Chapter 2:
Patterns, Conjecture, and Proof

Introduction

The assessments in Chapter 2 emphasize geometric thinking and spatial reasoning. "Students use geometric thinking to understand mathematical concepts and the relationships among them."* High School Geometry affords one of the first opportunities for students to explore the structure of mathematical systems and to understand what it means to prove mathematically that a conjecture is true. The emphasis of the problems in this chapter is on discovering this structure through investigations, observing patterns and formulating conjectures based on these experiences. Once the conjecture is formed, the students are asked to justify why it is or is not true.

Mad as a Hatter or Hat as a Madder assesses the students' understanding of logical reasoning. The remainder of the problems are grouped together to emphasize the process of having students investigate, make conjectures, and then justify their conjectures. For example Pizza Delivery Service Regions provides the opportunity for investigation and conjecture while More Pizza Delivery requires the verification of students' conjectures.

> Students use a variety of representations (concrete, pictorial, algebraic, and coordinate), tools, and technology, including, but not limited to, powerful and accessible hand-held calculators and computers with graphing capabilities to solve meaningful problems representing figures, transforming figures, analyzing relationships, and proving things about them. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999.)

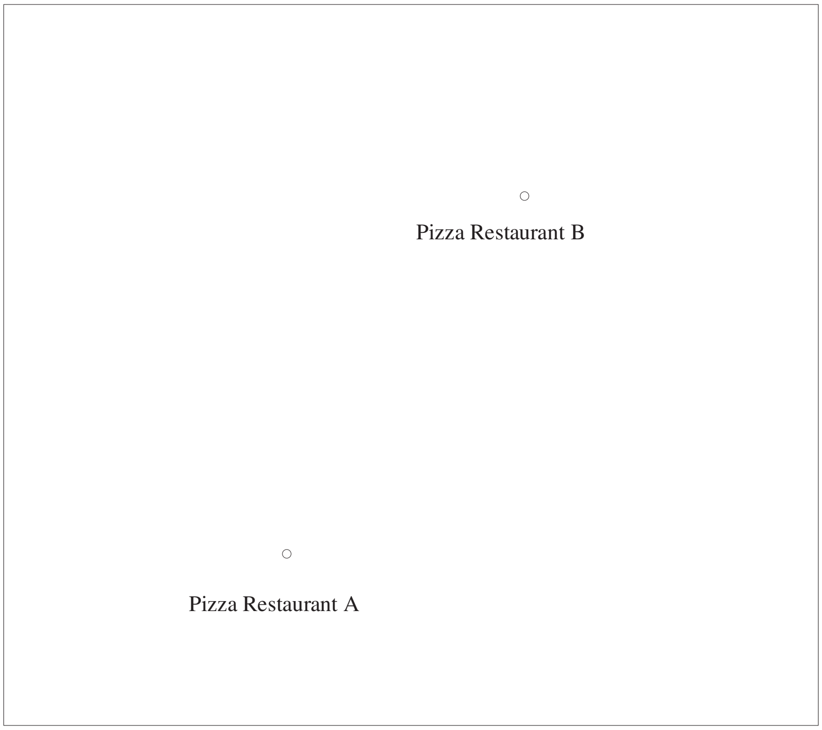
Pizza Delivery Service Regions

Problem 1

Two Restaurants

The rectangle below represents a map of a city, and the two points represent pizza restaurants. Your task is to accurately determine the delivery service region for each of the restaurants. A household is in a restaurant's delivery service region if it is located closer to that restaurant than to the other restaurant.

You will define each restaurant's delivery service region by using compass and straightedge to accurately construct the boundary between the two regions.



a) Write a few sentences explaining how you determined where to locate the boundary.

b) Based on your observations and construction, what do you think must be true about a household located on the boundary between service regions?

Complete the following conjectures: If a household is on the boundary between two service regions, then the household ________________________

____.

If a point is on a line segment's perpendicular bisector, then the point

____

____.

Problem 2

Three Restaurants

Determine the delivery service regions for pizza restaurants A, B, and C.

As in the previous task, a household is located in a delivery service region if it is closer to that restaurant than to either of the other two restaurants.

As before, you can define each restaurant's service region by using compass and straightedge to accurately construct the boundaries between each of the three regions.

a) Based on your observations and construction, what do you think must be true about a household located on all the boundaries between three service regions?

   Complete the following conjectures: If a household is on all the boundaries between three service regions, then the household

   ____

   ____.

The perpendicular bisectors of the three segments that form a triangle intersect at a point which is ____ ____.

Problem 3

Your Conjecture

Using your conjecture from Problem 1, write a few sentences explaining why your conjecture from Problem 2 must be true. In other words, if your conjecture about households on the boundary between two service regions is true, then use this fact to explain why your conjecture about households on the boundary between three service regions must, therefore, be true.

Teacher Notes

Materials:
One compass and straightedge per student.

Connections to Geometry TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(D) uses inductive reasoning to formulate a conjecture.

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

This set of problems is intended to determine if a student can make conjectures based on what they observe (use inductive reasoning). The next problem, More Pizza Delivery, will determine if the students can use deductive reasoning to prove the statement.

Student Response to Problem 1, Question 2:

"It means you can order service from either one, because the two are equal distance from you. I myself would buy pizza from the one that delivers faster and has better pizza."

Scaffolding Questions:

Problem 1

Two Restaurants

- If you mark a point at random on the map to represent a household, how can you use your compass to determine to which restaurant the household is closer?
- If you mark a point at random on the map to represent a household, how can you use your compass to determine the locations of other households at that same distance from either of the two restaurants?
- Use your compass to locate at least three households that are the same distance from both restaurants. Is it possible to find all the households that are the same distance from both restaurants?
- If you have not already done so, construct the perpendicular bisector of the line segment joining restaurant A and restaurant B. What appears to be true about all the points on the perpendicular bisector of the line segment joining restaurant A and restaurant B?

Problem 2

Three Restaurants

- Which construction could you use to draw the boundary between the service regions of restaurant A and restaurant C?
- Which construction could you use to draw the boundary between the service regions of restaurant C and restaurant B?
- What appears to be true about the three boundary lines you have constructed?

Problem 3

Your Conjecture

- What do you think is true about a household on the boundary line between restaurants A and B?
- What appears to be true about a point located on the perpendicular bisector of the segment joining restaurants A and B?
- What do you think is true about a household on the boundary line between restaurants A and C?
- What seems to be true about a point located onthe perpendicular bisector of the segment joining restaurants A and C?
- If your conjecture is proven to be true, what must be true about a household located on both boundary lines?
- If your conjecture is proven to be true, what must be true about a household located on both perpendicular bisectors?

Sample Solutions:

Problem 1

a) Construct the perpendicular bisector of the segment joining restaurant A and restaurant B. The compass setting represents the distance between a household and a restaurant. A circle with its center at one of

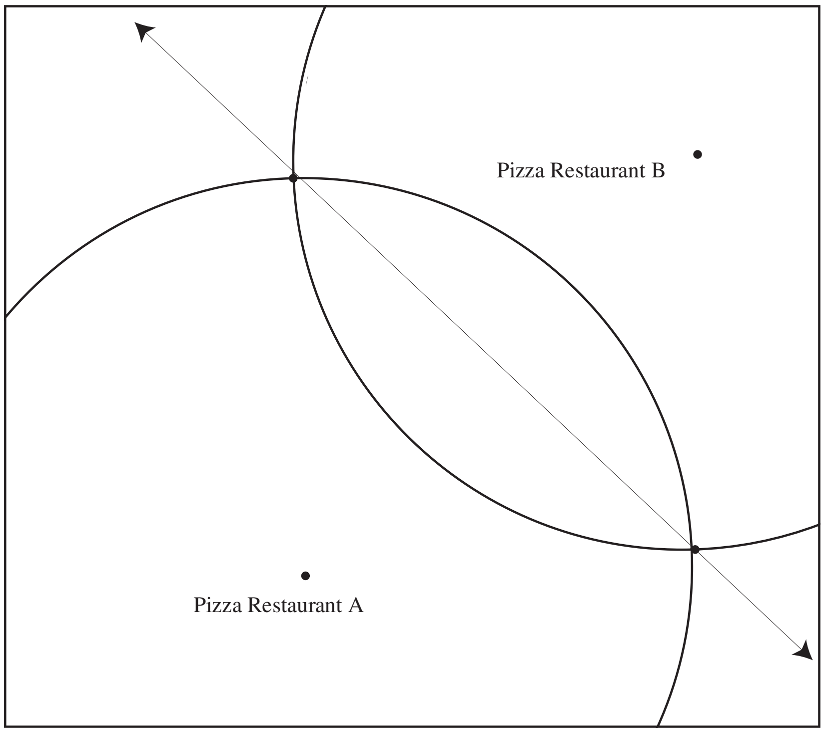
Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Bayou City Geometry

Teacher's Comment:

"To introduce this problem I explained they had all just been hired to work for the pizza company to decide on delivery areas and to be cost effective."

the restaurants, and a fixed compass setting as a radius, represents all households at that fixed distance from the restaurant. A household is the same distance from both restaurants only if its location is at the intersection of two circles with the same radii and centers, respectively, at each of the restaurants.



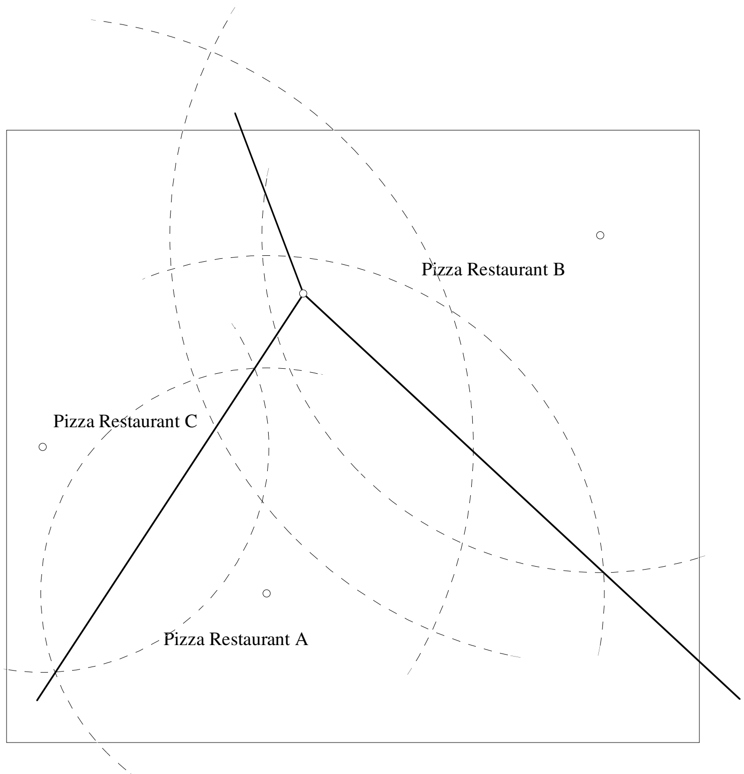
b) Complete the following conjectures: If a household is on the boundary between two service regions, then the household <u>is the same distance from restaurant A as it is from restaurant B</u>.

If a point is on a line segment's perpendicular bisector, then the point <u>is the same distance from each of the line segment's endpoints</u>.

Problem 2

Construct the three perpendicular bisectors of the line segments joining restaurants A, B, and C. Repeat the construction from problem 1 to find the service region boundaries between restaurants A and C and between restaurants B and C.

These boundary lines intersect at a single point. This point is the circumcenter of the triangle formed by joining the points representing the three restaurants.



a) Complete the following conjectures: If a household is on the intersection point that is the boundary between three service regions, then the household is the same distance from all three restaurants.

The perpendicular bisectors of the three segments that form a triangle intersect at a point which is the same distance from all three of the triangles' vertices.

Problem 3

Note: Students can be required to answer this question using both the terms specific to the problem's context (households, service regions, boundaries) as well as the more formal language of geometric abstraction (points, line segments, perpendicular bisectors).

A household located at the intersection of the three service region boundaries must be the same distance from all three restaurants because of the following:

- This household is on the boundary between restaurant A's and restaurant B's service regions, and because of the conjecture from Problem 1, it is, therefore, the same distance from restaurant A as it is from restaurant B.
- This household is also on the boundary between restaurant A's and restaurant C's service regions (or alternately, restaurant B's and restaurant C's service regions), and because of the conjecture from Problem 1, it is, therefore, the same distance from restaurant A as it is from restaurant C (or alternately, restaurant B from restaurant C).

This makes the household the same distance from all three restaurants.

A point located at the intersection of the perpendicular bisectors of a triangle must be the same distance from all three of the triangle's vertices because of the following:

This point is on the perpendicular bisector of the line segment joining vertices A and B, and because of the conjecture from Problem 1, it is, therefore, the same distance from vertex A as it is from vertex B.

This point is also on the perpendicular bisector of the line segment joining vertices A and C (or alternately, vertices B and C), and because of the conjecture from Problem 1, it is, therefore, the same distance from vertex A as it is from vertex C (or alternately, vertex B and vertex C).

Extension Questions:

- Suppose that the three restaurants are located on a coordinate grid at the points A (13,17), B (4,11), and C (3,21). Determine the equations of the boundary lines for the service regions and the circumcenter of the triangle ABC.

To determine the equation of the perpendicular bisector of $\overline{AB}$, find the midpoint of the segment, and determine the opposite reciprocal of the slope of $\overleftrightarrow{AB}$.

The midpoint of $\overline{AB}$ *is* $\left(\frac{13+4}{2},\frac{17+11}{2}\right)$ *or* $(8.5,14)$.

The slope of $\overleftrightarrow{AB}$ *is* $\frac{17-11}{13-4}=\frac{6}{9}=\frac{2}{3}$.

The slope of a perpendicular line is the opposite reciprocal, $-\frac{3}{2}$.

The equation of the perpendicular bisector is

$$y-14=-\frac{3}{2}(x-8.5)$$

$$y=-\frac{3}{2}x+26.75$$

To determine the equation of the perpendicular bisector of $\overline{AC}$*, find the midpoint of the segment, and determine the opposite reciprocal of the slope of* $\overleftrightarrow{AC}$*.*

The midpoint of $\overline{AC}$ *is* $\left(\frac{13+3}{2},\frac{17+21}{2}\right)$ *or* $(8,19)$.

The slope of $\overleftrightarrow{AC}$ *is* $\frac{17-21}{13-3}=\frac{-4}{10}=-\frac{2}{5}$.

The slope of a perpendicular line is the opposite reciprocal, $\frac{5}{2}$.

The equation of the perpendicular bisector is

$$y - 19 = \frac{5}{2}(x - 8)$$

$$y = \frac{5}{2}x - 1$$

To determine the equation of the perpendicular bisector of $\overline{BC}$, *find the midpoint of the segment and determine the opposite reciprocal of the slope of* $\overleftrightarrow{BC}$.

The midpoint of $\overline{BC}$ *is* $\left(\frac{3+4}{2}, \frac{21+11}{2}\right)$ *or* $(3.5, 16)$.

The slope of $\overleftrightarrow{BC}$ *is* $\frac{21-11}{3-4} = \frac{10}{-1} = -10$.

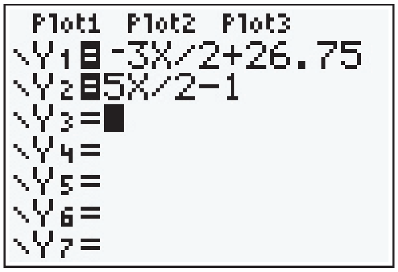
The slope of a perpendicular line is the opposite reciprocal, $\frac{1}{10}$.

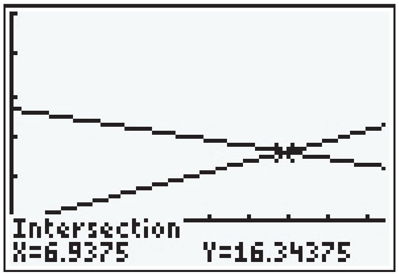
The equation of the perpendicular bisector is

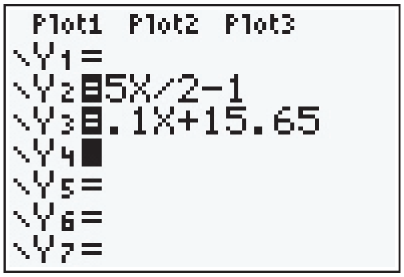
$$y - 16 = \frac{1}{10}(x - 3.5)$$

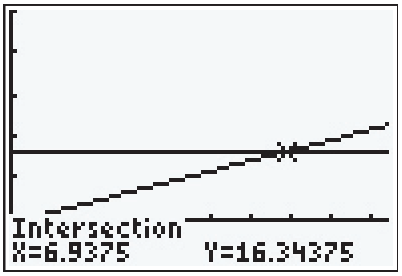
$$y = \frac{1}{10}x + 15.65$$

The intersection point of the three lines may be found by using a graphing calculator.



