know this area is equal to one-half of the area of the mile square, therefore

$$\frac{1}{2} = \frac{a+b}{2}$$
$$a+b=1$$

By substituting the slope $b = 2a$ into the area $1 = a + b$, we find that

$$1 = a + 2a$$
$$1 = 3a$$
$$\frac{1}{3} = a$$

The value of the slope for the equation $y = ax$ is $\frac{1}{3}$, so the equation of the cable line that bisects Jose's property is $y = \frac{1}{3}x$.

2. The length of the cable is represented by the distance from (1,*a*) to (2,*b*). The value of $a = \frac{1}{3}$, and the value of $b = 2 \bullet \frac{1}{3} = \frac{2}{3}$.

The coordinates of the vertices representing the opposite corners of Jose's land are $\left(1, \frac{1}{3}\right)$ and $\left(2, \frac{2}{3}\right)$.

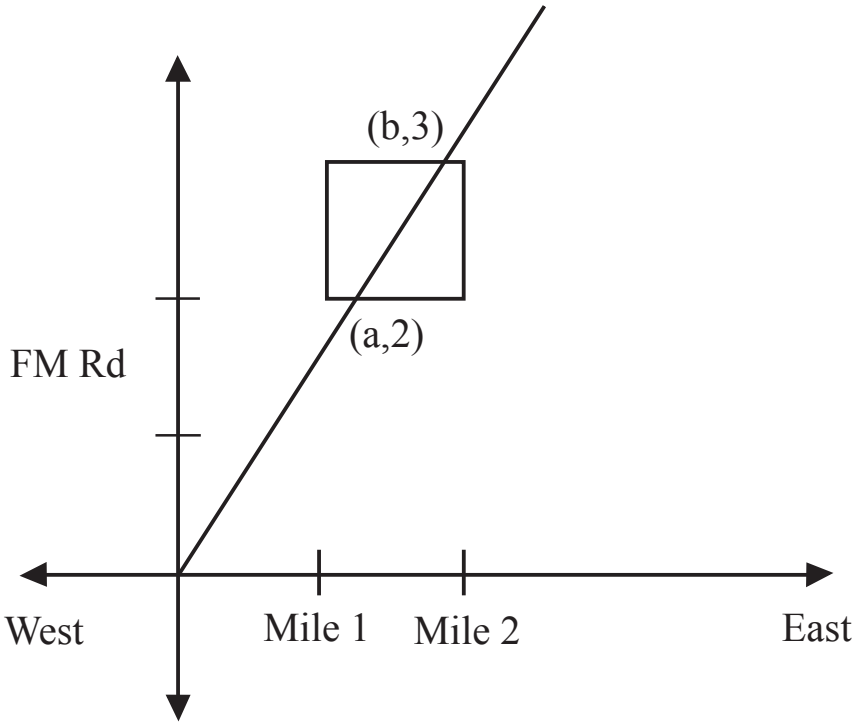
The distance can be found using the distance formula:

$$\sqrt{(2-1)^2 + \left(\frac{2}{3} - \frac{1}{3}\right)^2} = \sqrt{(1)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3} \text{ miles}$$

$$\approx 1.05 \text{ miles}$$

Extension Questions:

- Describe how this situation would change if the one square mile parcel of land were two miles north of West–East Road and one mile east of the Farm-to-Market Road. Justify your description with a diagram, and show your work.



The slope of the line can be represented as

$$\frac{3-0}{b-0} = \frac{3}{b} \text{ or}$$

$$\frac{2-0}{a-0} = \frac{2}{a}.$$

$$\frac{3}{b} = \frac{2}{a}$$

$$b = \frac{3}{2}a$$

Consider the portion to the left of the line. The height of the square is 1 unit. The two bases of the trapezoid may be expressed as a – 1 and b – 1. The area of the left half of the square parcel of land can be represented in this way:

$$\frac{1}{2} = \frac{1}{2}\left[(a-1)+(b-1)\right]\bullet 1$$
$$1 = a + b - 2$$
$$3 = a + b$$
$$b = \frac{3}{2}a$$
$$3 = a + \frac{3}{2}a$$
$$3 = \frac{5}{2}a$$
$$a = \frac{6}{5}$$
$$b = \frac{3}{2}a = \frac{3}{2}\left(\frac{6}{5}\right) = \frac{9}{5}$$

The points are $\left(\frac{6}{5}, 2\right)$ *and* $\left(\frac{9}{5}, 3\right)$.

The slope of the line is $\frac{3 - 2}{\left(\frac{9}{5} - \frac{6}{5}\right)} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$.

The equation of the line is $y = \frac{5}{3}x$.

To verify that the line divides the rectangle into two polygons of equal area, determine the area of the two trapezoids.

Left trapezoid:

$$\frac{1}{2}\left(\frac{6}{5} - 1 + \frac{9}{5} - 1\right)\bullet 1 = \frac{1}{2}\left(\frac{15}{5} - 2\right) = \frac{1}{2}\bullet 1 = \frac{1}{2}$$

Right trapezoid:

$$\frac{1}{2}\left(2 - \frac{6}{5} + 2 - \frac{9}{5}\right) \bullet 1 = \frac{1}{2}\left(4 - \frac{15}{5}\right) = \frac{1}{2} \bullet 1 = \frac{1}{2}$$

The land has been divided into two equal portions by the line $y = \frac{5}{3}x$.

Whitebeard's Treasure

Whitebeard, the notorious pirate of the West Bay, buried treasure on Tiki Island over 200 years ago. Archeologists recently discovered a map showing the location of the treasure. The location has generated quite a bit of media attention, much to the dismay of the archeologists. In order to allow both the media and archeologists to work together, officials have decided to erect two fences around the location, allowing the media access to the site, yet allowing the archeologists room to work. One fence encloses the actual area where the archeologists will work. Another fence surrounds the enclosed dig area.

Descriptions of the fencing locations have been provided to the media so they may indicate accessible areas for their employees. Use the given information to draw and label a quadrilateral on graph paper indicating the location of the two fences.

1. Corners of the first fence are located at points A(11,3), B(3,-11), C(-13,-9) and D(-5,9). The media must stay within this fenced area. Connect the points in alphabetical order, and then join point D to Point A.

2. Find and label the midpoints of each segment of quadrilateral ABCD, showing all work. Label the midpoints of the segments as follows:

   $\overline{AB}$ has midpoint Q,

   $\overline{BC}$ has midpoint R,

   $\overline{CD}$ has midpoint S,

   $\overline{DA}$ has midpoint T.

3. Connect the four midpoints in alphabetical order to create a new quadrilateral QRST. This quadrilateral represents the fence surrounding the archeological dig site.

4. Quadrilateral ABCD was an ordinary quadrilateral, but QRST is a special one. Determine the special name for quadrilateral QRST, and justify your answer using coordinate geometry in two different ways.

Teacher Notes

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(b.3) Geometric structure.
The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(d.2) Dimensionality and the geometry of location.
The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

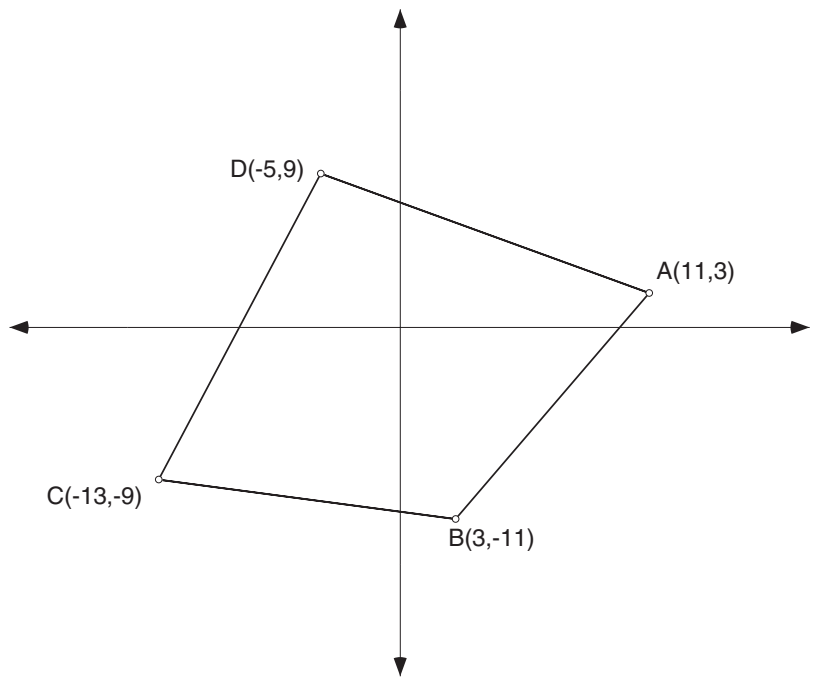
This problem addresses the same mathematical concepts as the problem Wearable Art. Whitebeard's Treasure gives the numerical coordinates. In Wearable Art the coordinates are given and the student must represent the situation using variable coordinates for the points. The teacher may choose to use one or both of these problems.

Scaffolding Questions:

- What is the formula for finding the midpoint of a line segment?
- Which of the quadrilaterals are special quadrilaterals?
- What are the characteristics of each special quadrilateral?
- What characteristics does quadrilateral QRST appear to possess that matches one of the special quadrilaterals?
- How can you prove these special characteristics?

Sample Solution:

Quadrilateral ABCD is graphed as shown. This is the outer fence.



To find the midpoint of each segment of quadrilateral ABCD, use the midpoint formula.

The midpoint of the segment with endpoints (x_1,y_1) and (x_2,y_2) is $\left(\frac{x_2+x_1}{2},\frac{y_2+y_1}{2}\right)$.

To find the midpoint of each segment, substitute the *x* and *y* values from the endpoints of the segment into the formula as follows:

Midpoint of $\overline{AB}$ (Point Q)

$$\left(\frac{11+3}{2},\frac{3+(-11)}{2}\right)=(7,-4)$$

Midpoint of $\overline{BC}$ (Point R)

$$\left(\frac{3+(-13)}{2},\frac{-11+(-9)}{2}\right)=(-5,-10)$$

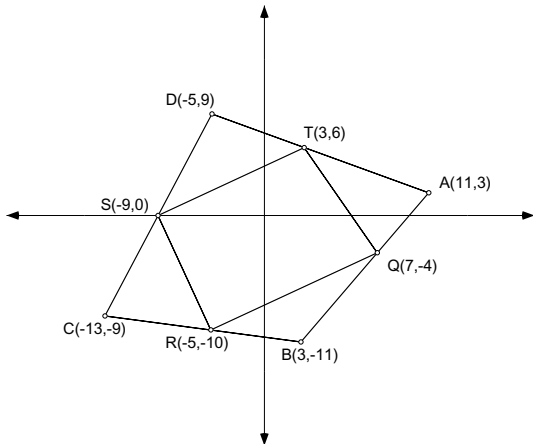
Midpoint of $\overline{CD}$ (Point S)

$$\left(\frac{-13+(-5)}{2},\frac{-9+9}{2}\right)=(-9,0)$$

Midpoint of $\overline{DA}$ (Point T)

$$\left(\frac{-5+11}{2},\frac{9+3}{2}\right)=(3,6)$$

Graph the midpoints and connect them in alphabetical order to form a new quadrilateral QRST.



Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Connections to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure:Midpoint Quadrilaterals

IV. Planar Figures: Investigating Quadrilateral Properties

Teacher's Comment:

" I had the students get out their notes before they started. I gave them two minutes to read the problem and think of questions they wanted to ask the class or me. Some of the scaffolding questions I asked were:

What types of quadrilaterals are there?

How do you know what type of quadrilateral it is?

How do you know when two lines are parallel?

How do you know when two lines are perpendicular?"

Quadrilateral QRST would be the fence that encloses the archeologists' dig site.

Quadrilateral QRST appears to be a parallelogram because the opposite sides of the newly formed quadrilateral appear to be parallel. One way to prove that a quadrilateral is a parallelogram is to prove that both pairs of opposite sides are parallel. Lines that have the same slope are parallel lines.

Use the slope formula:

The slope of the line through points (x_1,y_1) and (x_2,y_2) is $m = \frac{y_2 - y_1}{x_2 - x_1}$.

The slope of $\overleftrightarrow{RS}$ is $\frac{0-(-10)}{-9-(-5)} = \frac{10}{-4} = -\frac{5}{2}$.

The slope of $\overleftrightarrow{QT}$ is $\frac{6-(-4)}{3-7} = \frac{10}{-4} = -\frac{5}{2}$.

$\overleftrightarrow{RS} \,//\, \overleftrightarrow{QT}$ because both lines have the same slope.

The slope of $\overleftrightarrow{ST}$ is $\frac{6-0}{3-(-9)} = \frac{6}{12} = \frac{1}{2}$.

The slope of $\overleftrightarrow{RQ}$ = $\frac{-4-(-10)}{7-(-5)} = \frac{6}{12} = \frac{1}{2}$.

$\overleftrightarrow{ST} \,//\, \overleftrightarrow{RQ}$ because both lines have the same slope.

Quadrilateral QRST is a parallelogram by definition because both pairs of opposite sides are parallel.

Another way to show that QRST is a parallelogram is to prove that both sides of opposite sides are congruent (using the distance formula to find the lengths of each side).

$QR = \sqrt{(7-(-5))^2 + (-4-(-10))^2} = \sqrt{144+36} = \sqrt{180} = 6\sqrt{5}$

$ST = \sqrt{(-9-3)^2 + (0-6)^2} = \sqrt{144+36} = \sqrt{180} = 6\sqrt{5}$

$QR = ST = 6\sqrt{5}$

$RS = \sqrt{(-5-(-9))^2 + (-10-0)^2} = \sqrt{16+100} = \sqrt{116} = 2\sqrt{29}$

$QT = \sqrt{(7-3)^2 + (-4-6)^2} = \sqrt{16+100} = \sqrt{116} = 2\sqrt{29}$

$RS = QT = 2\sqrt{29}$

Both pairs of opposite sides of quadrilateral QRST are congruent, therefore it is a parallelogram.

Extension Questions:

- Use algebra to find the point of intersection of the diagonals of quadrilateral QRST .

To find the point where the diagonals intersect, the equations of lines $\overleftrightarrow{RT}$ and $\overleftrightarrow{SQ}$ must be identified and then used to find the point of intersection.

The slope of $\overleftrightarrow{RT}$ is $\frac{6-(-10)}{3-(-5)} = \frac{16}{8} = 2$

The slope of $\overleftrightarrow{SQ}$ is $\frac{0-(-4)}{-9-7} = \frac{4}{-16} = -\frac{1}{4}$

The equation of $\overleftrightarrow{RT}$ is $(y-6) = 2(x-3)$ or $y = 2x$

The equation of $\overleftrightarrow{SQ}$ is

$y = -\frac{1}{4}(x-(-9))$

$y = -\frac{1}{4}x - \frac{9}{4}$

The point where the diagonals intersect can be found by using linear combination.

$y = 2x$

$y = -\frac{1}{4}x - \frac{9}{4}$

$2x = -\frac{1}{4}x - \frac{9}{4}$

$8x = -1x - 9$

$9x = -9$

$x = -1$

$y = 2x = 2(-1) = -2$

The point of intersection is (-1,-2).

- Use coordinate geometry to prove the diagonals of quadrilateral QRST bisect each other.

The midpoint of QS is $\overline{QS}$ *is* $\left(\frac{7+(-9)}{2}, \frac{-4+0}{2}\right) = (-1,-2)$.

The midpoint of RT is $\overline{RT}$ *is* $\left(\frac{-5+3}{2}, \frac{-10+6}{2}\right) = (-1,-2)$.

The midpoints of the segment are the same point as the intersection point. The diagonals bisect each other.

Student Work Sample

Field Test Teacher's Comment:

> This was the first problem I had the students do using a poster. I enjoyed the poster and I feel most of the students did, too. I would like to have done this problem during the quadrilaterals section and will do so next year. One thing I did different this time was to have the students write on the back of their solution guide exactly what to put on their posters for the 3 criteria we emphasized.

Written on the back of one of her student's solution guide:

Shows an understanding of the relationships among elements.

- Statement showing how the elements are related.
- Can the history teacher understand your steps?

Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

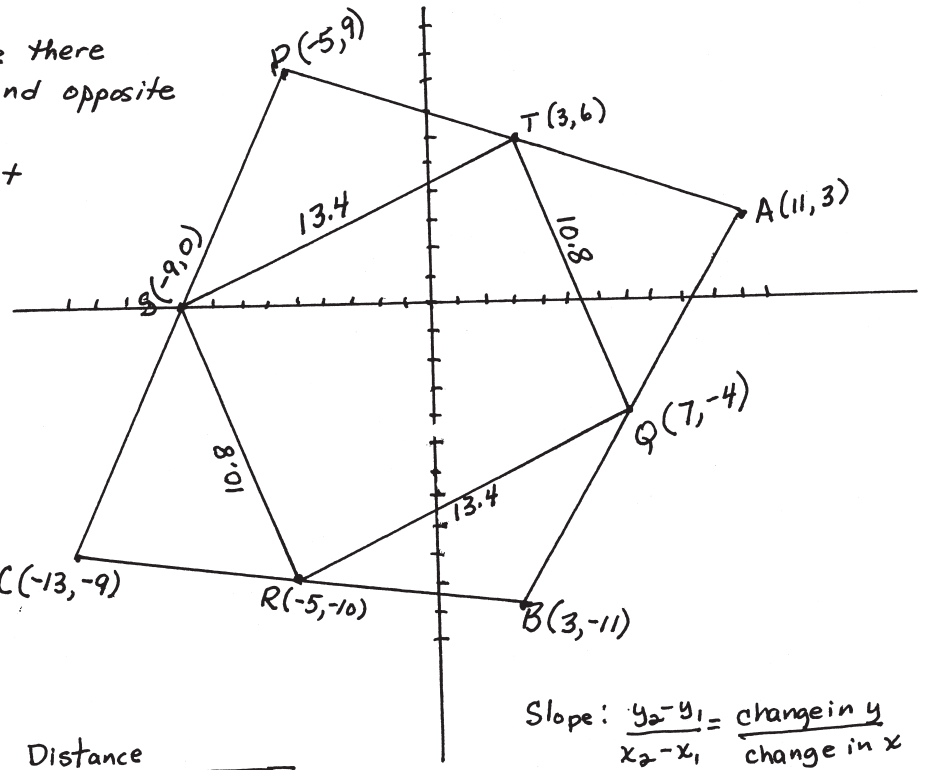
- Drawing the pictures
- Make appropriate markings on the picture

Communicates clear, detailed, and organized solution strategy.

- Step by step details that can be followed
- Don't plug in a number without showing why/how
- Must have justification
- Explaining your thinking!!!

A copy of the poster from this student's group appears on the next page.

A parallelogram - because there are two parallel sides and opposite sides are equal.
The diagonals are also not equal so it is not a rectangle or a square.



Midpoints on graph.

$RS = \sqrt{(-9-{}^-5)^2 + (0-{}^-10)^2}$

$= \sqrt{16+100} = \sqrt{116} \approx 10.8$

$TQ = \sqrt{(7-3)^2 + (-4-6)^2}$

$= \sqrt{16+100} = \sqrt{116} \approx 10.8$

opposite sides are equal.

$RS = TQ$

$QR = \sqrt{(-5-7)^2 + (-10-{}^-4)^2}$

$= \sqrt{144+36} = \sqrt{180} \approx 13.4$

$ST = \sqrt{(-9-3)^2 + (0-6)^2}$

$= \sqrt{144+36} = \sqrt{180} \approx 13.4$

Opposite sides are equal.

$QR = ST$

Distance

$RT = \sqrt{(-5-3)^2 + (-10-6)^2}$

$= \sqrt{64+256} = \sqrt{320} \approx 17.9$

$SQ = \sqrt{(-9-7)^2 + (0-{}^-4)^2}$

$= \sqrt{256+16} = \sqrt{272} \approx 16.5$

The diagonals are not equal.

Slope: $\frac{y_2-y_1}{x_2-x_1} = \frac{\text{change in } y}{\text{change in } x}$

$QR: \frac{-10-{}^-4}{-5-7} = \frac{-6}{-12} = \frac{1}{2}$

$ST: \frac{6-0}{3-{}^-9} = \frac{6}{12} = \frac{1}{2}$

$QR \parallel ST$ because the slopes are the same.

$RS: \frac{0-10}{-9-{}^-5} = \frac{10}{4} = \frac{5}{2}$

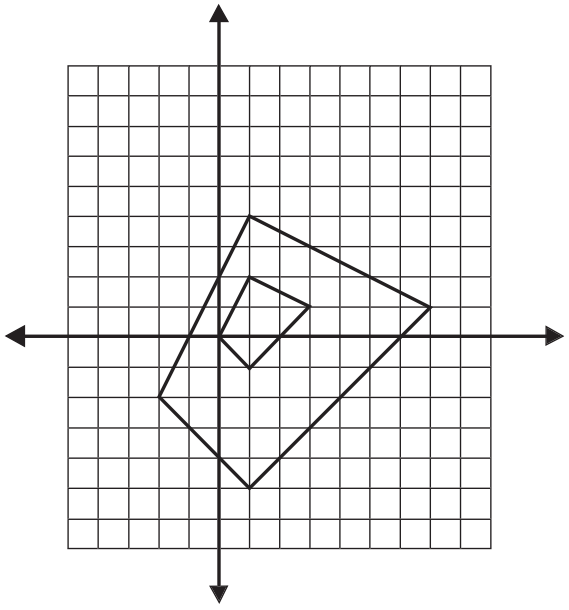
$QT: \frac{6-{}^-4}{3-7} = \frac{10}{-4} = \frac{5}{2}$

$RS \parallel QT$ because the slopes are the same.

Quadrilateral Quandary

The Seaside Hotel is going to install new landscaping on the hotel grounds. Plans that included the design and amounts of exact materials needed were purchased immediately after the landscaping project was approved. Due to recent water restrictions, however, the board of directors has decided to modify the project plans so that a smaller landscape bed will be constructed.

The hotel's landscape engineer has modified the size of the bed as shown. She must calculate the amount of the size reduction so that a detailed mathematical explanation can be presented to the board of directors, and the new costs can be calculated.



1. You have been hired to help the landscape engineer prepare her report and have been instructed to justify all of your calculations. You remember that the center of dilation is the intersection of 2 or more lines, each containing a point from the original figure and the corresponding point from the dilated figure. Use this definition and the figures above to calculate the center of dilation.

2. Find the scale factor for this dilation using the center of dilation and corresponding points of the quadrilaterals. Explain your process.

3. Verify that your scale factor is correct using the distance formula and corresponding parts of the figure.

Teacher Notes

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

(f) **Similarity and the geometry** of shape. The student applies the concepts of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures; and

(2) uses ratios to solve problems involving similar figures.

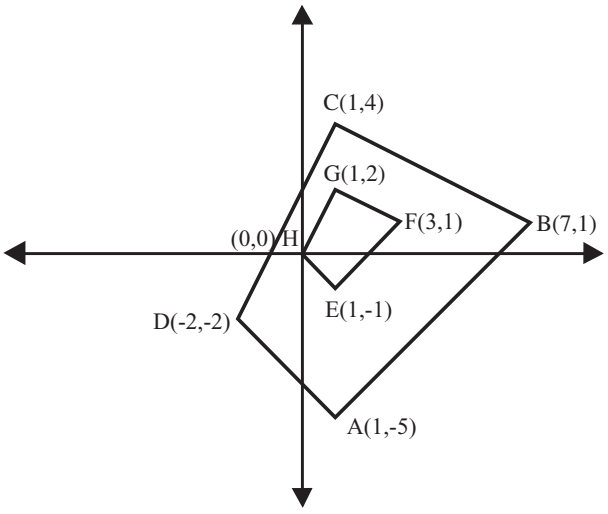
Scaffolding Questions:

- What kind of transformation is illustrated?
- Why is it necessary to calculate the center of dilation?
- Is the transformation rigid? Why or why not?
- What does the scale factor tell you about the size of the new figure?
- How does the value of the scale factor relate to the center of dilation?
- How do the dimensions of the new figure relate to the scale factor?

Sample Solutions:

1. The center of dilation is the intersection of 2 or more lines, each containing a point from the original figure and a corresponding point from the dilated figure. The strategy used will be to find the equations of the intersecting lines, enter them into the graphing calculator, and find the point of intersection. That point of intersection will be the center of dilation.

First, label the coordinates of the vertices of the quadrilaterals as follows: A(1,-5); B(7,1); C(1,4); D(-2,2) and E(1,-1); F(3,1); G(1,2); H(0,0).



Using the point-slope form, find the equation of the line that contains points A(1,-5) and E(1,-1) and an equation of the line containing D(-2,-2) and H(0,0).

The calculation for the slope of line $\overleftrightarrow{DH}$ is 1: $\frac{-2 - 0}{-2 - 0} = 1$.

The equation of line $\overleftrightarrow{DH}$ is found using point-slope form with H:

$y - 0 = 1\ (x - 0)$

$y = x$

The line $\overleftrightarrow{AE}$ is the vertical line $x = 1$. This line also passes through points C and G.

To find the intersection point of the line $y = x$ and $x = 1$ substitute 1 for x.

$y = 1$.

The intersection point of the two lines is (1,1). The vertical line through points B and F is $y = 1$. This line also passes through the point (1,1).

The center of dilation is (1,1). Label that point X on the diagram.

2. The center of dilation and the points A(1,-5) and E(1,-1) can be used in determining the scale factor. The ratio of the distances between the center of dilation and the corresponding points on the quadrilateral determines the scale factor. To calculate the scale factor, solve the equation:

$$\text{scale factor} = \frac{XE}{XA}$$

The distance formula is used to find the length of the segments.

$$XE = \sqrt{(1-1)^2 + (1-(-1))^2} = \sqrt{0 + 2^2} = 2$$

Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts of measurement and similarity.