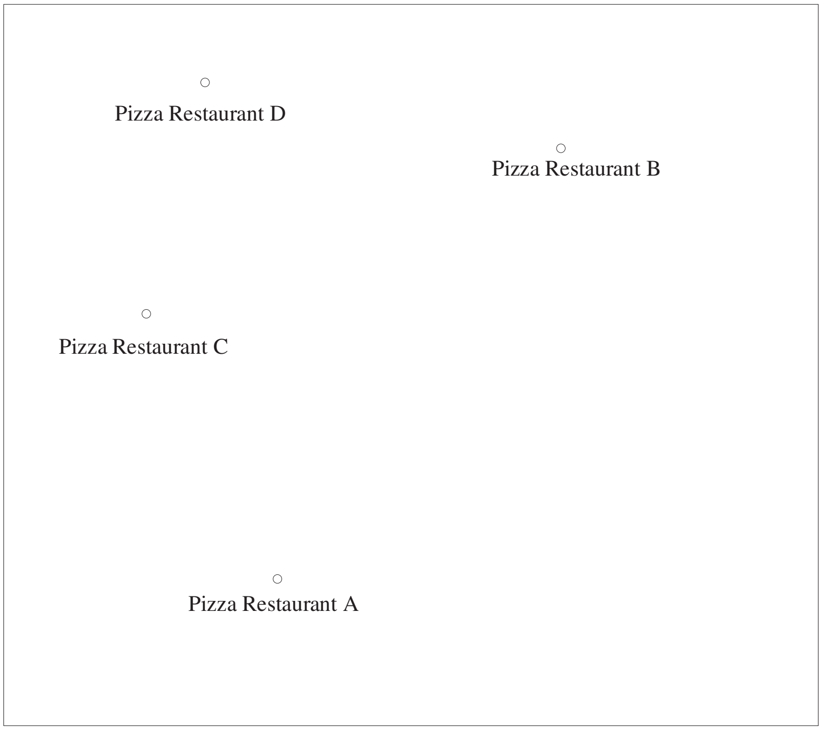


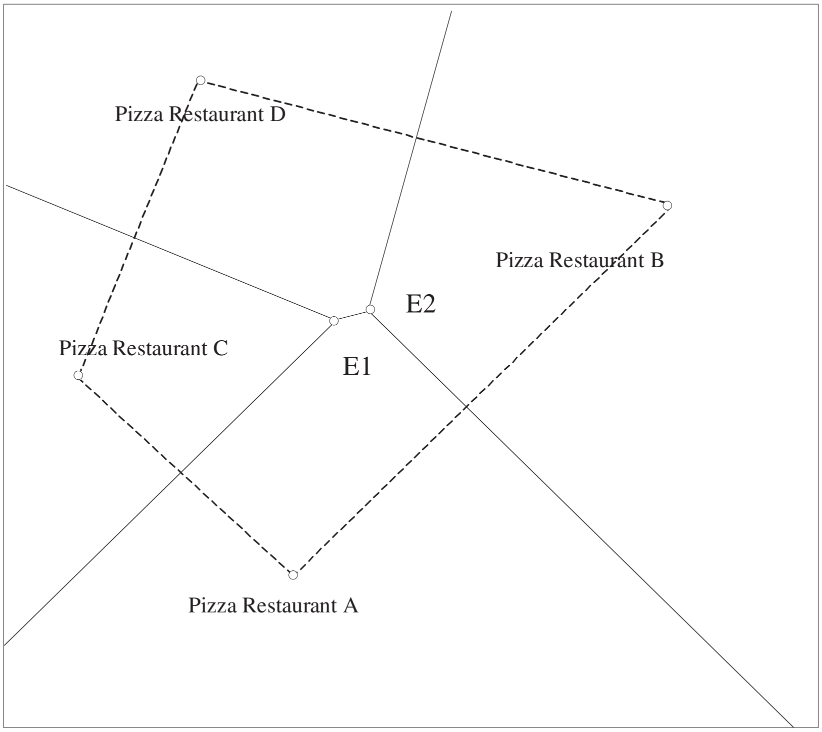




The intersection point is (6.9375, 16.34375).

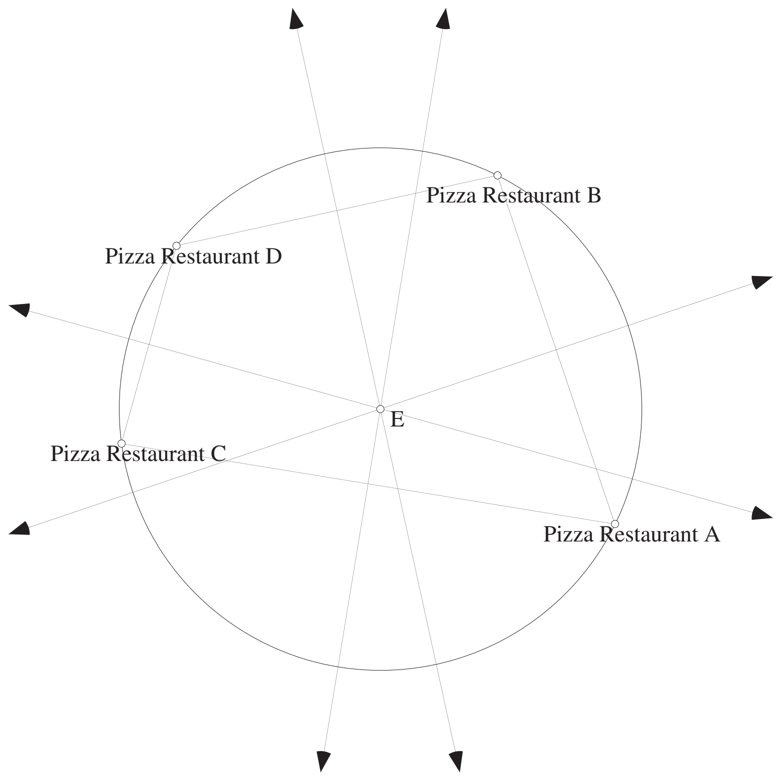
- Determine the pizza service boundary regions for restaurants A, B, C, D.





Point E1 is the circumcenter of the triangle formed by restaurants A, C, and D.
Point E2 is the circumcenter of the triangle formed by restaurants A, B, and D.

- Position the four restaurants so that there is a single household that is the same distance from all four restaurants. (Challenge: you must accomplish this task without positioning the restaurants so that they form the vertices of a square or a rectangle.)



The four restaurants must be located on a circle. The household represented by point E is equidistant from all four restaurants since it is the center of the circle. The four restaurants must be the vertices of a cyclic quadrilateral.

More Pizza Delivery

In the assessment, Pizza Delivery Service Regions, you used constructions to define the boundary between the delivery regions for two pizza restaurants.

You then used your construction to make the following conjecture:

> If a point is on a line segment's perpendicular bisector, then the point is the same distance from each of the line segment's endpoints.

Problem 1

Choose one of the following methods to verify the conjecture:

Axiomatic Approach

> Using the definitions, postulates, and theorems of your geometry course, write a deductive proof of the conjecture.

Coordinate Geometry Approach

> Given pizza restaurant A, with coordinates A(2,-7), and pizza restaurant B, with coordinates B(8,11):

a) Find the equation of the perpendicular bisector of $\overline{AB}$.

b) Use the equation you found to find the coordinates of a point on the perpendicular bisector of $\overline{AB}$.

c) Verify that the point you found satisfies the conditions of the conjecture.

Problem 2

Transformational Approach

Use transformational geometry to verify the converse of the conjecture:

> If a point is equidistant from the two endpoints of a line segment, then the point is on the segment's perpendicular bisector.

Given pizza restaurant A, with coordinates A(2,-7), and pizza restaurant B, with coordinates B(8,11). Also, given point D, with coordinates D(8,1), and point E, with coordinates E(2,3):

a) Use coordinate geometry formulas to verify: AD = BD and AE = BE.

b) Find the equation of the line containing points D and E.

c) Use transformations to show that this line is the perpendicular bisector of $\overline{AB}$.

Recall that a reflection takes a pre-image point and moves it across a *mirror line,* so that the *mirror line* is the perpendicular bisector of the segment connecting the point and its image.

Verify that point B is the reflection image of point A across line $\overleftrightarrow{DE}$. Thus, $\overleftrightarrow{DE}$ is the perpendicular bisector of $\overline{AB}$.

Teacher Notes

Materials:
Graph paper

Connections to Geometry TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

Scaffolding Questions:

Problem 1

Axiomatic Approach

- How would you set up a “given,” a “prove,” and a diagram that represents the conjecture?
- Which proof style would you like to use to write the proof: 2 column, flow chart, or paragraph?
- Depending on proficiency level of students, provide appropriate amounts of set up and/or proof. (See sample solution.)

Coordinate Geometry Approach

- What is the slope of $\overline{AB}$?
- What must be true about the slope of the perpendicular bisector of $\overline{AB}$?
- What are the coordinates of the midpoint of $\overline{AB}$?
- Use the slope and midpoint information to write the equation of the perpendicular bisector of $\overline{AB}$.
- What must be true about the coordinates of a point if it lies on this perpendicular bisector?
- What coordinate geometry formula could you use to show that the point you found is the same distance from A as from B?

Problem 2

Transformational Approach

- What coordinate geometry formula could you use to verify that these distances are the same?
- Find the slope of $\overleftrightarrow{DE}$.
- Use this information to write the equation of $\overleftrightarrow{DE}$ in slope-intercept form.

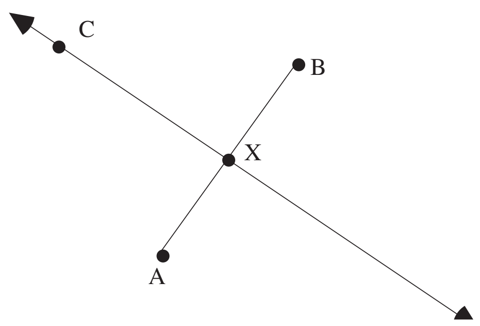
- What is the equation of the line that is perpendicular to $\overleftrightarrow{DE}$ and passes through point A?
- What are the coordinates of point F, the intersection point of the two perpendicular lines?
- What is the distance from point A to this intersection point?
- What are the coordinates of a point at distance AF, from point F, along $\overleftrightarrow{AF}$?

Sample Solution:

Problem 1

Axiomatic Approach

Given: Point C on $\overline{CX}$, the perpendicular bisector of $\overline{AB}$.
Prove: CA = CB



Proof:

AX = XB	(Definition of perpendicular bisector)
m∠AXC = m∠BXC = 90°	(Definition of perpendicular bisector)
CX = CX	(Reflexive property)
ΔAXC ≅ ΔBXC	(SAS)
Therefore CA = CB	(Corresponding parts of congruent triangles are congruent)

Coordinate Geometry Approach

a) The midpoint of $\overline{AB}$ is (5, 2).

The line passing through the points A(2,-7) and B(8,11) will have slope $\frac{11-(-7)}{8-2} = \frac{18}{6} = 3$.

(C) develops and uses formulas including distance and midpoint.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines.

Texas Assessment of Knowledge and Skills:

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Additional objectives are addressed by the Extension Questions that follow the sample solution:

Objective 4: The student will formulate and use linear equations and inequalities.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Geometric Structure

II. Transformations: Reflections

III. Triangles: Pythagorean Theorem

The slope of a perpendicular line is the opposite reciprocal, or $-\frac{1}{3}$.

The perpendicular bisector of line $\overleftrightarrow{AB}$ must have slope $-\frac{1}{3}$ and pass through (5, 2). Using point slope form, the equation of the perpendicular bisector is:

$$y-2=-\frac{1}{3}(x-5)$$

or

$$y=-\frac{1}{3}x+\frac{11}{3}$$

b) Choose a value for *x,* and solve for *y*.

$$x=2,\ y=-\frac{1}{3}(2)+\frac{11}{3}=3$$

Point D(2, 3) is a point on the perpendicular bisector of $\overline{AB}$.

c) Using the distance formula:

$$DA = \sqrt{(2-2)^2+(3-(-7))^2}=10$$

$$DB = \sqrt{(2-8)^2+(3-11)^2} = 10$$

Therefore DA = DB, and point D satisfies the conditions of the conjecture.

Note: Showing that one point works does not prove the statement in general. To prove in general select any point E(*x,y)* on the line.

$$y=-\frac{1}{3}x+\frac{11}{3}$$

$$EA=\sqrt{(2-x)^2+(-7-\left(-\frac{1}{3}x+\frac{11}{3}\right))^2}=$$

$$\sqrt{(2-x)^2+(-7+\frac{1}{3}x-\frac{11}{3})^2}=$$

$$\sqrt{(2-x)^2+\left(\frac{1}{3}x-\frac{32}{3}\right)^2}=$$

$$\sqrt{4-4x+x^2+\frac{1}{9}x^2-\frac{64}{9}x+\frac{1024}{9}}=$$

$$\sqrt{\frac{10}{9}x^2-\frac{100}{9}x+\frac{1060}{9}}$$

$$EB = \sqrt{(8-x)^2 + (11-\left(-\frac{1}{3}x + \frac{11}{3}\right))^2} =$$

$$\sqrt{(8-x)^2 + (11+\frac{1}{3}x - \frac{11}{3})^2} =$$

$$\sqrt{(8-x)^2 + \left(\frac{1}{3}x + \frac{22}{3}\right)^2} =$$

$$\sqrt{64 - 16x + x^2 + \frac{1}{9}x^2 + \frac{44}{9}x + \frac{484}{9}} =$$

$$\sqrt{\frac{10}{9}x^2 - \frac{100}{9}x + \frac{1060}{9}}$$

EA = EB for any point E on $\overleftrightarrow{CX}$.

Problem 2

Transformational Approach

a) Using the distance formula:

$$AD = \sqrt{(2-8)^2+(-7-1)^2} = 10$$

$$BD = \sqrt{(8-8)^2+(11-1)^2} = 10$$

$$AE = \sqrt{(2-2)^2+\left(3-(-7)\right)^2} = 10$$

$$BE = \sqrt{(2-8)^2+(3-11)^2} = 10$$

Therefore AD = BD, and AE = BE. This makes D and E both equidistant from A and B.

b) The slope of line $\overleftrightarrow{DE}$ is $\frac{3-1}{2-8} = \frac{2}{-6} = -\frac{1}{3}$.

Using point slope form, the equation of line $\overleftrightarrow{DE}$ is:

$$y-1=-\frac{1}{3}(x-8)$$

or

$$y=-\frac{1}{3}x+\frac{11}{3}$$

c) The line that is perpendicular to line $\overleftrightarrow{DE}$ and passes through A(2,-7) must have a slope that is the opposite reciprocal of $-\frac{1}{3}$. Using point slope form, the equation of that line is:

$y - (-7) = 3(x - 2)$. In slope intercept form: $y = 3x - 13$.

The point of intersection of the two lines can be found by algebraically solving both equations for both variables:

$$-\frac{1}{3}x+\frac{11}{3}=3x-13$$

$x = 5$ and $y = 2$

The intersection point has coordinates F(5, 2).

The distance AF $= \sqrt{(5-2)^2+\left(2-(-7)\right)^2} = \sqrt{90} = 3\sqrt{10}$.

We want the coordinates of the reflection image of point A. This point is $3\sqrt{10}$ units from point F along $\overleftrightarrow{AF}$. To find the coordinates of the point A(x,y), use the distance formula and the equation of $\overleftrightarrow{AF}$.

$$3\sqrt{10} = \sqrt{(x-5)^2 + (y-2)^2}$$

Since $y = 3x - 13$, $3\sqrt{10} = \sqrt{(x-5)^2 + (3x-13-2)^2}$

Solve this equation for x: $3\sqrt{10} = \sqrt{x^2 - 10x + 25 + 9x^2 - 90x + 225}$

$90 = 10x^2 - 100x + 250$

$0 = 10x^2 - 100x + 160$

$0 = 10(x^2 - 10x + 16)$

$0 = 10(x - 8)(x - 2)$

$x = 8$ or $x = 2$

When $x = 2$, $y = -7$. These are the coordinates of point A.

When $x = 8$, $y = 11$. These are the coordinates of point A', the reflection of point A across $\overleftrightarrow{DE}$.

But these are precisely the coordinates of point B. Point B is, therefore, the reflection image of point A across $\overleftrightarrow{DE}$. This makes $\overleftrightarrow{DE}$ the perpendicular bisector of $\overleftrightarrow{AB}$.

Extension Questions:

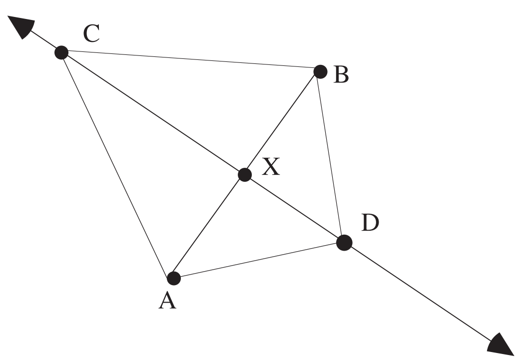
- Verify the converse of the conjecture using an Axiomatic Approach.

Converse of Conjecture: If a point is equidistant from the two endpoints of a line segment, then the point is on the segment's perpendicular bisector.

Given: $CB = CA$

$DB = DA$

Prove: Line $\overleftrightarrow{CD}$ *is the perpendicular bisector of segment* $\overline{AB}$.



Proof:

CA = CB and DA = DB	*(Given)*
This makes quadrilateral ACBD a kite	*(Definition of a kite)*
Then, $\overleftrightarrow{CX} \perp \overline{AB}$	*(Diagonals of a kite are perpendicular)*
This makes $right\ \Delta ACX \cong right\ \Delta BCX$	*(HL)*
and AX = XB	*(CPCTC)*
Therefore, $\overleftrightarrow{CX}$ *is the perpendicular bisector of segment* $\overline{AB}$.	*(Definition of perpendicular bisector)*

Another approach that does not use the properties of kites, involves proving that

$\Delta CAD \cong \Delta CBD$ *(SSS).*

Then $\angle ACX \cong \angle BCX$ *(CPCTC).*

Now, $\Delta ACX \cong \Delta BCX$ *(HA or SAS).*

Then establish AX = XB and $m\angle AXC = m\angle BXC = 90°$.

Line $\overleftrightarrow{CX}$ *is the perpendicular bisector of segment* $\overline{AB}$. *(Definition of perpendicular bisector)*

Conjecture as Discovery and Proof as Explanation

Problem 1

Triangle Midsegment Conjecture

Use paper, pencil, construction and measuring tools
or
appropriate geometry technology to complete this problem.

1. Sketch and label a $\triangle ABC$.
2. Find and label point D (the midpoint of side $\overline{AB}$) and point E (the midpoint of side $\overline{AC}$).
3. Draw midsegment $\overline{DE}$.
4. Take and record the following measurements in centimeters and degrees.