Connections to High School Geometry: Supporting the TEKS and TAKS Institute:

II. Transformations: Up to Scale

Teacher's Comment:

"Using the assessments has helped my students retain the concepts used. Before using the assessments my students would work problems and move on to more concepts losing retention. Now my students have real-world situations they relate the concepts with and retain the concepts used."

The length of XA is calculated as follows:

$$XA = \sqrt{(1-1)^2 + (1-(-5))^2} = \sqrt{0+6^2} = 6 \ .$$

$$\text{scale factor} = \frac{XE}{XA} = \frac{2}{6} = \frac{1}{3}$$

This scale factor is true for each pair of corresponding sides. Corresponding sides are $\overline{DA}$ and $\overline{HE}$, $\overline{DC}$, and $\overline{HG}$, $\overline{CB}$, and $\overline{GF}$, $\overline{AB}$ and $\overline{EF}$.

$$DA = \sqrt{(-2-1)^2 + (-2-(-5))^2} = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18}$$

$$HE = \sqrt{(0-1)^2 + (0-(-1))^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

The ratio of the lengths of sides is $\frac{HE}{DA} = \frac{\sqrt{2}}{\sqrt{18}} = \sqrt{\frac{2}{18}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

$$DC = \sqrt{(-2-1)^2 + (-2-4)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

$$HG = \sqrt{(0-1)^2 + (0-2)^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

The ratio of the lengths of sides is $\frac{HG}{DC} = \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

$$CB = \sqrt{(-1-7)^2 + (4-1)^2} = \sqrt{(-6)^2 + (3)^2} = \sqrt{45}$$

$$GF = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$$

The ratio of the lengths of sides is $\frac{GF}{CB} = \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

$$AB = \sqrt{(1-7)^2 + (-5-1)^2} = \sqrt{(-6)^2 + (-6)^2} = \sqrt{72}$$

$$EF = \sqrt{(1-3)^2 + (-1-1)^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$$

The ratio of the lengths of sides is $\frac{EF}{AB} = \frac{\sqrt{8}}{\sqrt{72}} = \sqrt{\frac{8}{72}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

Extension Question:

- Landscape timbers are sold in 8-foot lengths. Suppose the original project was designed for 850 feet of landscaping timber, at a cost of $1.39 per timber. Calculate the minimum amount of materials that will need to be ordered for the new bed and the cost of materials before tax. Explain how you arrived at your answers.

The scale factor for the new project is $\frac{1}{3}$ of the original design plan. If the original project was designed for 850 feet of landscaping timber, the new project will require $\frac{1}{3}$ as much.

$\frac{1}{3}$ (850 feet) = 283.3 feet.

If the timbers are 8 feet long, divide 283.3 by 8. The result is 35.4125 timbers. Timbers are sold in 8-foot lengths; therefore, a minimum of 36 will be needed.

The cost of the project per landscape bed is calculated by multiplying $1.39 by 36 timbers.

1.39 (36) = 50.04

The cost for each landscape bed will be $50.04.

Student Work Sample

The problem analysis on the next page was created by a group of students. The students were given the problem, told to work silently on the problem for at least three minutes, and asked to create a group poster of their analysis of the problem.

The work is a good example of:

- Shows a relationship among the elements.

  They demonstrated their understanding of which points corresponded to one another and used the appropriated labeling (A, A').

- Uses appropriate terminology and notation.

  The students used correct language for transformations (image, dilation, scale factor). They wrote and used the formulas for distance between two points.

The students neglected to give reasons for the steps and to explain why the point (1,1) is the center of dilation.

Quadrilateral Quandry

Original Image

A (1, 4) B (7, 1)

C (1, -5) D (-2, 2)

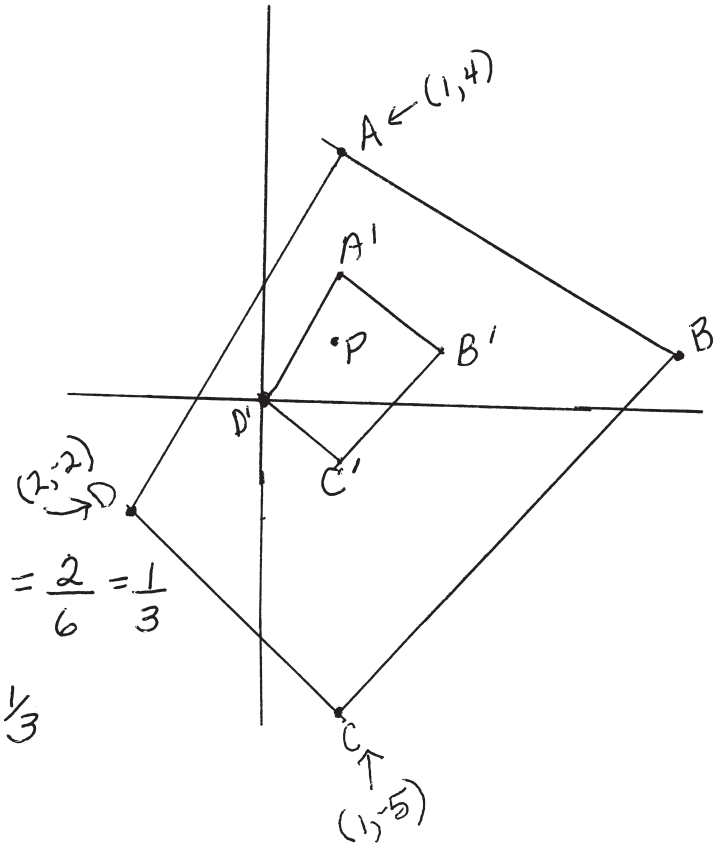
Revised Image

A' (1, 2) B' (3, 1)

C' (1, -1) D' (0, 0)

Center of Dilation

P (1, 1)



$\frac{PB'}{PB} = SF \quad \frac{PB'}{PB} = \frac{2}{6} = \frac{1}{3}$

Scale Factor $= \frac{1}{3}$

$$\frac{A'B'}{AB} = \frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}$$

$$= \frac{\sqrt{(2-1)^2 + (1-3)^2}}{\sqrt{(4-1)^2 + (1-7)^2}}$$

$$= \frac{\sqrt{1^2 + (-2)^2}}{\sqrt{3^2 + (-6)^2}}$$

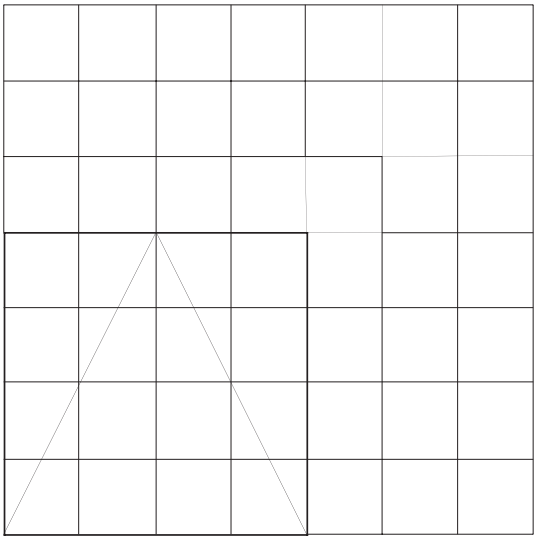
$$= \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}}$$

$$= \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Quite a Quilt

Ed has decided to enter a national geometry competition. The contest rules state that individuals must submit plans for a 4-inch quilt square design that will produce an octagonal region within the square whose area is no larger than 3 square inches. The entry must clearly explain and illustrate how the design is to be created, and it must prove that the inner octagonal area is within the contest guidelines.

Ed has designed his square so that the midpoint of each side of the square is joined to its two opposite vertices. The figure below shows Ed's beginning sketch.



1. Complete the quilt design, and label the figure representing Ed's octagon.
2. Determine the coordinates of the vertices of the octagon. Justify your answer.
3. Explain how you know whether or not the octagon is a regular octagon.
4. Does his design meet the contest's area criteria as outlined above? Justify your solution using coordinate geometry, and show all your work.

Teacher Notes

Materials:
One graphing calculator per student

Dynamic geometry computer program (optional)

Connections to Geometry TEKS:

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

(B) constructs and justifies statements about geometric figures and their properties;

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

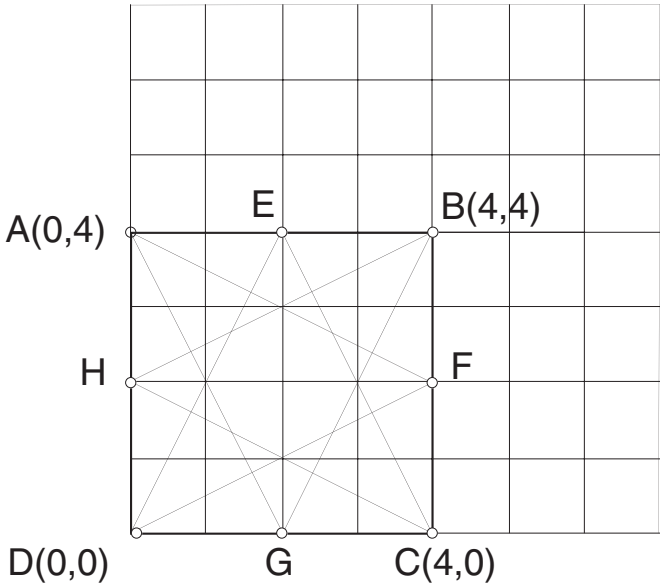
(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

Scaffolding Questions:

- Describe how to determine the coordinates of each vertex of the square.
- Determine the coordinates of the midpoint of each side of the square.
- Determine the coordinates of the center of the square.
- How can you mathematically determine these points? (Note: Students may investigate this problem using a dynamic geometry computer program.)
- If you connect the vertices of the octagon to the center of the square, what do you know about the 8 triangles that are formed?
- What two properties must be satisfied if the octagon is to be a regular octagon?
- How could you determine the area of one-fourth of the octagon?

Sample Solutions:

1.



The length of the side of the square is 4 inches. The coordinates of the midpoints will be:

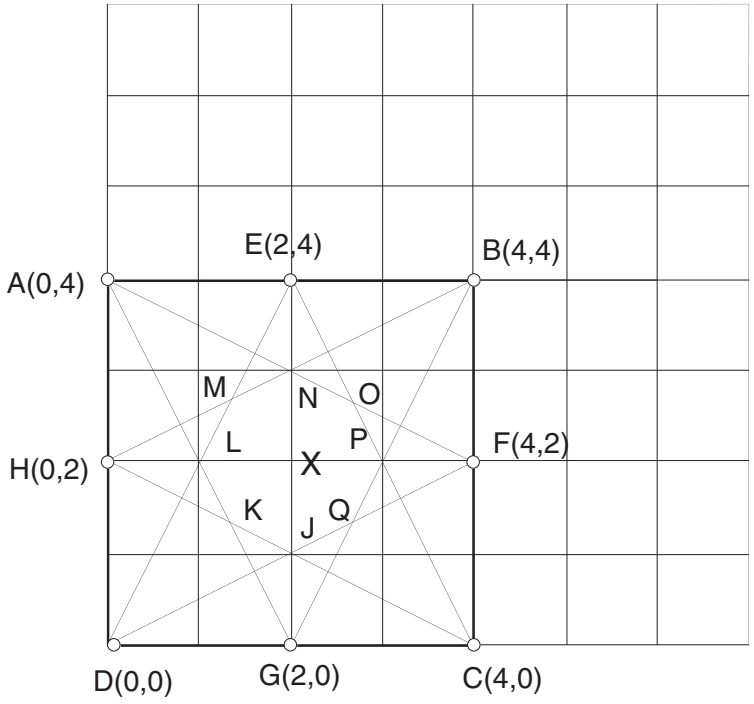
$$G = \left(\frac{0+4}{2}, \frac{0+0}{2}\right) = (2,0)$$

$$F = \left(\frac{4+4}{2}, \frac{0+4}{2}\right) = (4,2)$$

$$E = \left(\frac{0+4}{2}, \frac{4+4}{2}\right) = (2,4)$$

$$H = \left(\frac{0+0}{2}, \frac{0+4}{2}\right) = (0,2)$$

The center of the square will be at (2,2). Label the center point X. The center of the square is also the center of the octagon. Label the vertices of the octagon J, K, L, M, N, O, P, Q.



2. To determine the vertices, the equations of each of the lines can be determined. The slopes are determined using the graph.

(C) develops and uses formulas including distance and midpoint.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures.

Texas Assessment of Knowledge and Skills:

Objective 2: The student will demonstrate an understanding of properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Connections to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Investigating Quadrilateral Properties

IV. Planar Figures: Student Activity

$\overleftrightarrow{AG}$:

Slope: -2 Point A(0,4)

$y - 4 = -2(x - 0)$

$y = -2x + 4$

$\overleftrightarrow{AF}$:

Slope: $-\frac{1}{2}$ Point A(0,4)

$y = -\frac{1}{2}x + 4$

$\overleftrightarrow{BG}$:

Slope: 2 Point B(4,4)

$y - 4 = 2(x - 4)$

$y = 2x - 4$

$\overleftrightarrow{BH}$:

Slope: $\frac{1}{2}$ Point B(4,4)

$y - 4 = \frac{1}{2}(x - 4)$

$y = \frac{1}{2}x + 2$

$\overleftrightarrow{CH}$:

Slope: $-\frac{1}{2}$ Point H(2,0)

$y - 2 = -\frac{1}{2}(x - 0)$

$y = -\frac{1}{2}x + 2$

$\overleftrightarrow{CE}$:

Slope: -2 Point E(2,4)

$y - 4 = -2(x - 2)$

$y = -2x + 8$

$\overleftrightarrow{DF}$:

Slope: $\frac{1}{2}$ Point D(0,0)

$$y - 0 = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x$$

$\overleftrightarrow{DE}$:

Slope: 2 Point D(0,0)

$y - 0 = 2(x - 0)$

$y = 2x$

Using linear combination to solve the systems, the values of x and y can be found to determine the intersection of lines $\overleftrightarrow{HC}$ and $\overleftrightarrow{DF}$. The point of intersection will be labeled point J as in the diagram above.

$$y = -\frac{1}{2}x + 2$$

$$y = \frac{1}{2}x$$

Adding these 2 equations together produces the following:

$2y = 2$, therefore $y = 1$.

Substitute $y = 1$ into either equation to solve for x:

$$1 = \frac{1}{2}x$$

$2 = x$

Therefore the point of intersection is J(2,1).

The intersection of line $\overleftrightarrow{AG}$ and $\overleftrightarrow{HC}$ is the point K.

Using linear combination on the equation for lines $\overleftrightarrow{AG}$ and $\overleftrightarrow{HC}$ will allow the coordinates of point K to be found.

$$y = -2x + 4$$

$$y = -\frac{1}{2}x + 2$$

$$0 = -\frac{3}{2}x + 2$$

$$-2 = -\frac{3}{2}x$$

$$\frac{4}{3} = x$$

Substitute $x = \frac{4}{3}$ into either equation to find the coordinate of *y.*

$$y = -2\left(\frac{4}{3}\right) + 4$$

$$y = \frac{4}{3}$$

The coordinates of K are $\left(\frac{4}{3}, \frac{4}{3}\right)$.

Point L is the intersection of lines $\overleftrightarrow{AG}$ and $\overleftrightarrow{DE}$.

$\overleftrightarrow{AG}$: $y = -2x + 4$

$\overleftrightarrow{DE}$: $y = 2x$

$2x = -2x + 4$

$4x = 4$

$x = 1$

$y = 2x = 2(1) = 2$

The coordinates of L are (1,2).

Point M is the intersection of lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{BH}$.

$\overleftrightarrow{DE}$: $y = 2x$

$\overleftrightarrow{BH}$: $y = \frac{1}{2}x + 2$

$$2x = \frac{1}{2}x + 2$$
$$4x = x + 4$$
$$x = \frac{4}{3}$$
$$y = 2x = 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

The coordinates of point M are $\left(\frac{4}{3}, \frac{8}{3}\right)$.

Point N is the intersection of lines $\overleftrightarrow{AF}$ and $\overleftrightarrow{BH}$.

$\overleftrightarrow{AF}$: $y = -\frac{1}{2}x + 4$

$\overleftrightarrow{BH}$: $y = \frac{1}{2}x + 2$

$$-\frac{1}{2}x + 4 = \frac{1}{2}x + 2$$
$$-1x = -2$$
$$x = 2$$
$$y = \frac{1}{2}x + 2 = 1 + 2 = 3$$

The coordinates of N are (2,3).

Point O is the intersection of lines $\overleftrightarrow{AF}$ and $\overleftrightarrow{CE}$.

$\overleftrightarrow{AF}$: $y = -\frac{1}{2}x + 4$

$\overleftrightarrow{CE}$: $y = -2x + 8$

$$-\frac{1}{2}x + 4 = -2x + 8$$

$$-1x + 8 = -4x + 16$$

$$x = \frac{8}{3}$$

$$y = -2x + 8 = -2\left(\frac{8}{3}\right) + 8 = \frac{8}{3}$$

The coordinates of O are $\left(\frac{8}{3}, \frac{8}{3}\right)$.

Point P is the intersection of lines $\overleftrightarrow{CE}$ and $\overleftrightarrow{BG}$.