   DE = __________ BC = _____________

   $\angle$ ADE = ________ $\angle$ AED = __________

   $\angle$DBC = ________ $\angle$ECB = ___________

5. Repeat steps 1 – 4 to complete the sketch, and take measurements on at least two more triangles that are different from your original triangle.

   If you are working with a group, you may compare your triangle measurements with the other group members.

   If you are using geometry technology, you may drag the vertices of the original triangle to generate new triangles and sets of measurements.

6. Based on your drawings and observations, complete the following conjecture:

The midsegment of a triangle is ___________________________ to

one side of the triangle, and it measures ______________________________

of that side.

Problem 2

Why is it true?

If you completed Problem 1, you discovered two important characteristics of a triangle's midsegment:

a) The midsegment is parallel to a side of the triangle.

b) The midsegment is $\frac{1}{2}$ the length of the side of the triangle it is parallel to.

You might have already known about these properties from previous lessons, or you might have even guessed what they were without drawing or measuring. But can you explain why they are true?

In order for your explanation to be fully convincing from a mathematical standpoint, it must satisfy three requirements. First, it must be logical. Second, it must consist of facts, definitions, postulates, or theorems that have been previously proven or accepted as true. Third, it must apply to all cases.

Write an explanation of why the first property of triangle midsegments is true. Your explanation must satisfy all of the above requirements. Use your knowledge of postulates and theorems about parallel lines and angle relationships to help with your explanation.

Problem 3

A Different Look

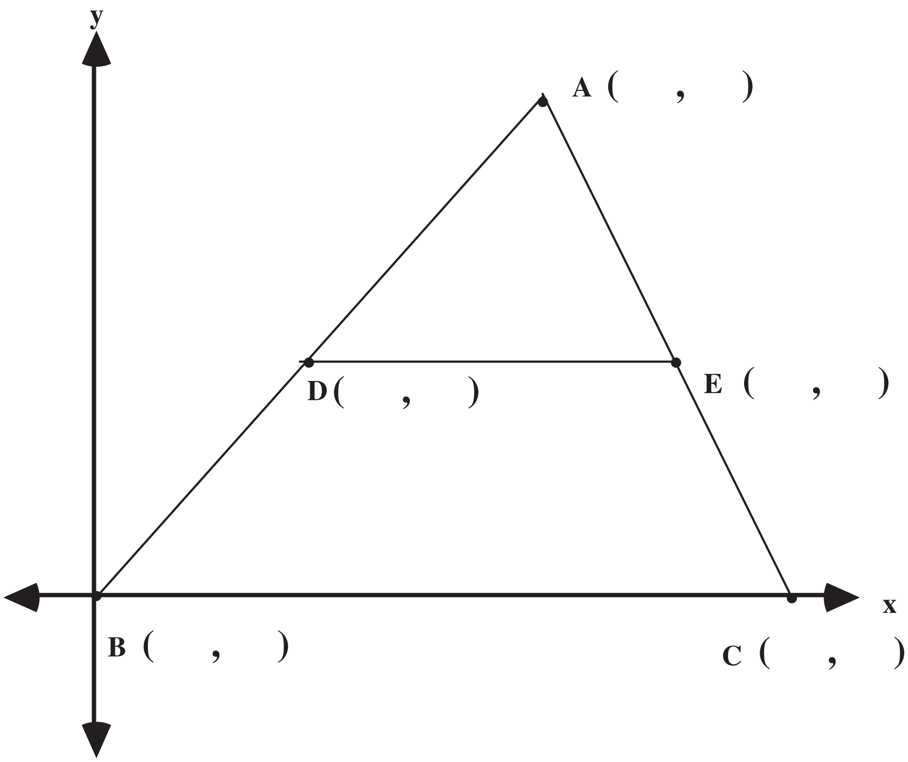
Mathematicians call explanations similar to the one you wrote in the previous problem "proofs." In order to prove the second property about triangle midsegments, it is helpful to represent the situation in a different mathematical context.

1. Draw and label triangle a ΔABC on graph paper. Label and record the numerical coordinates of points A, B, and C.
2. Use the midpoint formula to find and label the following: point D (the midpoint of segment $\overline{AB}$) and point E (the midpoint of segment $\overline{AC}$).
3. Use the distance formula to calculate the length of $\overline{DE}$ and to calculate the length of $\overline{BC}$.
4. What is the relationship between the length of $\overline{DE}$ and the length of $\overline{BC}$?
5. Do the diagram and calculations above constitute a proof of the second property of triangle midsegments? Write a few sentences discussing why or why not.

Problem 4

The General Situation

Any triangle may be rotated and translated so that one vertex is at the origin and another vertex is on the positive *x*-axis.



1. Fill in the coordinates of points A, B, and C in the above diagram using variables to represent any triangle.
2. Use the midpoint formula to calculate and fill in the coordinates of point D (the midpoint of $\overline{AB}$) and point E (the midpoint of $\overline{AC}$).
3. Use the distance formula to calculate the length of $\overline{DE}$ and to calculate the length of $\overline{BC}$. Show all work and calculations.
4. What is the relationship between the length of $\overline{DE}$ and the length of $\overline{BC}$?
5. Do the diagram and calculations above constitute a proof of the second property of triangle midsegments? Write a few sentences discussing why or why not.

Materials:
Graph paper

One pencil, ruler, protractor per student and/or

Appropriate geometry technology

Connections to Geometry TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

Teacher Notes

This problem asks the students to investigate and then prove important characteristics about a triangle's midsegment. The problem, Extending the Triangle Midsegment Conjecture, investigates the midsegment for other polygons.

Scaffolding Questions:

Problem 1

- Step 1: ΔABC can be any type of triangle. If students are working in groups, encourage different group members to draw different types of triangles.
- Step 3: What is the definition of a triangle's midsegment?
- Steps 5 and 6: What types of triangles seem to provide more convincing evidence in support of the conjecture? Why?

Problem 2

- What type of angles are $\angle ADE$ and $\angle DBC$?
- What type of angles are $\angle AED$ and $\angle ECB$?

Problem 3

- Steps 1 – 4: Students will orient their triangles on the graph paper many different ways and will soon find out that different triangle placements result in "easier" or "harder" numbers to work with.
- Is it all right to place your triangle on the graph paper in the most convenient way for the calculations to work out?
- Step 4: Can you write the relationship as a mathematical statement?
- Step 5: It can be helpful to point out that in fact the first two requirements of a mathematical proof have been fulfilled in this case. The formulas used are accepted

as true, and the diagram as well as the sequence of calculations constitute the manipulation of symbols according to logical rules. Many students do not realize this.

Problem 4

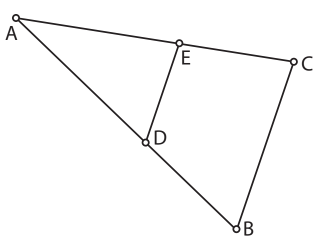
- After problem 3, students will have realized that their numerical calculations do not constitute a proof.
- What can we replace the numerical coordinates with so that the calculations will apply to any triangle?
- Why does this particular placement of the triangle make it easier to fill in the coordinates than other possible placements?
- What are the fewest number of variables that can be used to accurately label all the required coordinates?
- Step 2: What could we do to the coordinates of A, B, and C to make the calculations easier?
- What effect does multiplying a coordinate variable by a constant have on subsequent calculations? Is it mathematically all right to do this?
- Encourage students to simplify the complicated-looking expressions as much as possible. Review of notation and symbol manipulation under the radical symbol may be necessary.

Sample Solution:

Problem 1

1 – 5 This is one possible solution.

Length (Segment AD) = 3.26 cm
Length (Segment DB) = 3.26 cm
Length (Segment AE) = 2.83 cm
Length (Segment EC) = 2.83 cm
Angle (ADE) = 59°
Angle (AED) = 81°
Angle (DBC) = 59°
Angle (ECB) = 81°



6. The midsegment of a triangle is <u>parallel</u> to one side of the triangle and measures <u>one-half the length</u> of that side.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures; and

(C) develops and uses formulas including distance and midpoint.

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(B) justifies and applies triangle congruence relationships.

Texas Assessment of Knowledge and Skills:

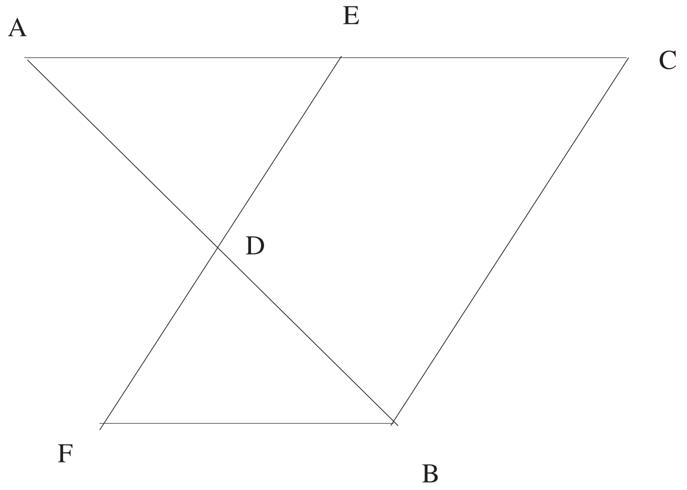
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Midpoint Triangle

Problem 2

A possible axiomatic proof:



Given: D is the midpoint of $\overline{AB}$, E is the midpoint of $\overline{AC}$.

Prove: DE $= \frac{1}{2}$BC and $\overline{DE}$ is parallel to $\overline{BC}$.

Extend $\overline{ED}$ to point F such that $\overline{ED} \cong \overline{DF}$.

$\overline{AE} \cong \overline{EC}$, $\overline{AD} \cong \overline{DB}$ by definition of midpoint

$\angle ADE \cong \angle BDF$ because vertical angles are congruent.

$\Delta ADE \cong \Delta BDF$ because two triangles are congruent if two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle.

$\angle DBF \cong \angle A$ and $\overline{AE} \cong \overline{FB}$ because corresponding parts of congruent triangles are congruent.

$\overline{AC}$ is parallel to $\overline{BF}$ because two lines are parallel if the alternate interior angles are congruent.

$\overline{EC} \cong \overline{FB}$ by the transitive property of congruence.

BCEF is a parallelogram because the two sides of the quadrilateral are parallel and congruent.

EF = BC because the opposite sides of a parallelogram are equal in length.

FD + DE = BC by segment addition

2 DE = BC by substitution

$DE = \frac{1}{2}BC$ by division

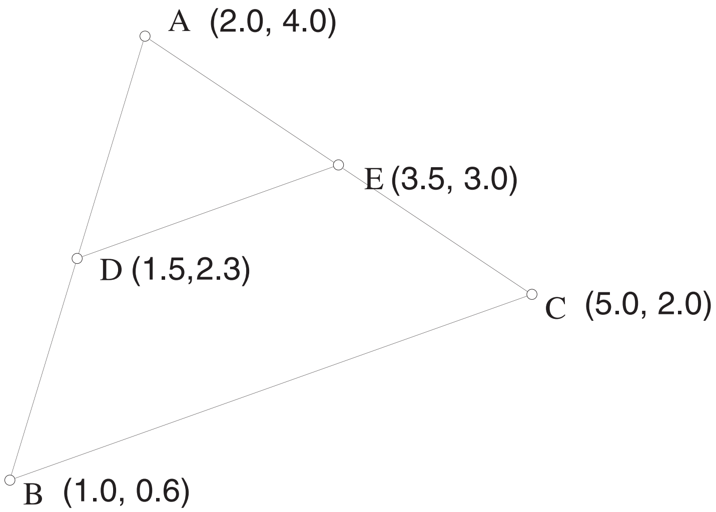
$\overline{DE}$ is parallel to $\overline{BC}$ because opposite sides of a parallelogram are parallel.

This argument applies to the midsegment of any triangle.

Problem 3

One possible coordinate representation:

1 and 2.



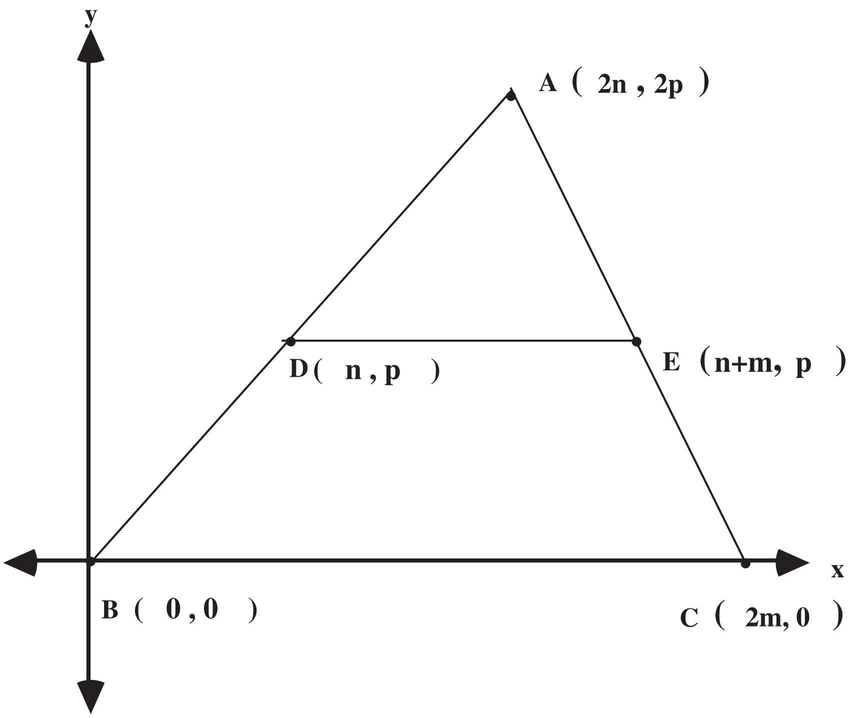
3. $DE = \sqrt{(3.5-1.5)^2+(3.0-2.3)^2} = \sqrt{4+0.49} \approx 2.12$

   $BC = \sqrt{(5.0-1.0)^2+(2.0-0.6)^2} = \sqrt{16+3.7636} \approx 4.44$
4. $DE = \frac{1}{2}BC$
5. The diagram and calculations establish that the midsegment is one-half the length of the side it is parallel to only for a particular triangle with specific coordinates. They do not establish that this property is true for the midsegments of all triangles.

Problem 4

For example:

1 and 2.



3. $DE = \sqrt{(n+m-n)^2 + (p-p)^2} = \sqrt{m^2} = |m| = m$ because $m > 0$.

   $BC = \sqrt{(2m-0)^2 + (0-0)^2} = \sqrt{4m^2} = |2m| = 2m$ because $m > 0$.

4. $DE = \frac{1}{2}BC$

5. The diagram and the calculations do constitute a proof of the second property of triangle midsegments. The formulas used are accepted as true, and the diagram, as well as the sequence of calculations, constitute the manipulation of symbols according to logical rules. Finally, using variables instead of specific numbers for coordinates makes the relationship true for all triangle midsegments.

Extension Questions:

- Write a few sentences detailing why the postulates and theorems about parallel lines and angle relationships **cannot** be used to explain why the second property of triangle midsegments described in Problem 2 (b) is true.

  The postulates and theorems about parallel lines and angle relationships establish that lines are parallel given that certain angle relationships are true, or that certain angle relationships must be true given that parallel lines are cut by a transversal. The second property of triangle midsegments deals with the relationship between the lengths of the parallel segments. These postulates and theorems don't provide any information about the lengths of parallel lines.

- Using the diagram and the coordinates from problem 4, prove the first property of the triangle midsegments: the midsegment of a triangle is parallel to a side of the triangle.

  The lines are parallel since it can be shown that they have the same slope.

  Slope of DE = $\frac{p-p}{n+m-n} = 0.$

  Slope of AB = $\frac{0-0}{2m-0} = 0.$

Student Work Sample

The work on the next two pages shows a student's approach to problems 1, 2 and 3.

This work is a good example of the criteria

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

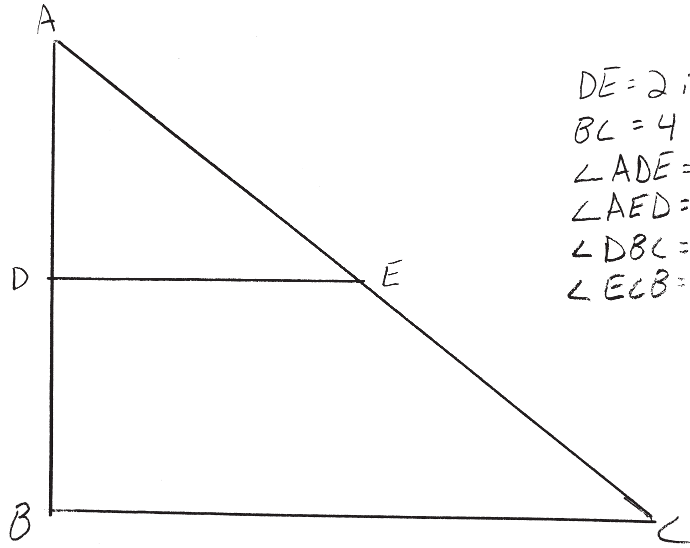
  In problems 2 and 3, this student explains his reasons for the statements he makes. (Since line a is the midsegment of triangle B, it bisects AB and AC….).

- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

Note that the student does not explain how he got the measurements.

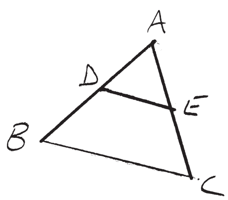
Conjecture as Discovery and Proof as Explanation

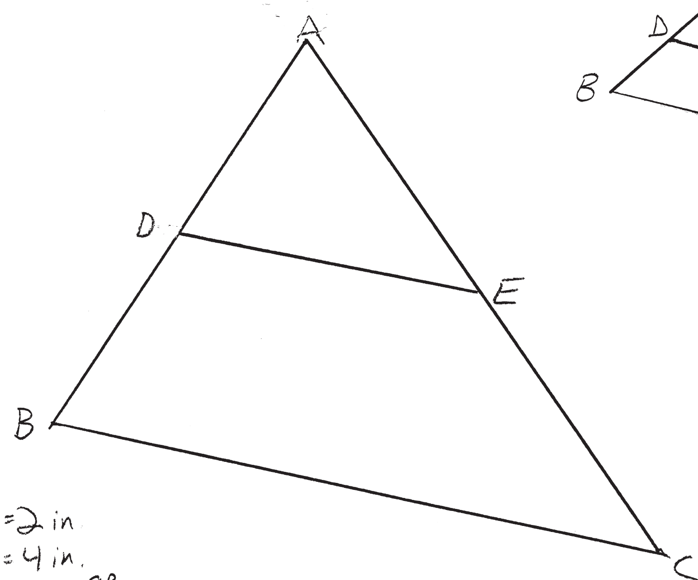
Problem 1.



DE = 2 in.
BC = 4 in.
∠ADE = 90°
∠AED = 32°
∠DBC = 90°
∠ECB = 32°

DE = ½ in.
DC = 1 in.
∠ADE = 50°
∠AED = 58°
∠DBC = 50°
∠ECB = 58°

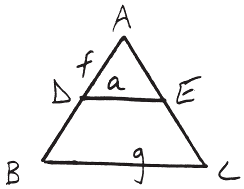




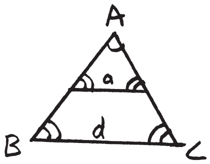
DE = 2 in
BC = 4 in.
∠ADE = 69°
∠AED = 44°
∠DBC = 69°
∠ECB = 44°

The midsegment of a triangle is proportional to one side of the triangle and measures ½ of that side.

Problem 2.

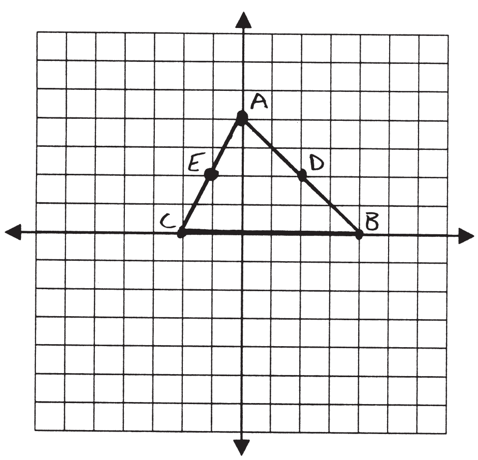


Since line a is the mid-segment of triangle B, it bisects AB and AC. Therefore, $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$.
△ABC and △ADE share ∠A, so by the SAS theorem, △ABC △ADE



Since they are ~, their angles are ≃
If a line intersects 2 other lines and corresponding angles are ≃, then the lines are ||, so line a and line d are ||.

3.



A = (0,4)
B = (4,0)
C = (-2,0)
D = (2,2)
E = (-1,2)

$\overline{DE} = \sqrt{(2 - -1)^2 + (2-2)^2} = \sqrt{9} = 3$

$\overline{CB} = \sqrt{(4 - -2)^2 + (0-0)^2} = \sqrt{36} = 6$

$\overline{DE} = \frac{1}{2} CB$

The diagram does not constitute a proof because it is only one situation. For it to be a theorem, it must work in all situations.

Extending the Triangle Midsegment Conjecture

These problems investigate the relationship between a polygon's diagonals and the polygon formed by connecting the midpoints of its sides. We begin the investigation by considering a quadrilateral.

Problem 1

Use paper, pencil, construction and measuring tools
or
appropriate geometry technology to complete this problem.

1. Sketch and label Quadrilateral ABCD.
2. Draw all possible diagonals for Quadrilateral ABCD. In this case there are 2 diagonals, $\overline{AC}$ and $\overline{BD}$.
3. Create a polygon (in this case a quadrilateral) by connecting the midpoints of the sides of quadrilateral ABCD.
4. Take appropriate measurements, and write a conjecture that relates the perimeter of the midpoint quadrilateral to the diagonals of the original quadrilateral.
5. Repeat steps 1 – 4 to complete the sketch and take measurements on at least two more quadrilaterals that are different from your original quadrilateral.

   If you are working with a group, you may compare your quadrilateral measurements with the other group members.

   If you are using geometry technology, you may drag the vertices of the original quadrilateral to generate new quadrilaterals and sets of measurements.

Problem 2

In the Assessment, Conjecture as Proof and Proof as Explanation, you discovered and proved the Triangle Midsegment Conjecture:

> The midsegment of a triangle is parallel to one side of the triangle and measures half the length of that side.

Since you have proved this conjecture, you can use it to help prove and explain why other conjectures may or may not be true.

Use the Triangle Midsegment Conjecture to write an explanation of why your quadrilateral conjecture is true. Provide a diagram with your explanation.

Problem 3

Do you think your conjecture will be true for other polygons besides quadrilaterals? Why or why not?

Sketch, measure, and investigate the relationship between a polygon's diagonals and the polygon formed by connecting the midpoints of its sides. Start with a pentagon, then a hexagon, etc.

Problem 4

Modify your original conjecture (or write new conjectures) to take into account other polygons besides quadrilaterals.

Teacher Notes

Materials:
Paper and graph paper

One pencil, ruler, and protractor per student, and/or

Appropriate geometry technology

Connections to Geometry TEKS:

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(d.2) **Dimensionality and the geometry of location**. The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(C) develops and uses formulas including distance and midpoint.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The problem, Conjecture as Discovery and Proof as Explanation, asks the students to investigate and then prove important characteristics about a triangle's midsegment. This problem investigates the midsegment for other polygons. The next problem, Why Doesn't My Conjecture Always Work?, asks students to explain the results of their investigations for the different polygons.

Scaffolding Questions:

Problem 1

- If students need additional structure during the investigation, tell them to restrict their measurements to the lengths of segments and to ignore angle measures.

Problem 2

- This explanation does not need to be a formal, 2-column proof. Encourage students to examine a diagram of the triangle midsegment theorem and to see how this diagram applies to their quadrilateral diagram.

Problem 3

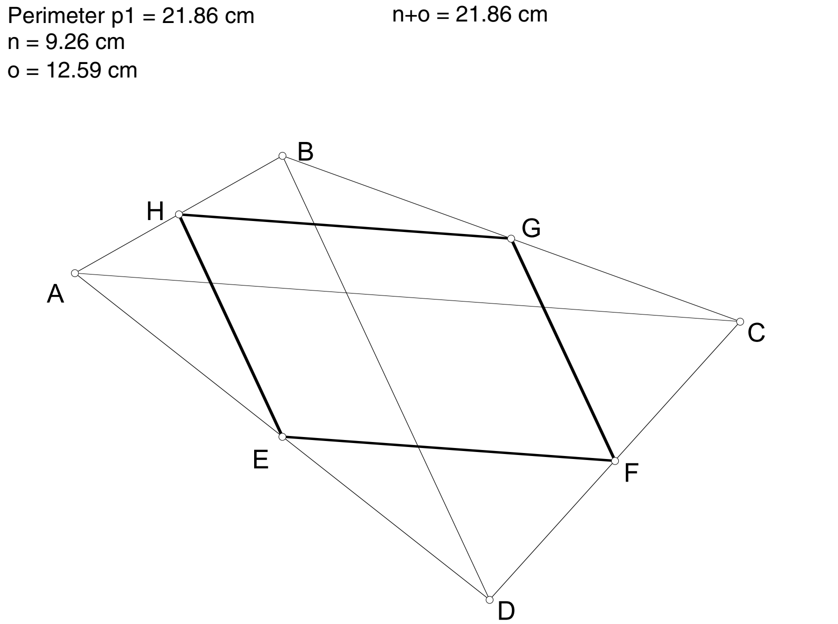
- Many students will assume that since other polygons have diagonals, the conjecture should "work the same way" no matter how many sides the polygon has. These investigations are a good way to get students to "listen to the math," and to see how the geometry changes as the number of sides of the polygon increase.

Problem 4

- Students should see that the initial conjecture doesn't hold for pentagons; moreover, the conjecture for pentagons doesn't hold for hexagons. Encourage students to write separate conjectures for quadrilaterals, pentagons, and polygons with more than 5 sides.

Sample Solution:

Problem 1



Conjecture: The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

Problem 2

By the Triangle Midsegment Conjecture:

$GF = \frac{1}{2} BD \quad HE = \frac{1}{2} BD \quad HG = \frac{1}{2} AC$

$EF = \frac{1}{2} AC$,

so

$GF + HE + HG + EF = \frac{1}{2} BD + \frac{1}{2} BD + \frac{1}{2} AC + \frac{1}{2} AC$

$GF + HE + HG + EF = BD + AC$.

Therefore the perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

(2) uses ratios to solve problems involving similar figures.

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

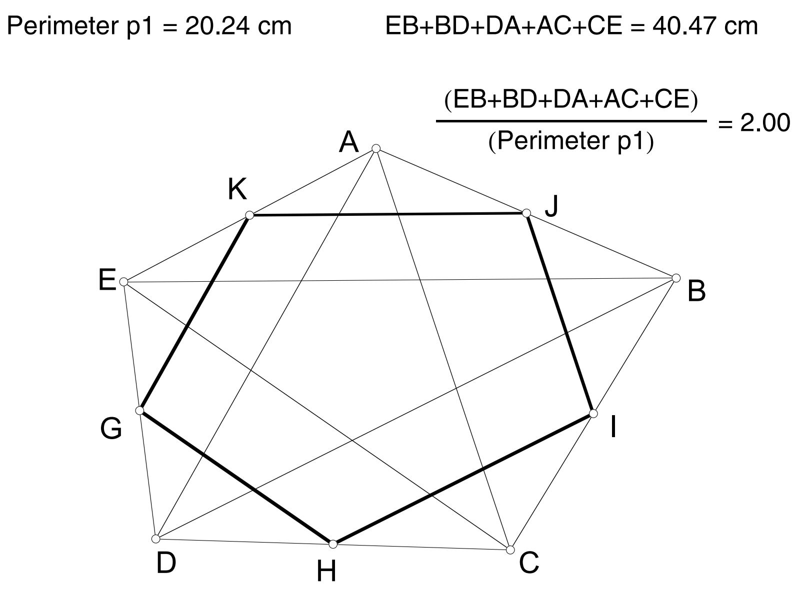
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

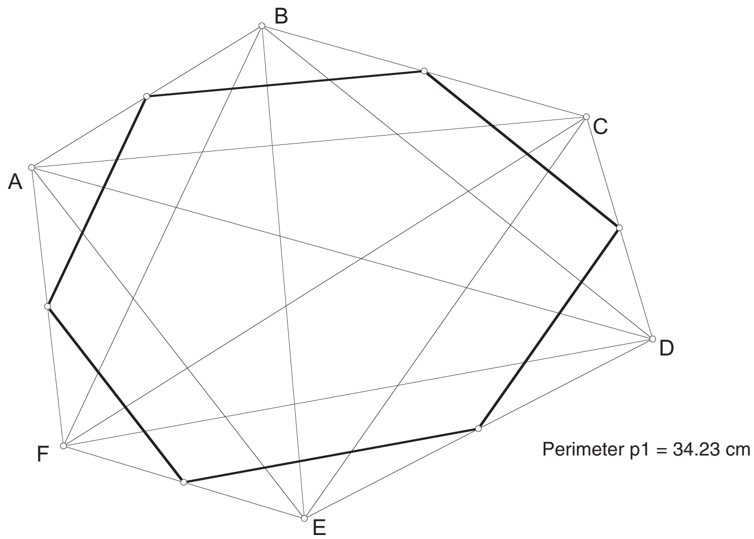
I. Structure: Geometric Structure: Midpoint Quadrilateral

Problem 3

For a pentagon:



For a hexagon, and for polygons with more than six sides:



AC+AD+AE+BD+BE+BF+CF+CE+DF=107.80 cm

There is no relationship; the ratio between perimeter of midpoint polygon and sum of diagonals changes with different hexagons.

Problem 4

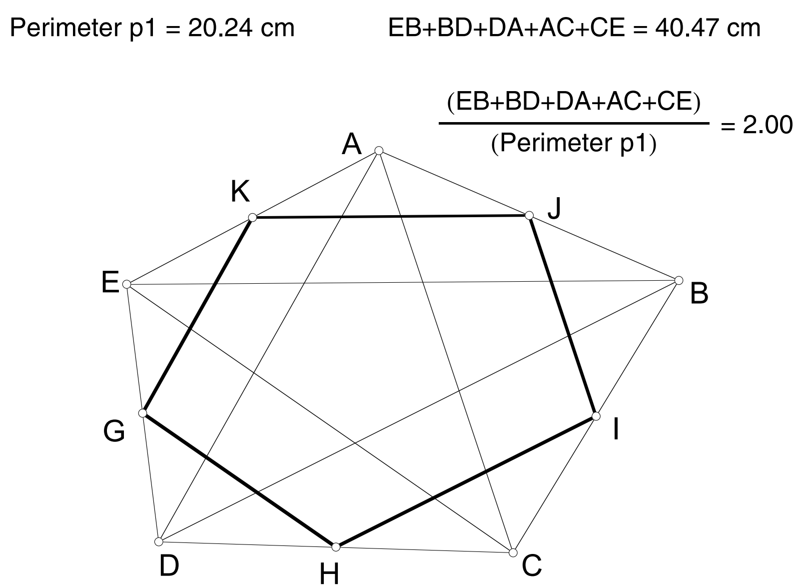
Conjecture: The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

Conjecture: The perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

Conjecture: There does not appear to be any relationship between the perimeter of the midpoint polygon and the sums of the lengths of the diagonals for polygons with more than 5 sides.

Extension Questions:

- Use the Triangle Midsegment Conjecture to explain why your conjecture for pentagons must be true.



By the Triangle Midsegment Conjecture:

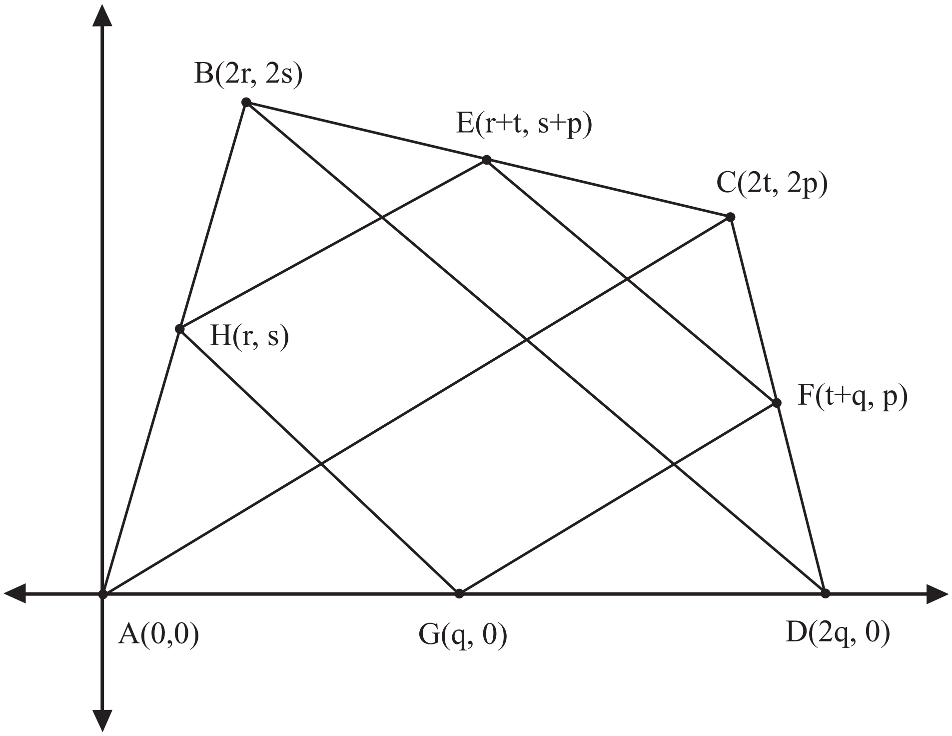
$KJ = \frac{1}{2}\ EB \qquad JI = \frac{1}{2}\ AC \qquad IH = \frac{1}{2}BD \qquad GH = \frac{1}{2}\ EC \qquad GK = \frac{1}{2}\ AD,$

so

$KJ + JI + IH + GH + GK = \frac{1}{2}\ EB + AC + BD + EC + AD.$

Therefore the perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

- Use Coordinate Geometry to prove the quadrilateral conjecture.



$$HE = \sqrt{(r+t-r)^2 + (s+p-s)^2} = \sqrt{t^2 + p^2}$$

$$FG = \sqrt{(t+q-q)^2 + p^2} = \sqrt{t^2 + p^2}$$

$$EF = \sqrt{(r+t-t-q)^2 + (s+p-p)^2} = \sqrt{(r-q)^2 + s^2}$$

$$GH = \sqrt{(r-q)^2 + s^2}$$

$$BD = \sqrt{(2(r-q))^2 + (2s)^2} = \sqrt{4(r-q)^2 + 4s^2} = \sqrt{4((r-q)^2 + s^2)}$$
$$= 2\sqrt{((r-q)^2 + s^2)}$$

$$AC = \sqrt{(2t)^2 + (2p)^2} = \sqrt{4t^2 + 4p^2} = \sqrt{4(t^2 + p^2)} = 2\sqrt{(t^2 + p^2)}$$

$$HE + GF + EF + GH = 2\sqrt{t^2 + p^2} + 2\sqrt{(r-q)^2 + s^2}$$

$$BD + AC = 2\sqrt{t^2 + p^2} + 2\sqrt{(r-q)^2 + s^2}$$

Therefore, the perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

Why Doesn't My Conjecture Always Work?

In the assessment, Extending the Triangle Midsegment Theorem, you investigated the relationship between a polygon's diagonals and the perimeter of the polygon formed by connecting the midpoints of its sides.

You discovered that there is no single mathematical connection between these two things that is true for all polygons; in fact, the relationship changes as the number of the polygon's sides increases.