$\overleftrightarrow{CE}$: $y = -2x + 8$

$\overleftrightarrow{BG}$: $y = 2x - 4$

$2x - 4 = -2x + 8$

$4x = 12$

$x = 3$

$y = 2x - 4 = 2(3) - 4 = 2$

The coordinates of P are (3,2).

Point Q is the intersection of lines $\overleftrightarrow{DF}$ and $\overleftrightarrow{BG}$.

$\overleftrightarrow{DF}$: $y = \frac{1}{2}x$

$\overleftrightarrow{BG}$: $y = 2x - 4$

$$\frac{1}{2}x = 2x - 4$$
$$x = 4x - 8$$
$$-3x = -8$$
$$x = \frac{8}{3}$$
$$y = \frac{1}{2}x = \frac{1}{2}\left(\frac{8}{3}\right) = \frac{4}{3}$$

The coordinates of Q are $\left(\frac{8}{3}, \frac{4}{3}\right)$.

3. A regular octagon has 8 congruent sides and 8 congruent angles.

It can be demonstrated that the octagon is not a regular octagon by determining the distances of LK, KJ, XK, XJ, and XL.

$$LK = \sqrt{\left(1 - \frac{4}{3}\right)^2 + \left(2 - \frac{4}{3}\right)^2} = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{5}{9}}$$

$$KJ = \sqrt{\left(2 - \frac{4}{3}\right)^2 + \left(1 - \frac{4}{3}\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = \sqrt{\frac{5}{9}}$$

$$XK = \sqrt{\left(2 - \frac{4}{3}\right)^2 + \left(2 - \frac{4}{3}\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{8}{9}}$$

$$XL = \sqrt{(2-1)^2 + (2-2)^2} = 1$$

$$XJ = \sqrt{(2-2)^2 + (2-1)^2} = 1$$

$\overline{XK}$ is not equal in length to $\overline{XL}$ or $\overline{XJ}$.

Triangles XKJ and XLK are congruent triangles, but they are not isosceles triangles because the sides are of three different lengths.

XL = XJ = 1

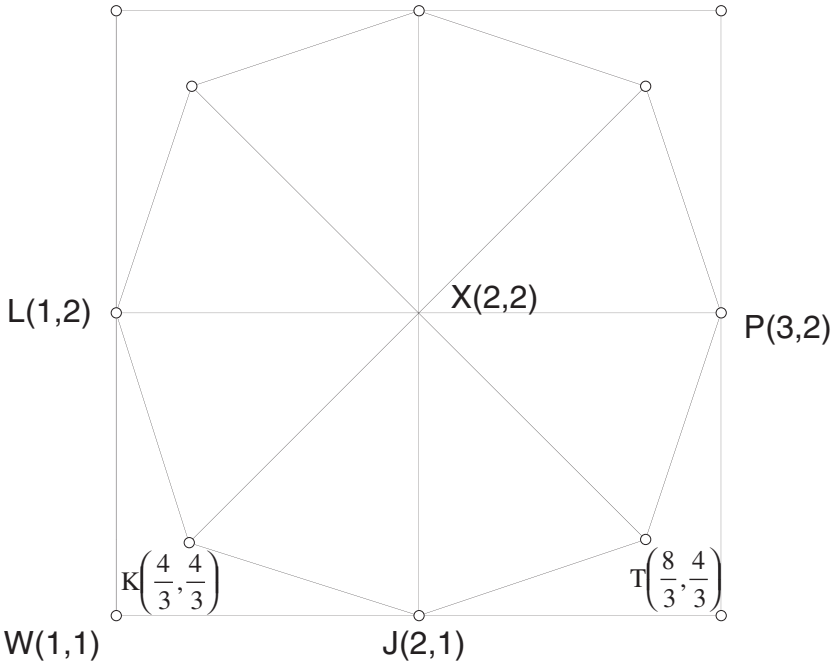
$LK = KJ = \sqrt{\frac{5}{9}}$

The length of the common side $\overline{XK}$ is $\sqrt{\frac{8}{9}}$.

Thus, the octagon is not a regular octagon.

4. It is still possible to find the area of the octagon as it can be divided into 4 parts that are congruent. Each part of the bottom half is composed of the two congruent triangles.

For example, one-fourth of the octagon, LKJX, is composed of the two congruent triangles LXK and JXK.



The area of square LXJW is 1 square unit.

Draw $\overline{WK}$. Triangle WJK has a base, $\overline{WJ}$, that measures one unit.

The height from K to $\overline{WJ}$ is $\frac{4}{3} - 1 = \frac{1}{3}$.

The area of the triangle is $\frac{1}{2} \bullet 1 \bullet \frac{1}{3} = \frac{1}{6}$.

Similarly, the area of triangle LWK is $\frac{1}{2} \bullet 1 \bullet \frac{1}{3} = \frac{1}{6}$.

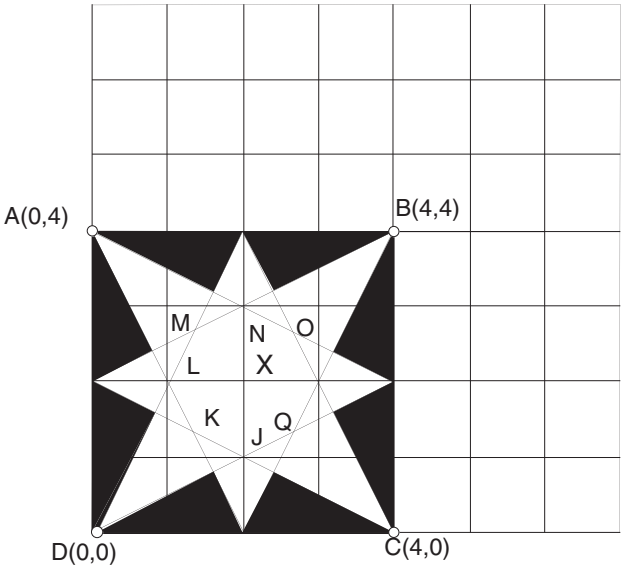
The area of LKJX is the area of LXJW minus the area of the two triangles LWK and WKJ, or $1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}$.

The area of the octagon is four times the area of LKXJ, or $4\left(\frac{2}{3}\right) = \frac{8}{3} = 2\frac{2}{3}$ square inches.

Ed's design meets the guidelines because the total area is no larger than 3 square inches.

Extension Question:

- Determine the area of the white star.



Call the intersection of the lines $\overleftrightarrow{DF}$ and $\overleftrightarrow{AG}$ *point W.*

$\overleftrightarrow{DF}$: $y = \frac{1}{2}x$

$\overleftrightarrow{AG}$: $y = -2x + 4$

$$\frac{1}{2}x = -2x + 4$$

$$x = -4x + 4$$

$$x = \frac{8}{5}$$

$$y = \frac{1}{2}x = \frac{1}{2}\left(\frac{8}{5}\right) = \frac{4}{5}$$

Triangle DWG has a base of 2 inches and a height $\frac{4}{5}$ *inches.*

The area of triangle DWG is $\frac{1}{2} \bullet 2 \bullet \frac{4}{5} = \frac{4}{5}$ *square inches.*

The area of the star is the area of the large square minus 8 of the triangles that are congruent to triangle DWG.

$$16 - 8\left(\frac{4}{5}\right) = 16 - \frac{32}{5} = 16 - 6\frac{2}{5} = 9\frac{3}{5} \text{ square inches.}$$

Student Work Sample

The student work displayed on the next page was completed using geometry computer software.