For quadrilaterals:

> The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

For pentagons:

> The perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

For polygons with more than 5 sides:

> There does not appear to be any relationship between the perimeter of the midpoint polygon and the sums of the lengths of the diagonals for polygons with more than 5 sides.

You also used the previously proven Triangle Midsegment Conjecture to explain why the conjectures for quadrilaterals and for pentagons must be true.

Problem 1

Use what you have learned in previous investigations to explain why the relationship between diagonals and midpoint polygon is different for pentagons than it is for quadrilaterals.

Provide a written explanation with diagrams.

Problem 2

Use what you have learned in previous investigations to explain why there is no relationship between diagonals and midpoint polygons for shapes with more than 5 sides.

Provide a written explanation with a diagram(s).

Teacher Notes

Materials:
One pencil, ruler, and protractor per student and/or appropriate geometry technology

Connections to Geometry TEKS:

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(b.4) **Geometric structure**. The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns**. The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

Scaffolding Questions:

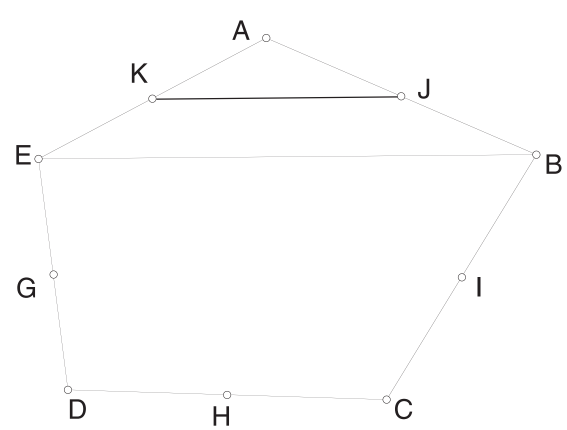
- What determines the number of times you can apply the Triangle Midsegment Conjecture to a shape?
- What is the relationship between the segment drawn between the midpoints of consecutive sides and the diagonal drawn between two endpoints of consecutive sides?
- Complete the chart:

number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4		
5		
6		

Sample Solution:

Key Concepts:

Every time a segment is drawn between the midpoints of consecutive sides of a polygon, it creates the midsegment of a triangle. This triangle's two sides are the consecutive sides of the polygon, and its third side is the diagonal drawn between the endpoints of the two consecutive sides.



For example, segment $\overline{KJ}$ is the midsegment of the triangle formed by consecutive sides $\overline{AE}$ and $\overline{AB}$ and diagonal $\overline{EB}$.

The Triangle Midsegment Conjecture applies to the geometry of this situation, since it states that KJ = $\frac{1}{2}$ EB.

When a midpoint polygon is formed, each of its sides is the midsegment of a triangle, and its length is therefore one-half the length of one of the original polygon's diagonals.

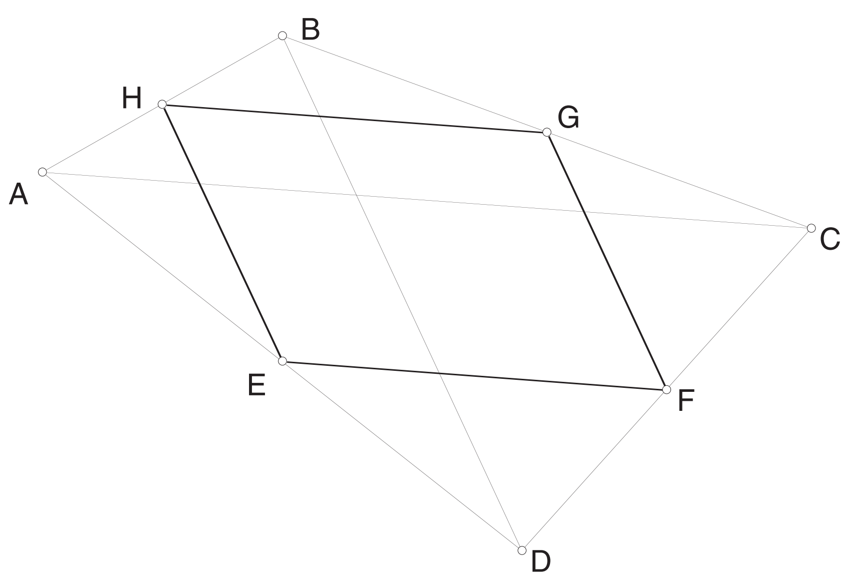
The reason that the relationship between the perimeter of a midpoint polygon and length of diagonals does not stay constant is that the number of times a diagonal is matched to a midsegment does not stay constant as the number of sides of the polygons increases. For some polygons there are more diagonals than sides to the midpoint polygon.

Problem 1

In a quadrilateral:

Perimeter HGFE = 21.86 cm
m $\overline{BD}$ = 9.26 cm
m $\overline{AC}$ = 12.59 cm

(m $\overline{BD}$) + (m $\overline{AC}$) = 21.86 cm



There are two diagonals and four triangle midsegments. So each diagonal gets matched with two different triangle midsegments. For example, diagonal $\overline{AC}$ is matched with midsegment $\overline{HG}$ and midsegment $\overline{EF}$.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines; and

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts.

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Midpoint Quadrilateral

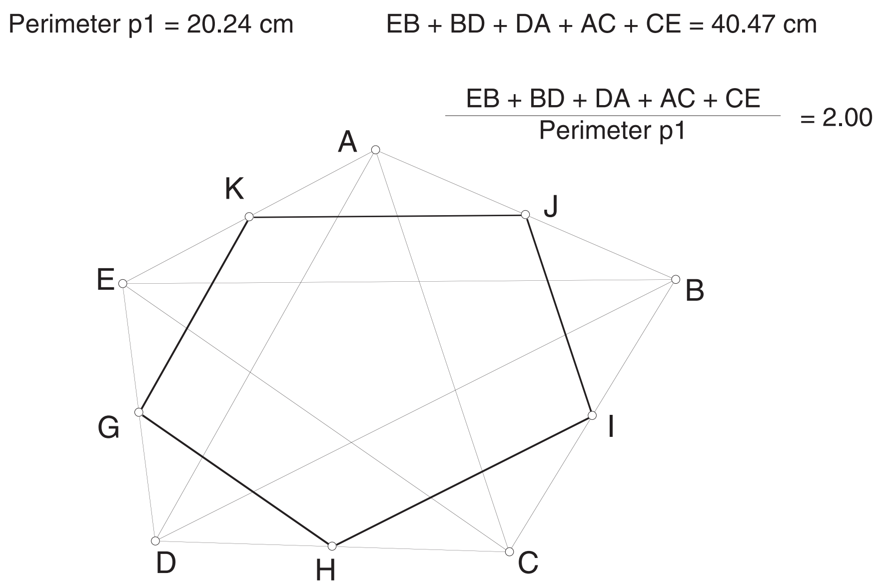
II. Structure: Midpoint Triangle

In a quadrilateral, the Triangle Midsegment Conjecture is applied to 4 different triangles corresponding to the lengths of only 2 different diagonals. Each diagonal, therefore, is counted twice.

Thus, in a quadrilateral:

perimeter of the midpoint polygon = $2\left(\frac{1}{2}\right)$(sum of diagonal lengths).

In a pentagon, there are five diagonals and exactly five triangle midsegments. Each midsegment gets matched with one and only one diagonal.



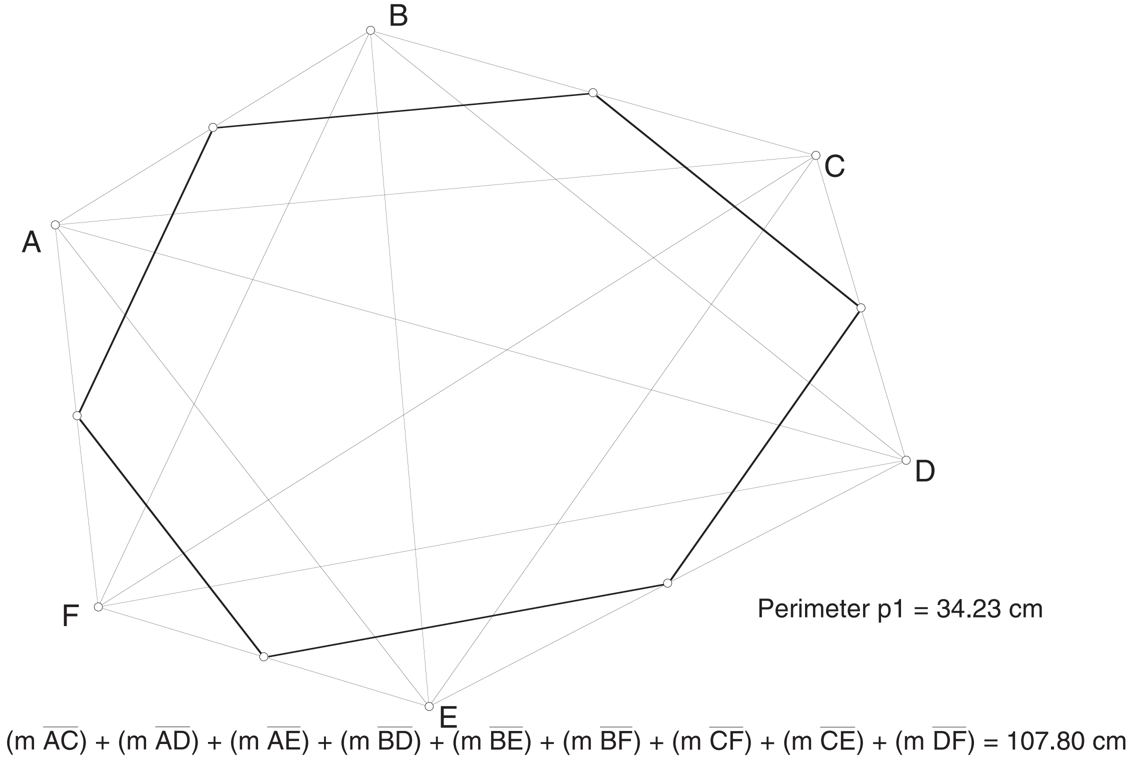
Therefore:

perimeter of the midpoint polygon (p1) = $\frac{1}{2}$ (sum of diagonal lengths.)

Problem 2

For hexagons and polygons with more than 5 sides:

Since the number of diagonals is more than the number of sides of the midpoint polygon, some diagonals get matched with a triangle midsegment, and some don't.



In the hexagon above, diagonals AC, BD, CE, DF, and AE are matched to triangle midsegments. The remaining three diagonals, $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are not matched to a diagonal.

By the Midsegment Conjecture:

perimeter of the midpoint polygon = $\frac{1}{2}$ (AC + BD + CE + DF + AE)

The conjecture, however, has nothing to say about the remaining four diagonals. Therefore, it cannot be used to establish a relationship between the perimeter of the midpoint polygon and the sum of the length of the polygon's diagonals.

The situation is the same for polygons with more sides than hexagons.

Extension Questions:

- Modify your conjecture about the relationship between the perimeter of a midpoint polygon and the sum of the length of the polygon's diagonals so that it applies to polygons with 4 or more sides.

For polygons with 5 or more sides:

The perimeter of the midpoint polygon is one-half the length of the sum of the lengths of the diagonals formed by connecting the endpoints of consecutive sides of the original polygon.

- Prove your conjecture for problem 1.

By the Triangle Midsegment Conjecture, each segment connecting two midpoints is one-half the length of the diagonal connecting the endpoints of 2 consecutive sides. Since there are the same number of midpoint segments as there are diagonals:

perimeter of the midpoint polygon = $\frac{1}{2}$ *(sum of lengths of consecutive sides diagonals.)*

- Complete the chart to find a formula for the number of diagonals of a polygon in terms of *n*, the number of sides.

number of sides of polygon	**number of sides of midpoint polygon**	**number of diagonals**
4		
5		
6		
n		

number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4	4	2
5	5	5
6	6	9
n		

number of diagonals $= \frac{n(n-1)}{2} - n$

Steiner's Point

A developer wants to locate sites for three campgrounds and to build a system of roads to connect them. To accomplish this economically, he needs to find the shortest path between the three campgrounds.

A mathematician named Jakob Steiner investigated problems like this during the nineteenth century. He found that the minimum path connecting three nonlinear points is a line segment from each of the vertices of the triangle formed by the three points. These three line segments meet at a single point in the interior of the triangle. Steiner proved that each segment of the minimum path forms a 120° angle with the other segments it intersects.

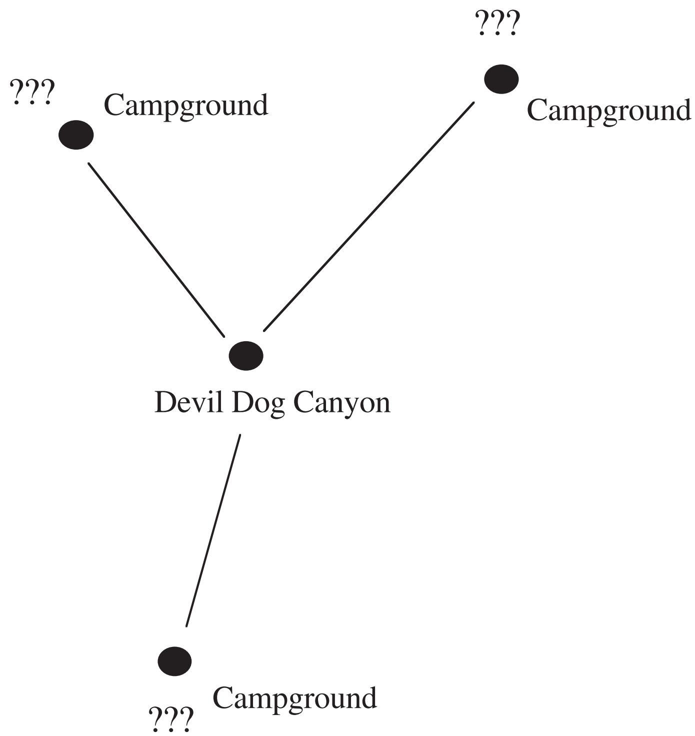
The Situation

A developer wants to locate sites for three campgrounds so that campers have access to Devil Dog Canyon, a major tourist attraction, by the shortest possible route. Bill Steiner, the developer, has a budget which will allow him to build 18 miles of roads connecting the campgrounds to the canyon. Campground B, his high-end luxury camping resort, will have trams shuttling tourists to Devil Dog Canyon, so Steiner wants the road leading to it to be the shortest. Campground A, on the other hand, is a primitive camping area with plenty of privacy, so the road to this campground should be the longest. Finally, budgeting, payroll, and scheduling are all much simpler if the roads from Devil Dog Canyon to the campgrounds are each an integer number of miles in length.

Problem 1

Ruler and Protractor

Where should Steiner locate the campgrounds so that all the conditions are satisfied?



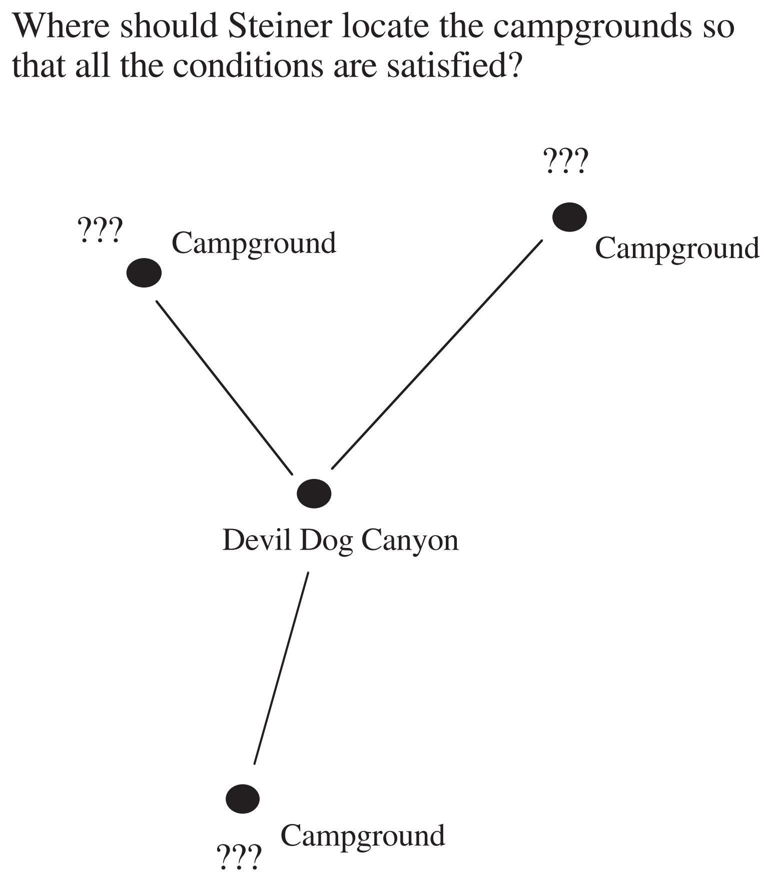
a) Use a ruler and protractor to develop a possible model to represent this situation.

b) Mark the position of Devil Dog Canyon and of the three campgrounds on your model.

c) Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.

Problem 2

Compass and Straightedge



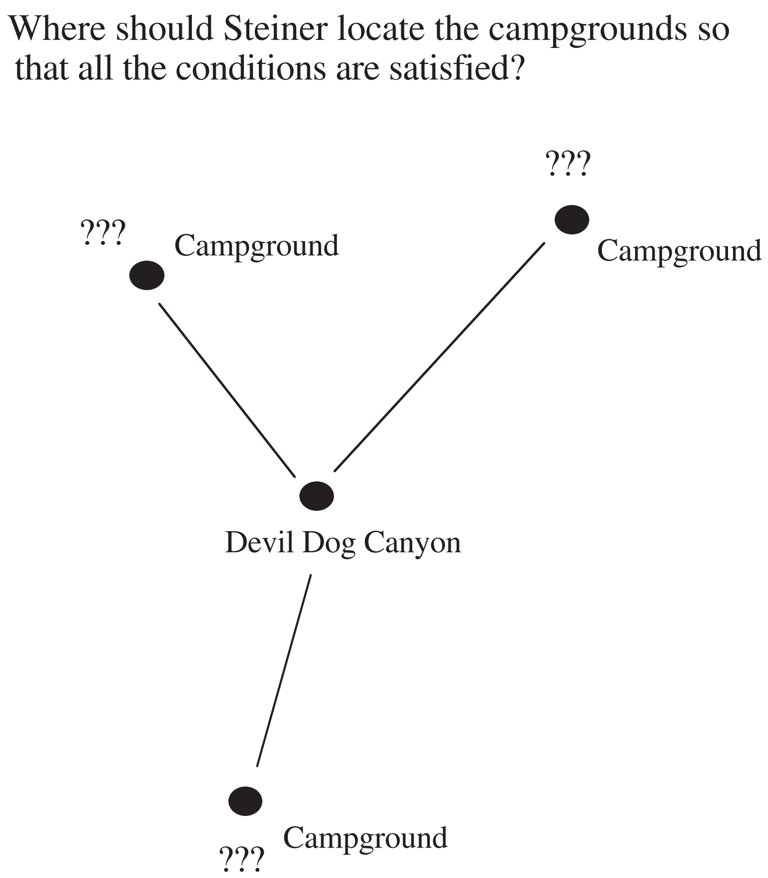
a) Use a compass and straightedge to develop a possible model to represent this situation

b) Mark the positions of Devil Dog Canyon and of the three campgrounds on your model.

c) Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.

Problem 3

Graph Paper and a Coordinate System



a) Develop a model of this situation using graph paper and an *x,y* coordinate system.

b) Locate the positions, and label the numerical coordinates of Devil Dog Canyon and of the three campgrounds on your model. (Note: where appropriate, express coordinates as exact values.)

c) Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.

Teacher Notes

Materials:
Paper and graph paper

One ruler, protractor, compass, and straightedge per student

Connections to Geometry TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Jakob Steiner (1796-1863) was a well-known geometer. He was born in Bern, Switzerland, but spent most of his life in Germany and was a professor at the University of Berlin. He is most famous for his contributions to projective geometry. Another one of his famous theorems shows that only one given circle and a straight edge are required for Euclidean constructions. It is said that Steiner did not like algebra and analysis and felt that calculation replaces thinking while geometry stimulates thinking.[1]

Scaffolding Questions:

Problem 1

- What are some possible integer lengths of the roadways leading from the canyon to each of the campgrounds?
- Why does AD + BD + DC represent the minimum path between the three campgrounds?

Problem 2

- What construction(s) could you use to make three 120° angles at point D?
- What construction(s) could you use to mark off distances from the canyon (point D) to the three campgrounds so that BD is the shortest and DA is the longest?
- In marking off these distances to meet the problem's requirements, which segment should you use as your unit length? Why?
- Why does AD + BD + DC represent the minimum path between the three campgrounds?

Problem 3

- Which point in the model would be the best choice to locate at the origin? Why?

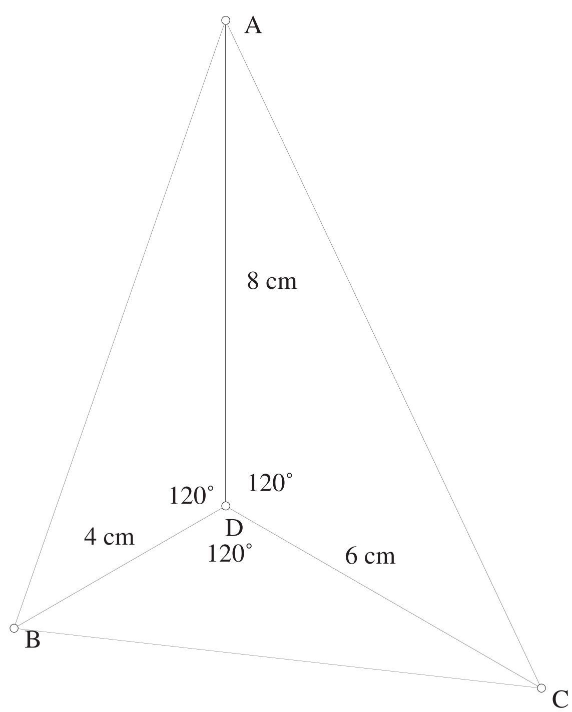
1. University of St. Andrews, St. Andrews Scotland. (1996, December). Retreived August 8, 2002 from http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Steiner.html.

- Is it possible to orient the vertices of your triangle so that the distance from one of the campgrounds to the canyon lies along one of the axes?
- (If above is accomplished) What are the angle measures between the *x* axis and the segments from the canyon to each of the two other canyons?
- How can you use the properties of 30-60-90 right triangles to determine the coordinates of points B and C?
- For campgrounds B and C, which coordinate, *x* or *y*, corresponds to the short leg of the 30-60-90 right triangle? Which coordinate, *x* or *y*, corresponds to the long leg of the 30-60-90 right triangle?
- Why does AD + BD + DC represent the minimum path between the three campgrounds?

Sample Solutions:

Problem 1

There are many possible solutions.



(Note: Distances from D to triangle vertices can be any three integers that sum to 18, such that DB< DC< DA.)

(d.2) **Dimensionality and the geometry of location**. The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representation of geometric relationships and shapes.

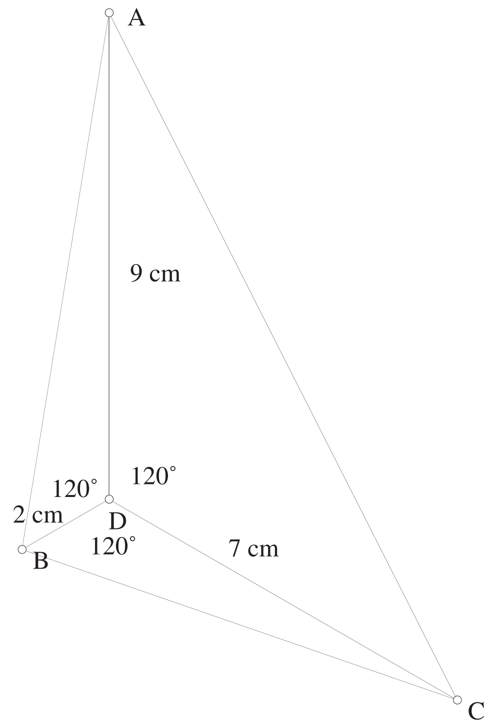
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

II. Transformations: Rotations

III. Triangles: Pythagorean Theorem

Another possible solution:

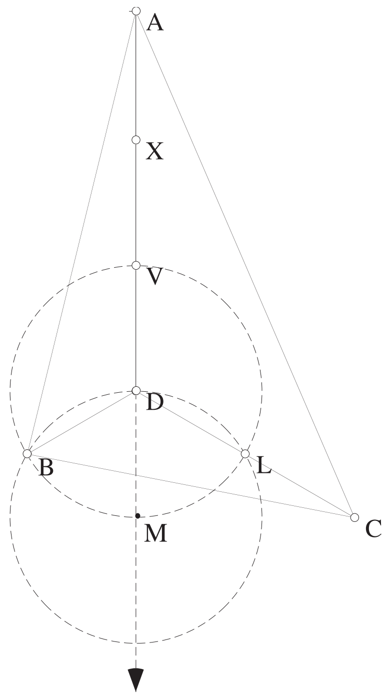


c) Here is one possible explanation for this last example:

$\overline{DA}$ was drawn with length 9 centimeters. At point D the two 120-degree angles were measured. The third angle will also be 120 degrees. $\overline{DB}$ was measured 2 centimeters, and $\overline{DC}$ was measured 7 centimeters. The sum of the segments is 9 + 2 + 7 or 18 centimeters. Point D meets the requirements for Steiner's minimum path because each path forms a 120 degree angle at D.

Problem 2

One possible solution:



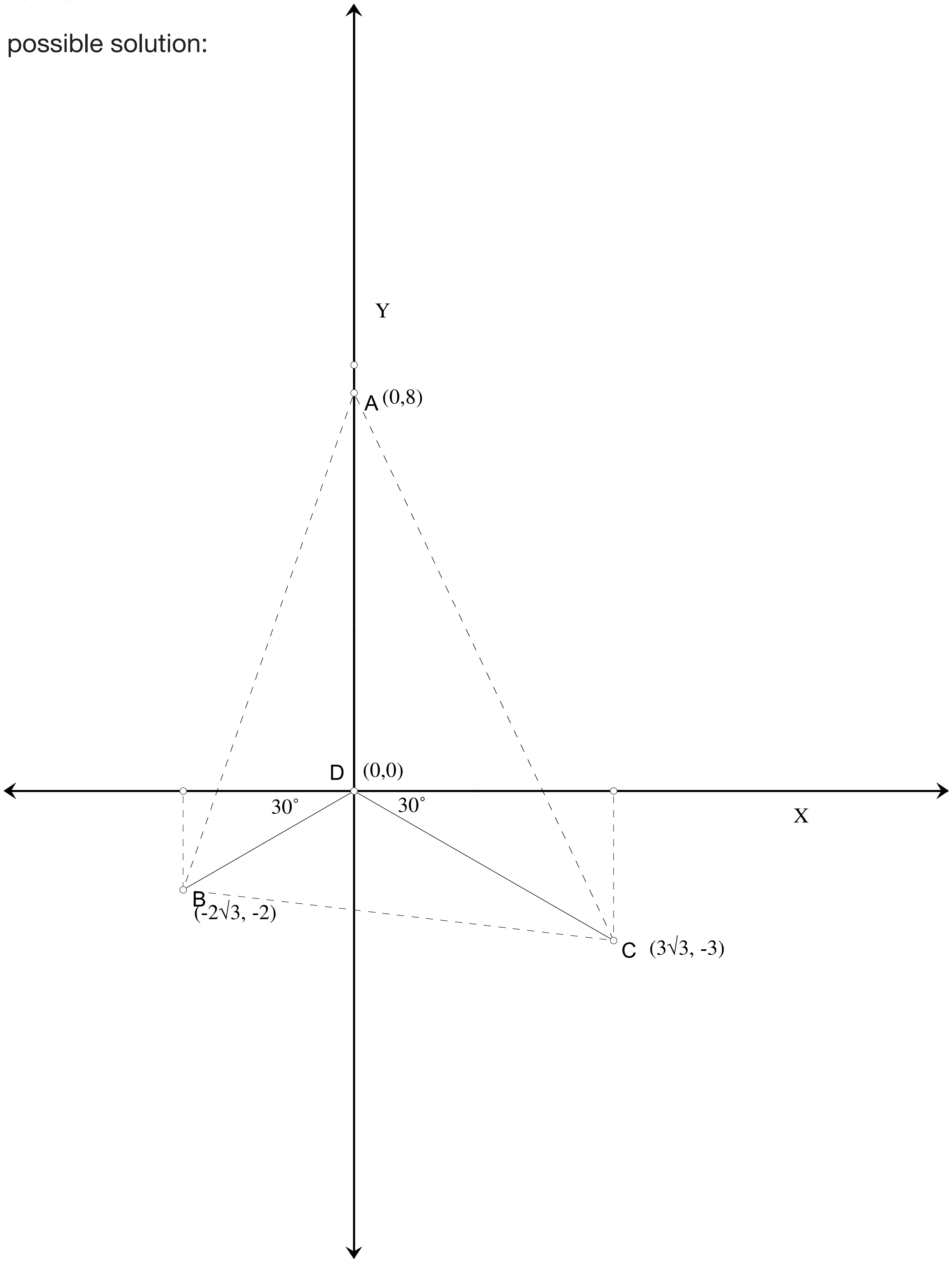
(Note: By construction DC = 2BD, and DA = 3BD. In the model BD = 3 units, DC = 6 units, and DA = 9 units)

c) Begin with a point, D. Draw a line, $\overleftrightarrow{DX}$. Choose a radius to represent 3 miles. Use that radius to construct a circle with center D. The circle will intersect $\overleftrightarrow{DX}$ at point V and M. Use M as a center to construct another circle with the same radius. The two circles intersect at points B and L. Draw $\overline{BD}$ and $\overline{DL}$. Extend $\overline{DL}$ to point C, such that DC = 2BD. Extend $\overline{DV}$ to point A so that DA = 3BD.

   Triangle DMB would be an equilateral triangle with three sides congruent to $\overline{BD}$. Therefore, angles BDM and LDM measure 60 degrees. Angle BDL measures 120 degrees. Angle BDX measures (180 – 60) degrees or 120 degrees. The point D meets the conditions of Steiner's minimum path.

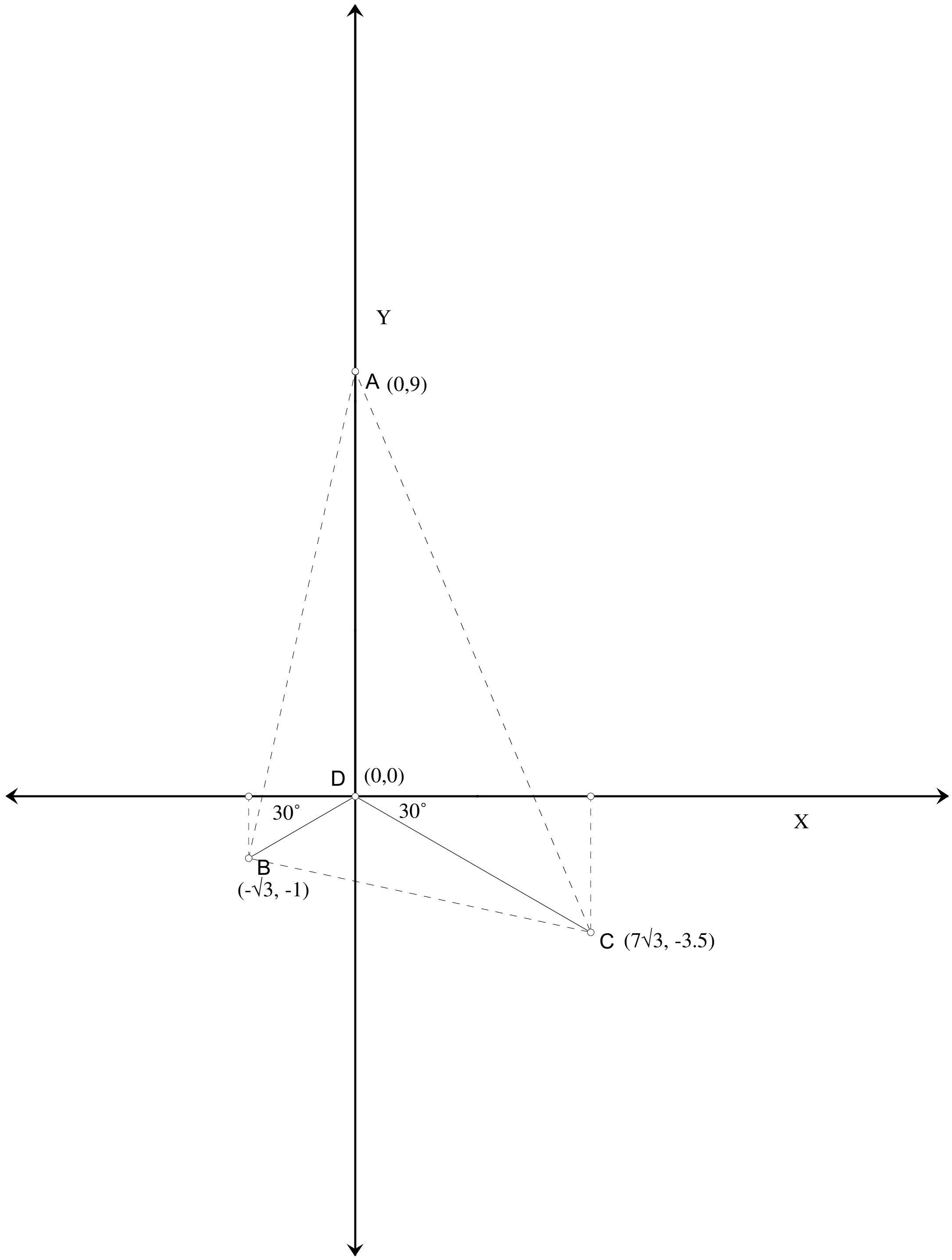
Problem 3

One possible solution:



$(-2\sqrt{3}, -2) \approx (-3.5, -2)$ $(3\sqrt{3}, -3) \approx (5.2, -3)$

Another possible solution:

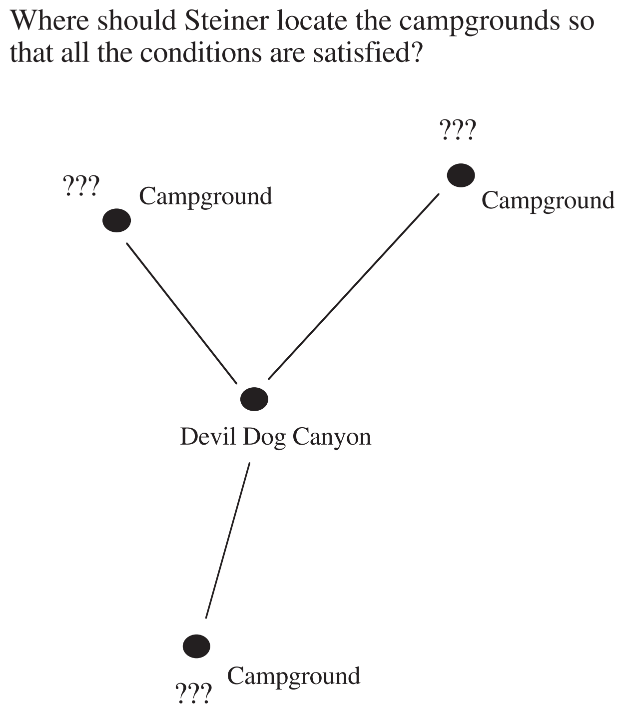


c) Consider the first solution.