The criteria of the Geometry Solution Guide that are exemplified in this example are the following:

- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

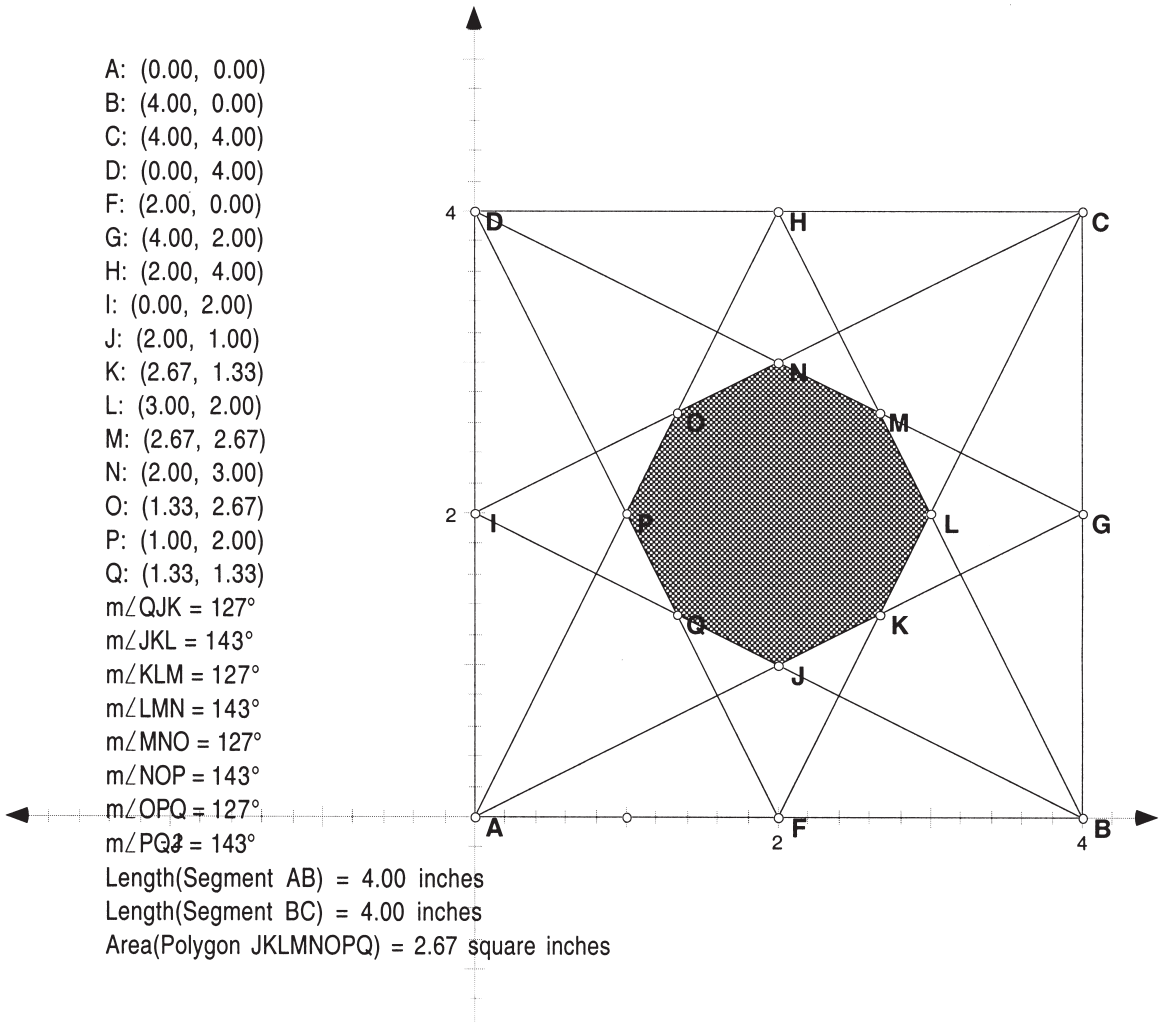
  The student has clearly and correctly labeled points. The measurements have been taken that will allow the student to answer the questions about the quilt design.

- Uses appropriate tools.

  The problem did not require that the measurements or calculations be done using algebraic methods. The use of the geometric software demonstrates the solution, but note that it does constitute a geometric proof.

These are my coordinates for each design point. From each point to another point the points are listed. The coordinates are accurate on the graph.

A: (0.00, 0.00)
B: (4.00, 0.00)
C: (4.00, 4.00)
D: (0.00, 4.00)
F: (2.00, 0.00)
G: (4.00, 2.00)
H: (2.00, 4.00)
I: (0.00, 2.00)
J: (2.00, 1.00)
K: (2.67, 1.33)
L: (3.00, 2.00)
M: (2.67, 2.67)
N: (2.00, 3.00)
O: (1.33, 2.67)
P: (1.00, 2.00)
Q: (1.33, 1.33)
m∠QJK = 127°
m∠JKL = 143°
m∠KLM = 127°
m∠LMN = 143°
m∠MNO = 127°
m∠NOP = 143°
m∠OPQ = 127°
m∠PQJ = 143°
Length(Segment AB) = 4.00 inches
Length(Segment BC) = 4.00 inches
Area(Polygon JKLMNOPQ) = 2.67 square inches



The octagon is not a regular octagon. The angles are not the same.

This quilt matches the guidelines to enter the contest. The quilt is a 4 in quilt and the shape is not larger than 3 square inches.

Wearable Art

Lorraine's graphics arts class has been assigned a t-shirt design project. Each student is to create a design by drawing any quadrilateral, connecting the midpoints of the sides to form another quadrilateral, and coloring the regions. Lorraine claims that everyone's inner quadrilateral will be a parallelogram. Use coordinate geometry to determine if she is correct. Show all of your work, and explain your reasoning.

Teacher Notes

Materials:
One graphing calculator per student

Computer drawing utility (optional)

Connections to Geometry TEKS:

(b.2) **Geometric structure**. The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure**. The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(d.2) **Dimensionality and the geometry of location**. The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

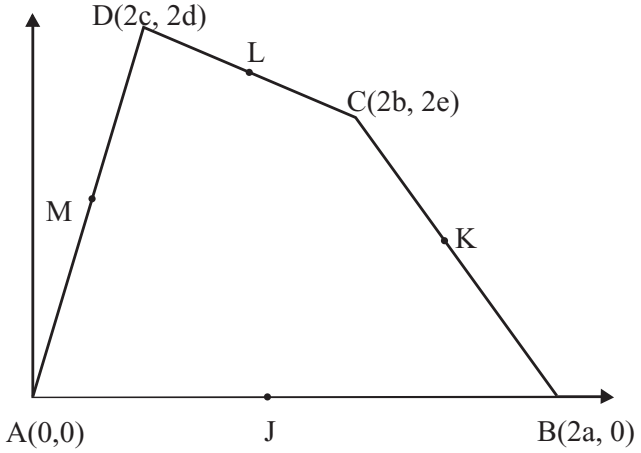
This problem addresses the same mathematical concepts as the problem Whitebeard's Treasure that gives the numerical coordinates. In Wearable Art the coordinates are given and the student must represent the situation using variable coordinates for the points. The teacher may choose to use one or both of these problems.

Scaffolding Questions:

- How will using randomly selected coordinates for the vertices of the quadrilateral help prove Lorraine's claim?
- How could you place your quadrilateral on the coordinate plane so that the coordinates of your vertices will be "easy" to work with?
- How can you select coordinate values that will be "friendly" when figuring midpoints?

Sample Solution:

Draw a quadrilateral on a coordinate plane. Locate one vertex at the origin and one side on the x-axis. Remembering that midpoints will be needed, select coordinates that are multiples of two. Label the vertices of the quadrilateral A, B, C, and D and the midpoints J, K, L, and M.



Find the coordinates of the midpoints as follows:

The midpoint of $\overline{AB}$, J, is the point

$$\left(\frac{0+2a}{2}, \frac{0+0}{2}\right) = (a, 0) .$$

The midpoint of $\overline{BC}$, K, is the point

$$\left(\frac{2a+2b}{2}, \frac{0+2e}{2}\right) = (a+b, e) .$$

The midpoint of $\overline{CD}$, L, is the point

$$\left(\frac{2b+2c}{2}, \frac{2d+2e}{2}\right) = (b+c, d+e) .$$

The midpoint of $\overline{DA}$, M, is the point

$$\left(\frac{2c+0}{2}, \frac{2d+0}{2}\right) = (c, d) .$$

Draw line segments $\overline{JK}$, $\overline{KL}$, $\overline{LM}$, and $\overline{MJ}$. These segments form the sides of the inner quadrilateral.

If the inner quadrilateral is a parallelogram, then opposite sides must be parallel. Using the slope formula, it can be shown that the lines are parallel because parallel lines have the same slope. $\overline{JK}$ must be parallel to $\overline{LM}$, and $\overline{KL}$ must be parallel to $\overline{MJ}$ if the figure is a parallelogram.

The slope of $\overline{JK} = \dfrac{e-0}{(a+b)-a} = \dfrac{e}{b}.$

The slope of $\overline{LM} = \dfrac{(d+e)-d}{(b+c)-c} = \dfrac{e}{b}.$

Segments $\overline{JK}$ and $\overline{LM}$ both have the same slope, therefore they are parallel.

The slope of $\overline{MJ} = \dfrac{d-0}{c-a} = \dfrac{d}{c-a}.$

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Connections to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Midpoint Quadrilaterals

IV. Planar Figures: Investigating Quadrilateral Properties

The slope of $\overline{KL} = \frac{(d+e)-e}{(b+c)-(a+b)} = \frac{d}{c-a}$.

Segments $\overline{MJ}$ and $\overline{KL}$ both have the same slope, therefore they are parallel.

By definition quadrilateral JKLM is a parallelogram (both pairs of opposite sides are parallel). Lorraine's claim will be true for any quadrilateral that joins the midpoints of the sides of any convex quadrilateral.

Extension Questions:

- Mark claims that the inner quadrilateral created by joining the midpoints of the sides will be a rhombus. Prove or disprove his conjecture. Be sure to show all of your work.

A drawing utility can be used to test the conjecture. A rhombus has perpendicular diagonals and 4 congruent sides. In this case $\overline{MK}$ is not perpendicular to $\overline{JL}$ because the slopes of the diagonals are not opposite reciprocals of one another.

The slope of $\overline{MK}$ is $\frac{e-d}{a+b-c}$.

The slope of $\overline{JL}$ is $\frac{d+e-0}{b+c-a}$.

$$-\frac{1}{\frac{d+e-0}{b+c-a}} = -\frac{b+c-a}{d+e-0} \neq \frac{e-d}{a+b-c}.$$

The slopes of the two lines are not opposite reciprocals. Therefore, the quadrilateral is not a rhombus.

There is no need to check the 4 congruent sides because the first condition of a rhombus was not met.

- Can the quadrilateral formed by joining the midpoints be a rectangle? A square? Write your conjectures and test them using a drawing program. Justify your answers.

If the quadrilateral is a rectangle, then $\overline{MJ}$ must be perpendicular to $\overline{JK}$. That would mean that the slope of $\overline{MJ}$ is equal to the opposite reciprocal of the slope of $\overline{JK}$.

$$\frac{d}{c-a} = \frac{-1}{\frac{e}{b}}$$

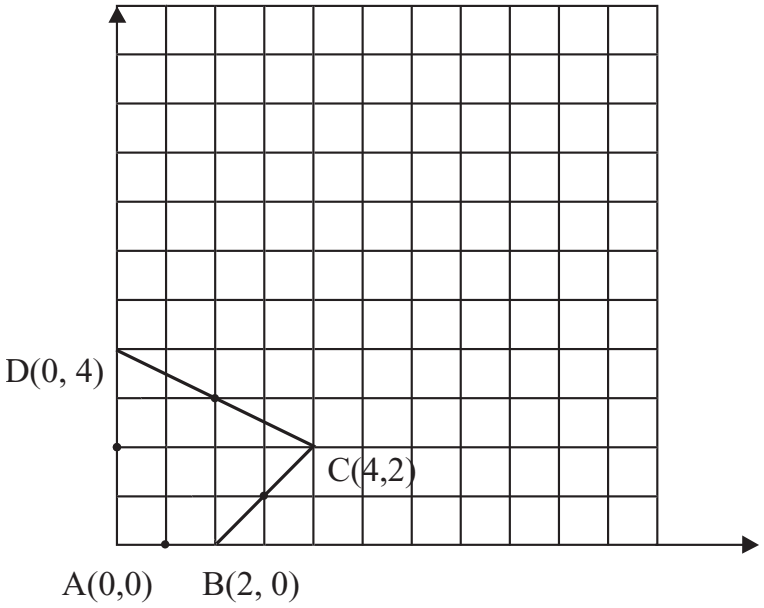
$$\frac{d}{c-a} = \frac{-b}{e}$$

$$de = -bc + ab$$

Choose any values for a, b, c, d, and e that make this a true statement. One set of values is d = 2, e = 1, a = 1, c =0, b = 2.

de = -bc+ab
2(1)=-2(0)+1(2)

This choice results in a rectangle.



The midpoints are J(1,0); K(3,1); L(2,3); and M(0,2).

Each side of the rectangle is the hypotenuse of a right triangle with legs measuring 1 unit and 2 units. Therefore, by the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1^2 + 2^2} = \sqrt{5}$.

The resulting figure, JKLM, is also a square because it is a rectangle with four congruent sides.

Sea Quest

The Coastal Marine Institute is going to gather data in Galveston Bay. A new computer device will be submerged just outside the ship channel fairway. In order to get the most accurate readings, the device must be located at a point such that the distance from the bay floor to the device is equal to one-third of the distance between the computer device and the top of the antenna. The depth of the bay at the selected location is 18.5 feet. The height of the antenna will be 21.5 feet above the water at mean tide.

Draw a diagram of the situation, and determine the depth at which the new computer device will be located at mean tide. Justify your solution.

Teacher Notes

Materials:
One graphing calculator per student.

Connections to Geometry TEKS:

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(C) develops and uses formulas including distance and midpoint.

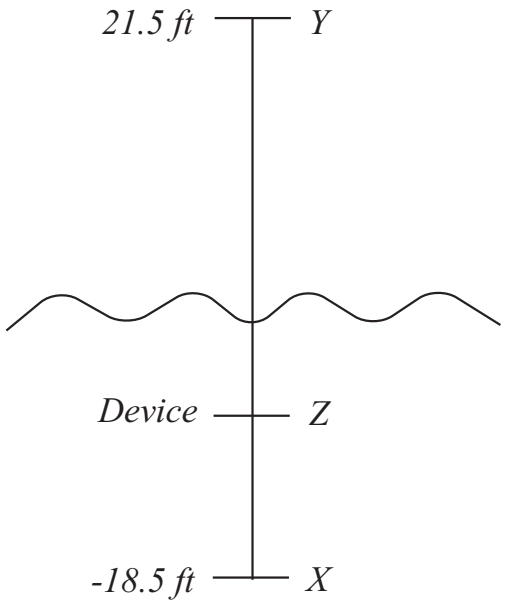
Scaffolding Questions:

- If you assign numerical values to the objects, where will the zero value be?
- What type of diagram would best help illustrate this situation?
- How can you find the total distance between the floor of the bay and the top of the antenna?

Sample Solution:

If the depth of the water is 18.5 feet, and the height of the antenna is 21.5 feet, both positive and negative numbers will need to be used. The depth will correspond to -18.5 feet because the distance is below the water. The height of the antenna will correspond to a positive 21.5 feet.

A vertical number line can be used to illustrate this situation.



The distance between X and Z is equal to $\frac{1}{3}$ the distance between Z and Y. Using segment addition, it can be shown that XZ + ZY = XY, and we are told that XZ = $\frac{1}{3}$(ZY). The distance between X and Y is |-18.5 – 21.5| or 40 feet.

To find the values of XZ and ZY, the system of 2 equations may be solved by substitution.

$XZ + ZY = XY$

$$XZ = \frac{1}{3}(ZY)$$

$$\frac{1}{3}(ZY) + ZY = 40$$

$$\frac{4}{3}(ZY) = 40$$

$$\frac{3}{4} \bullet \frac{4}{3}(ZY) = \frac{3}{4} \bullet 40$$

ZY = 30 and XZ = 10 because XZ + ZY = 40.

The depth at point Z must be 10 feet from the floor of Galveston Bay. If the depth at the selected location is 18.5 ft, the computer device will be located at a depth of 8.5 feet. (-18.5 + 10 = -8.5).

Texas Assessment of Knowledge and Skills:

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

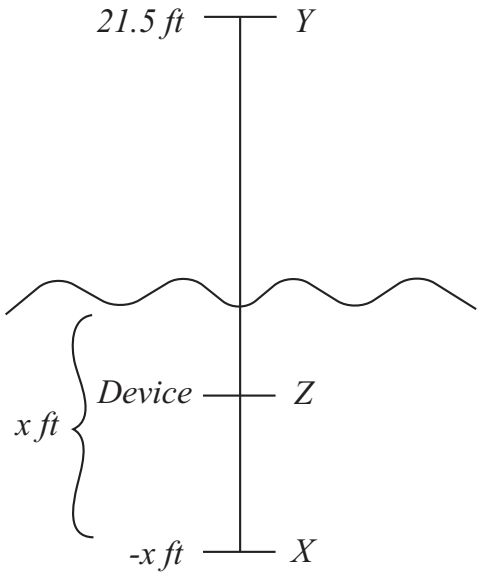
Connection to High School Geometry: Supporting TEKS and TAKS Institute:

III. Triangles: Pythagorean Theorem

Extension Questions:

- How deep would the bay have to be in order to locate the computer device at half the depth of the bay? Assume the device still gives the most accurate readings when the distance from the bay floor to the device is one-third the distance from the device to the top of the antenna, 21.5 feet above the water.

Let x represent the depth of the bay. The coordinate –x would represent the depth. The location of the device would be represented by $-\frac{1}{2}x$.



The distance from the top of the antenna to the device would be presented by $21.5-\left(-\frac{1}{2}x\right)$.

Half of the depth is $\frac{1}{2}x$ *which must be equal to one-third of the distance from the top of the antenna to the device.*

$$\frac{1}{2}x=\frac{1}{3}(21.5-(-\frac{1}{2}x))$$
$$\frac{1}{2}x=\frac{21.5}{3}+\frac{1}{6}x$$
$$3x=43+x$$
$$2x=43$$
$$x=21.5$$

The depth of the bay would have to be 21.5 feet.