Decimal approximations may be used to graph the points A(0,8), B (-3.5,-2), and C(5.2,-3). The measure of angle ADB is (90 + 30) or 120 degrees. The measure of angle ADC is (90 + 30) or 120 degrees. The length of $\overline{DB}$ is 2(2) or 4. The length of $\overline{DC}$ is 2(3) or 6. The sum of the distances BD + CD + AD is 4 + 6 + 8 or 18 units. The points A, B, and C meet the requirements of Steiner's minimum path.

Steiner's Point Revisited

In your work on the Steiner's Point assessment, you developed geometric models for locating campground sites. There you used ruler and protractor, compass and straightedge, and coordinate geometry.



The mathematician George Steiner proved that the minimum path connecting three points is a line segment from each of the vertices of the triangle formed by the three points. In addition, these three line segments form a 120° angle with each other at their point of intersection in the triangle's interior.

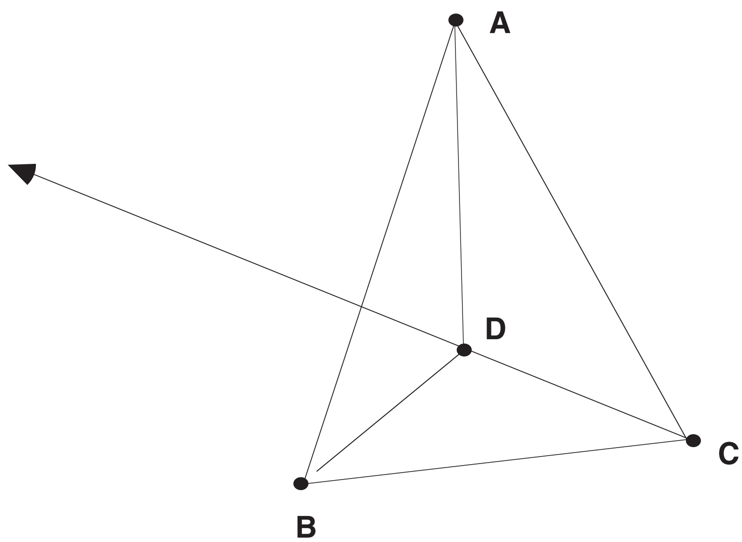
You may want to experiment by having the three segments intersect at angles other than 120° (representing the campground problem using geometry software is useful for this). After some trial and error, it should be evident that the 18-unit path with lines intersecting at 120° is indeed the minimum path between the three campgrounds.

Your task is to use your models to explain why.

The Setup

Use your Ruler and Protractor Model and your Coordinate Geometry Model.

Draw a ray from one of the vertex points of the triangle, making the ray intersect the point in the triangle's interior, which represents the canyon, and then continuing through the triangle's exterior.



Problem 1

Ruler and Protractor Model

a) Use transformations to locate and label point F on the ray so that it satisfies the following requirement:

   The distance from F to the ray's endpoint = Distance from Campground A to canyon + Distance from Campground B to canyon + Distance from Campground C to canyon.

   (Note: FC = 18 units. How can you use transformations to get the images of $\overline{BD}$ and $\overline{AD}$ to lie adjacent to each other on the ray?)

b) Provide a written explanation of how you used transformations to locate point F.

Problem 2

Explain Your Thinking

Write a paragraph explaining how the path between the campgrounds and the canyon must be the minimum path between these points. Use your findings from the previous problem as support for your argument.

Problem 3

Coordinate Geometry Model

a) Use your knowledge of special right triangles to locate and label the exact numerical coordinates of point F on the ray so that it satisfies the following requirement:

   The distance from F to the ray's endpoint = Distance from Campground A to canyon + Distance from Campground B to canyon + Distance from Campground C to canyon.

b) Provide a written explanation of how you used special right triangles to find the coordinates of point F.

Teacher Notes

Materials:

Graph paper

One pencil, ruler, protractor, and compass per student

Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Scaffolding Questions:

Problem 1

- If we know that FC = 18 cm, how is it possible to use transformations to get images of segments $\overline{BD}$ and $\overline{AD}$ to lie adjacent to each other on segment $\overline{FC}$?
- Why would it be advantageous to do this?
- What shape contains both segment $\overline{BD}$ and segment $\overline{AD}$?
- How many degrees, and around what point, would ∆ADB have to rotate in order for the image of segment $\overline{AD}$ to lie on ray $\overline{CD}$?
- Why is ∆BDD' an equilateral triangle?

Problem 2

- What is the shortest path between points F and C?
- Express $\overline{FC}$ as the sum of the lengths of the three segments that compose it.
- Which 2 of the 3 segments composing $\overline{FC}$ are transformed images of segments representing roads from the canyon to the campgrounds?
- Express $\overline{FC}$ as the sum of the lengths of the three segments leading from the canyon to the campgrounds.

Problem 3

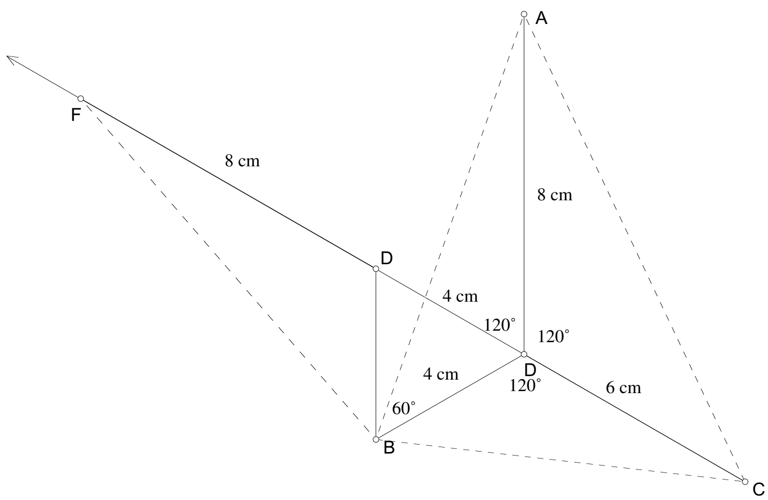
- What must the length of segment $\overline{FD}$ be?
- Explain how to determine the measure of angle FDM.
- For point F, what coordinate, *x* or *y*, corresponds to the short leg of 30-60-90 right ∆FDM? What coordinate, *x* or *y*, corresponds to the long leg of 30-60-90 right ∆FDM?

Sample Solution:

Problem 1

This is one possible solution.

a)

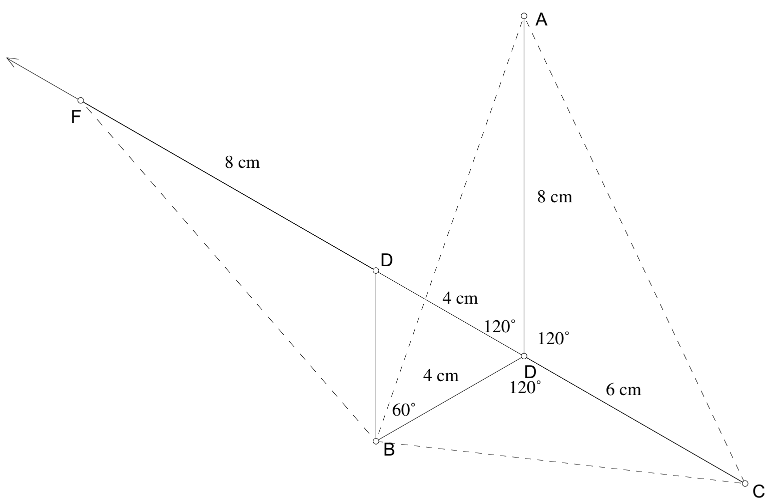


b) Rotate Δ ADB 60° around point B. FD' = AD = 8 cm
BD = BD' = 4 cm This makes ΔBDD' equilateral.
Therefore DD' = 4 cm.

FC = AD + DB + DC = 18 cm.

Problem 2

For example, using the Ruler and Protractor Model, the minimum distance path between points F and C is the straight line segment FC.



Texas Assessment of Knowledge and Skills:

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

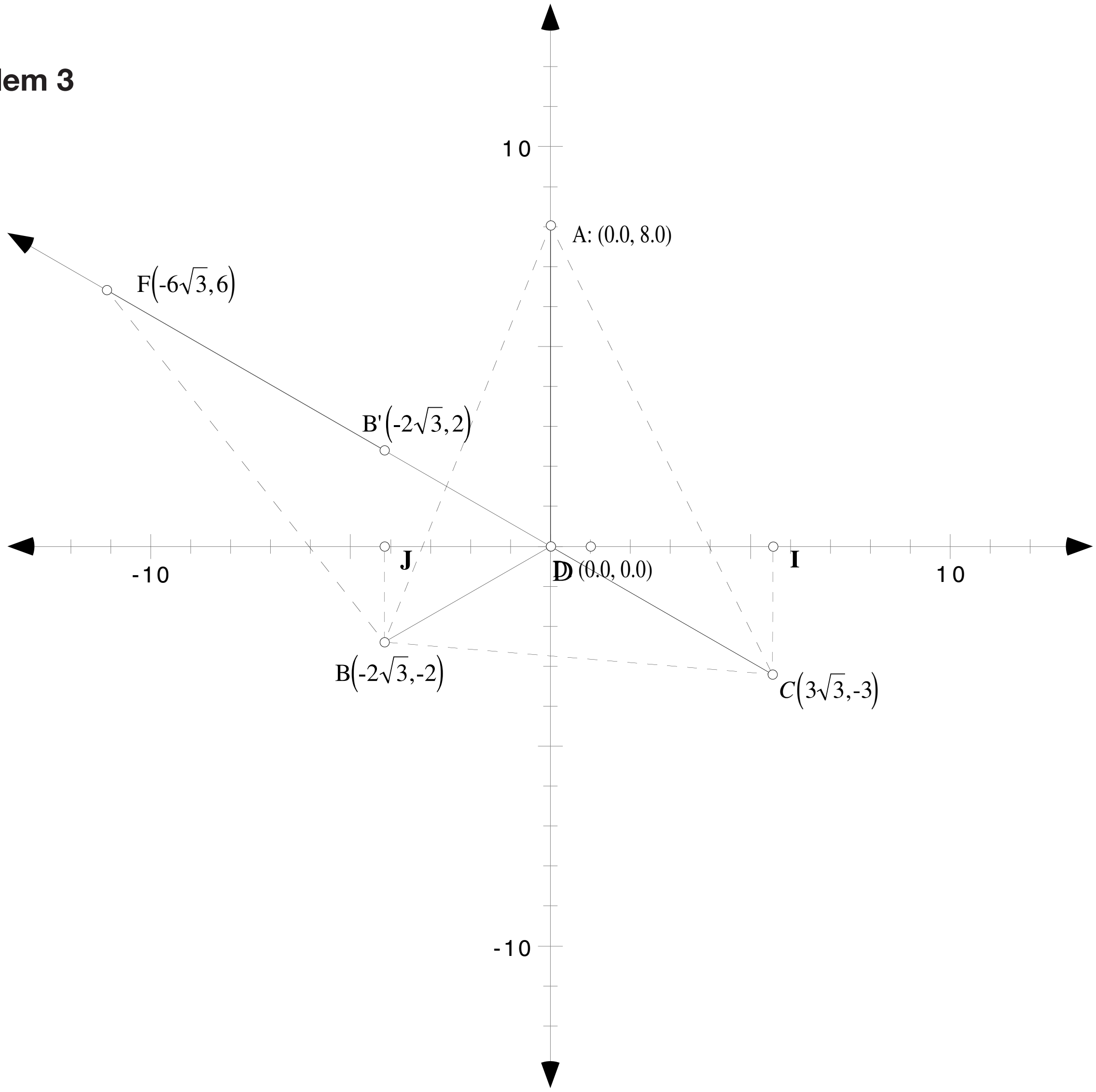
Connection to High School Geometry: Supporting TEKS and TAKS Institute:

II. Transformations: Rotations

FC = FD' + D'D + DC.
But by preservation of length under transformations,
FD' = AD and D'D = BD, so by substitution,
FC = AD + BD + DC.
Since FC is the shortest distance between F and C, it therefore, must be the minimum path between points A, B, and C.

Problem 3

a)



b) FD = FC – DC = 18 – 6 = 12. $\overline{FD}$ is the hypotenuse of 30-60-90 right triangle DMF. The short leg of this triangle corresponds to the *y* coordinate of point F and the long leg corresponds to the *x* coordinate of point F. Since the hypotenuse is 12, the short leg is 6, and the long leg is $6\sqrt{3}$. Therefore the coordinate of point F is F($-6\sqrt{3}$, 6).

Extension Questions

- Use the Compass and Straightedge Model

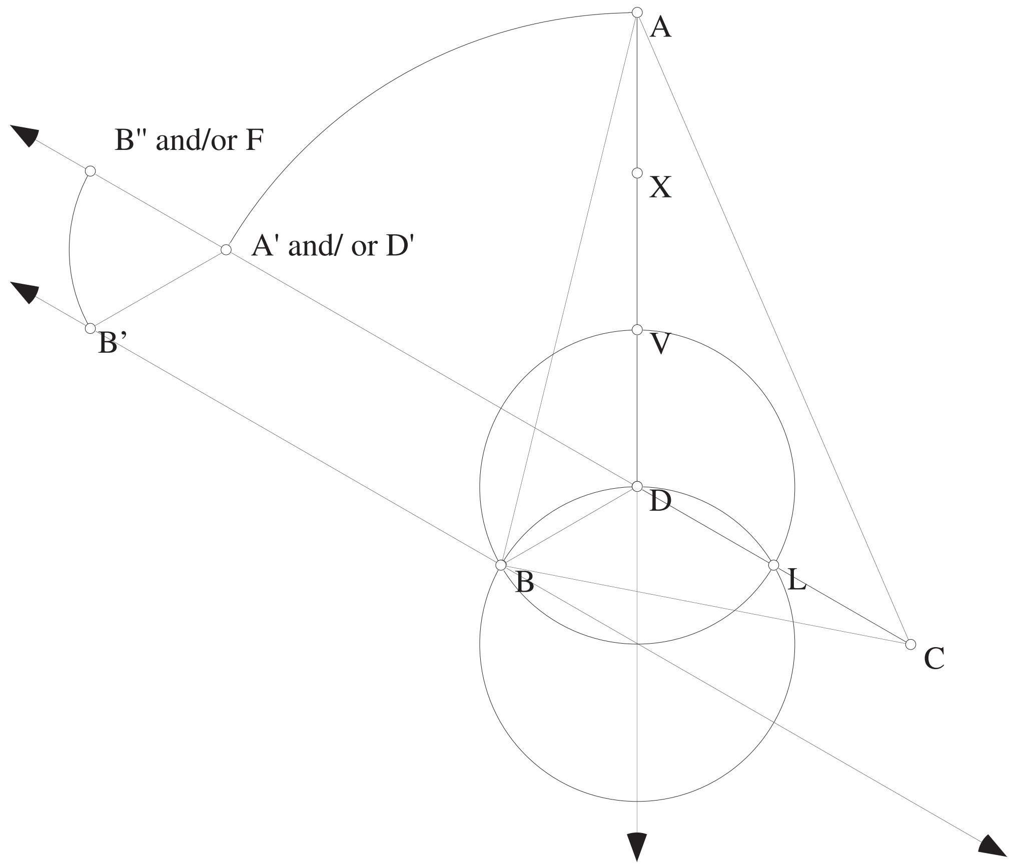
a) Use transformations to locate and label point F on the ray so that it satisfies the following requirement:

The distance from F to the ray's endpoint = Distance from Campground A to canyon + Distance from Campground B to canyon + Distance from Campground C to canyon.

Note: For this model you are limited to your construction tools as you perform and describe the transformations.

b) Provide a written explanation of how you used transformations to locate point F.

a)



b) Rotate point A about point D until the rotated point intersects the ray. Label the intersection A'. DA = DA'.

Construct a line parallel to ray $\overrightarrow{DC}$ through point B. Translate point D distance DA' along segment $\overline{DA'}$. Translate point B distance DA' the parallel line through point B. B'D' = BD.

Rotate point B' about point A' until the rotated point intersects the ray.

Mark the intersection point F. FA' = B'D' = BD.

FC = AD + BD + DC.

- In the solution to the first Extension Question, how is it possible to get the transformed image of segment BD to lie adjacent to segment DA' without constructing a line parallel to ray $\overrightarrow{CD}$ through point B?

  Rotate point B about point D until B', the image point, intersects ray $\overrightarrow{CD}$. Then translate B' distance DA' along ray $\overrightarrow{CD}$.

Walking the Archimedean Walk

Most geometry students know where the value of π comes from—their calculators. Hopefully, most geometry students also realize that the number their calculator gives them is really an approximation of the value of π—the constant ratio between a circle's circumference and its diameter:

$$\pi = \frac{C}{d}$$

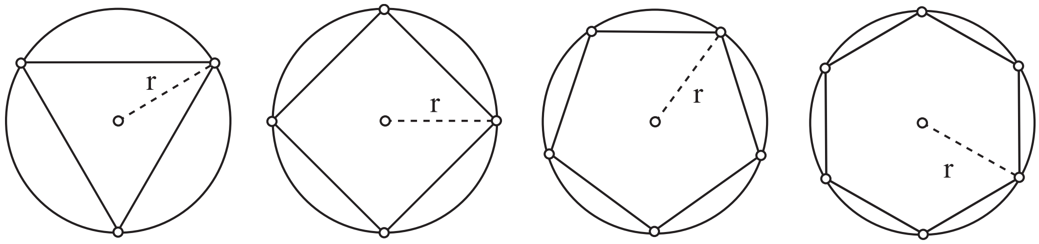
During the course of human history, diverse cultures throughout the world were aware of this constant ratio. The attempt to fix its exact value has been a vexing problem that has occupied many mathematical minds over the centuries. (See, for example, *A History of Pi* by Petra Beckman, St. Martin's Press, 1971.)

In our Western cultural tradition, the historical record tells us that Archimedes was the first person to provide a mathematically rigorous method for determining the value of π.

In this assessment you will have to retrace his footsteps in order to demonstrate a solid understanding of where that number comes from when you push the "π" button on your calculator.

A logical starting place for determining π is to measure the circumferences and diameters of many circles and then calculate the ratio C/d based upon those measurements. You may have done a measurement activity like this in your geometry class. Archimedes realized that this method would always be limited by the precision of the people doing the measuring and by the accuracy of the measuring devices they were using. He sought a way to fix the value of π that was based upon direct calculation rather than upon measurement.

Archimedes' approach involved inscribing regular polygons in circles. He then considered what "π" would be for each of the inscribed regular polygons. You will model his approach by examining the figures below in the problems for this assessment.



Problem 1

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe an equilateral triangle in a circle with a radius of 2 units.

2. Make appropriate calculations (no measuring allowed!) in order to determine the value of π based upon an inscribed equilateral triangle.

Problem 2

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe a square in a circle with a radius of 2 units.

2. Make appropriate calculations (no measuring allowed!) in order to determine the value of π based upon an inscribed square.

Problem 3

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe a regular hexagon in a circle with a radius of 2 units.

2. Make appropriate calculations (no measuring allowed!) in order to determine the value of π based upon an inscribed regular hexagon.

Problem 4

1. Complete the table below, and summarize your findings.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter	Diameter	Approximation for π
3			4	
4			4	
6			4	

2. Does this method overestimate or underestimate the value of π? Will this method result in an exact value for π?

3. Write a few sentences and provide a diagram to answer question 2.

Problem 5

Write a few sentences explaining the basics of the method Archimedes used.

Teacher Notes

Materials:
Construction tools, circular geoboards, circular dot paper, or geometry software

Connections to Geometry TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(B) through the historical development of geometric systems, recognizes that mathematics is developed for a variety of purposes.

(c) **Geometric patterns**. The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

The assessment, Talk the Archimedean Talk, is an extension of this problem. It requires the student to repeat this activity for a dodecagon.

Scaffolding Questions:

Problems 1, 2, and 3

- What special right triangle is formed by the radius and the apothem of the inscribed polygon?

Problem 4

- Will the perimeter of the inscribed polygon be greater than, less than, or equal to the circumference of the circle?
- Will the ratio $\frac{\text{perimeter}}{\text{diameter}}$ be greater than, less than, or equal to the ratio $\frac{\text{circumference}}{\text{diameter}}$?
- Will the perimeter of the inscribed polygon ever be equal to the circumference of the circle?

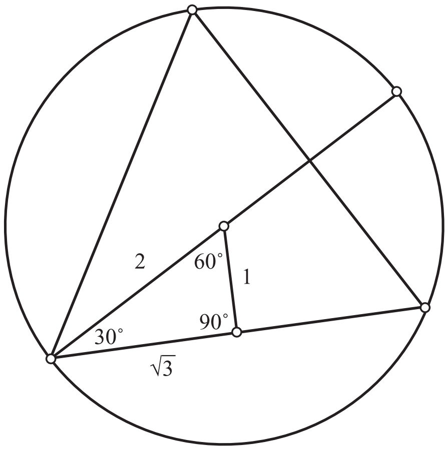
Problem 5

- If you consider the inscribed regular polygon to be a “primitive circle,” what measure in the polygon corresponds to the circumference of the circle?
- If you consider the inscribed regular polygon to be a “primitive circle,” what measure in the polygon corresponds to the diameter of the circle?
- What will happen to the ratio $\frac{\text{perimeter}}{\text{diameter}}$ as the number of sides of the polygon increases?

Sample Solutions:

Problem 1

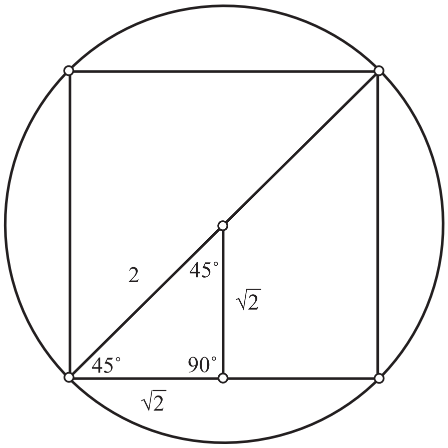
1.



2. $\dfrac{\text{perimeter}}{\text{diameter}} = \dfrac{6\sqrt{3}}{4} = \dfrac{3\sqrt{3}}{2} \approx 2.5981$

Problem 2

1.



2. $\dfrac{\text{perimeter}}{\text{diameter}} = \dfrac{8\sqrt{2}}{4} = 2\sqrt{2} \approx 2.8284$

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

Teacher's Comment:

"I introduced it to my students as a Gold Medal problem (which they knew required a written, detailed response). They worked on it independently for 30 minutes, and then they had an opportunity to work in a group setting.

Many students extended the problem naturally to see how close they could come to π.

I used geoboard circular dot paper and that worked well. The students saw the natural progression to a dodecagon."

Student's Comment:

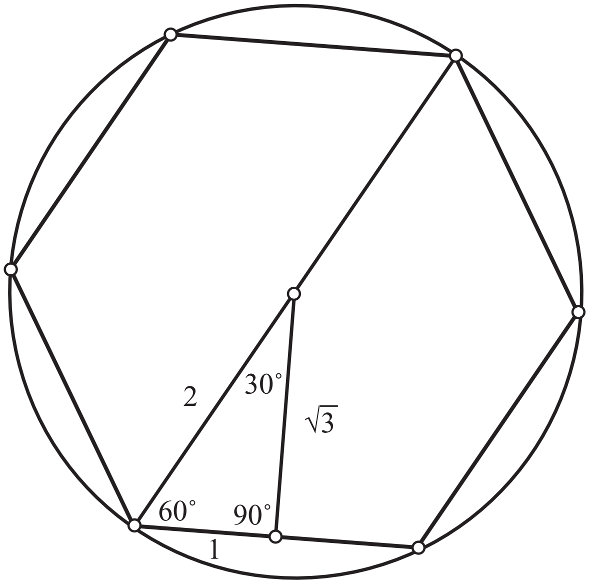
"This problem looked very complex before I started, but when I began working, it helped create itself and turned out to be relatively simple with the proper procedures. I learned to think differently and used sub-problems to meet my conclusions. I don't believe that I've ever done a similar problem, but I know that if I do, I will know how to start. The sub-problems really made it easy for me to follow where the problem led."

Student's Comment:

"Archimedes must have been a really smart man and must have had a lot of time on his hands to figure out this problem, especially without calculators."

Problem 3

1.



2. $\dfrac{\text{perimeter}}{\text{diameter}} = \dfrac{12}{4} = 3$

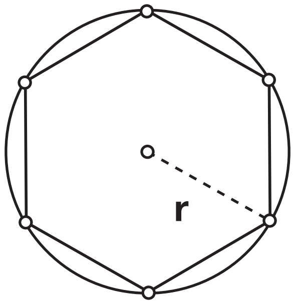
Problem 4

1.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter	Diameter	Approximation for π
3	120˚	$6\sqrt{3}$	4	2.5981
4	90˚	$8\sqrt{2}$	4	2.8284
6	60˚	12	4	3.000

2. This method underestimates the value of π. The perimeter of the inscribed regular polygon is always less than the circumference of the circle it is inscribed within.

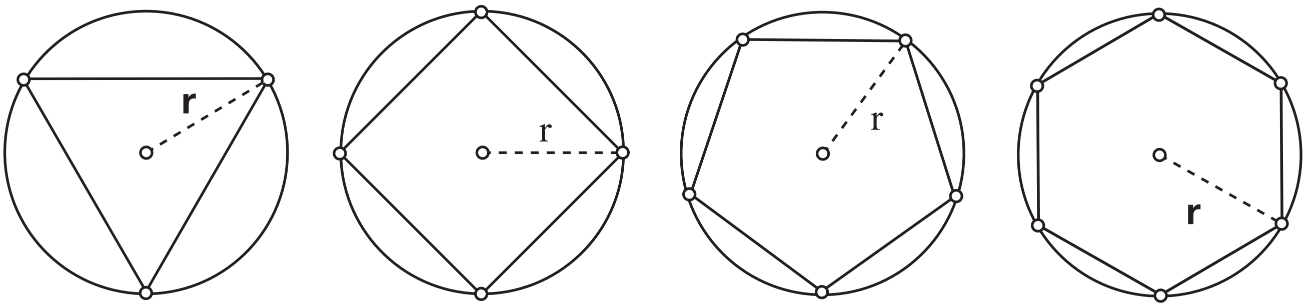
3.



The perimeter of the polygon is always less than the circumference of the circle, but the diameters for both polygon and circle are equal. This means that the ratio $\frac{\text{perimeter}}{\text{diameter}}$ is less than the ratio $\frac{\text{circumference}}{\text{diameter}}$.

As the number of sides of the polygon increases, the ratio $\frac{\text{perimeter}}{\text{diameter}}$ will get closer and closer to the value of π. It will never result in the exact value for π because no matter how many sides the inscribed polygon has, its perimeter will always be less than the circumference of the circle with the same center and radius.

Problem 5



Each inscribed regular polygon can be considered an approximation of the circle it is inscribed within. Consider the diameter of a polygon to be twice the radius of the inscribed circle. Therefore, the ratio of the polygon's perimeter to its "diameter" can be considered an approximation of the ratio of the circle's circumference to its diameter, π. As the number of sides in the regular polygon increases, the ratio; perimeter/diameter, gets closer and closer to the value of π. The method yields successively more accurate approximations of π.

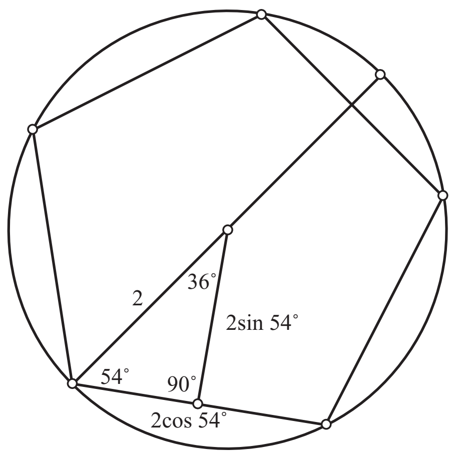
Extension Questions:

- Based on the chart in Problem 4, between what two values should an estimate of π, based on a pentagon with radius 2, fall?

The value should be between the values for a quadrilateral and a hexagon, or between 2.8284 and 3.000.

- Use construction tools or geometry software to construct a regular pentagon inscribed in a circle of radius 2. Then calculate an estimate for π based on the inscribed pentagon.

Geometry software was used to construct the figure.



Use trigonometry to solve this problem.

$$\frac{perimeter}{diameter} = \frac{20\cos 54}{4} = 5\cos 54 \approx 2.9398$$

The approximation for π is 2.9398.

Student Work Sample

The students in this class were allowed to report their findings in a variety of ways. This student's work satisfies many of the criteria on the solution guide.

For example:

- Uses geometric and other mathematical principles to justify the reasoning and analyze the problem.

He uses his prior knowledge to explain the procedures he uses. "because the hypotenuse is the leg times $\sqrt{2}$ *".*

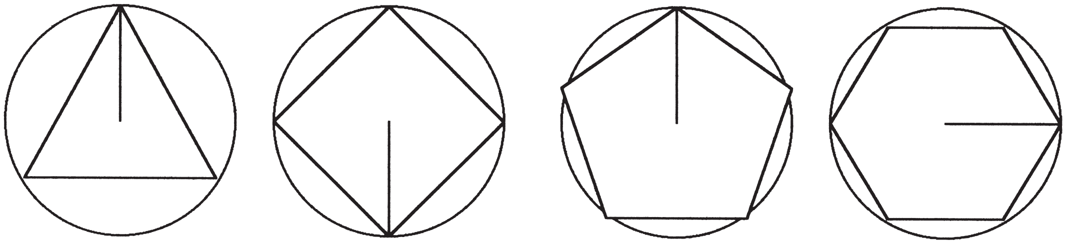
- Communicates a clear, detailed, and organized solution strategy.

The student's narrative explains his strategy for determining the perimeter of each triangle.

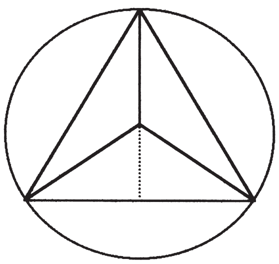
The student's work includes appropriate and accurate representations of the problem with correct diagrams that are not labeled. However, labeling may not be necessary because his narrative explains the process and the figures.

Archimedean Walk

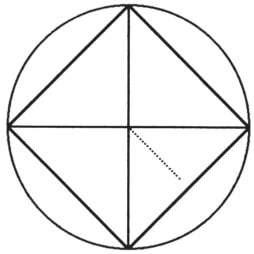
One way to find what π (pi) equals is by measuring the circumference of a circle and dividing it by the diameter. Archimedes realized that this would always differ by the accuracy of that person. He wanted to find away to fix the value of π that was based on calculations and not on measurement. He decided to put regular polygons inside circles, figure out their perimeters, and find π according to that shape. Archimedes decided to find π this way because the more sides the polygon had the closer he would get to finding the actual value of π.



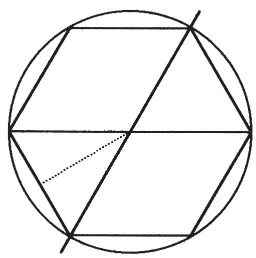
This diagram shows that the closer you get to the circumference of a circle, when you divide by the diameter, which is two, you can get closer and closer to the value for π. To answer the question about finding the approximation for the first, second, and fourth shape, I used trigonometry equations and special triangles. When you dissect the figures as follows you can use special triangles to solve.



When you divide the equilateral triangle in half (dotted line) you get a right triangle and a hypotenuse of two, also the radius of the circle. When we divided the triangle in half that made the central angle, which we found out using the equation to find the value for each angle in a regular polygon, divide in half also to equal 60°. With this knowledge you can conclude that the last angle is 30° and that the triangle is a 30:60:90 triangle. Since this is a special triangle we can figure out the lengths of the two legs. Knowing that the hypotenuse in two, we know that the short leg is half of that or one, and since the short leg is one, the long leg is $1\sqrt{3}$ or just $\sqrt{3}$. That leaves us with half of the original triangle, and with this information we know the whole side of the triangle is $2\sqrt{3}$. To find the perimeter of the triangle we just multiply the value for the side times 3 to get $6\sqrt{3}$ or ≈10.39 units. We then take the perimeter and divide it by the diameter of the circle to find the approximation of π, which is ≈2.598 units. This is the way in which to solve the other polygons hereafter.



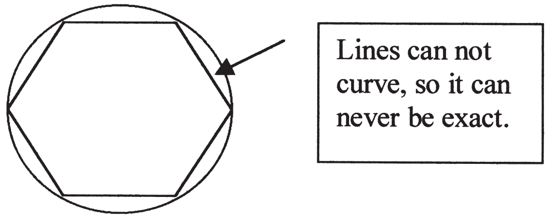
Although the triangle made a 30:60:90 triangle, the square makes a 45:45:90 triangle because when the central angle is divided by two the angle becomes a 45°, which makes the other a 45° angle also. With a 45:45:90 triangle we know that the legs are going to be the same. To find the legs we are going to divide the hypotenuse by √2 because the hypotenuse is a leg times the √2, but since we already have the hypotenuse, we can reverse the equation. Using the equation, 2/√2, we know that the two legs of the triangle are ≈1.41 units. This means that one side of the square equals ≈2.23 units. With this information we know that the perimeter of the square is ≈11.31 and that divided by 4, the diameter, equals, for the approximation of π, ≈2.83 units.



The last shape, a hexagon, makes a 30:60:90 triangle when dissected. Since the angles and hypotenuse are the same as the triangles, we know that the short leg is one. That means that the length of one complete side is two units. When we multiply that by six for the perimeter, we get twelve. Then twelve divided by the diameter, four, equals three units.

# of sides of the inscribed shape	Measure of the central angle	Perimeter	Diameter	Approximation for π
3	120°	≈10.39 units	4	2.598
4	90°	≈11.31 units	4	2.83
6	60°	12 units	4	3

In conclusion the more sides you add the closer the shape get to the circumference of the circle and the closer you get to the circle, when you divide by the diameter, the closer you get to the real value of π. This method will always be an underestimate for the value of π. Since you can never make an exact circle, you can not overestimate the answer. No matter how many sides you add, you can never make a circle, on reason because the sides curve in a circle, but it can come very close, yet never an exact number.



Talking the Archimedean Talk

In the assessment, Walking the Archimedean Walk, you modeled Archimedes method of inscribed polygons to arrive at increasingly accurate approximations for the value of π.

The table shows the results of that investigation.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter	Diameter	Approximation for π
3	120°	$6\sqrt{3}$	4	2.5981
4	90°	$8\sqrt{2}$	4	2.8284
6	60°	12	4	3.000

In reality, Archimedes was versed well enough in basic geometry to know that he could start his method for approximating π with an inscribed hexagon. From there, he refined his approximation by doubling the sides of the hexagon and calculating the ratio $\frac{\text{perimeter}}{\text{diameter}}$ for the resulting dodecagon.

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to construct a regular hexagon inscribed in a circle of radius 2 units.

2. Using this construction, double the number of sides to construct a regular dodecagon inscribed in a circle of radius 2 units.

3. Make appropriate calculations (no measuring allowed!) in order to determine the value of π based upon an inscribed regular dodecagon.

Teacher Notes

Materials:
Construction tools, circular geoboards, circular dot paper, or geometry software

Connections to Geometry TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(B) through the historical development of geometric systems, recognizes that mathematics is developed for a variety of purposes.

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

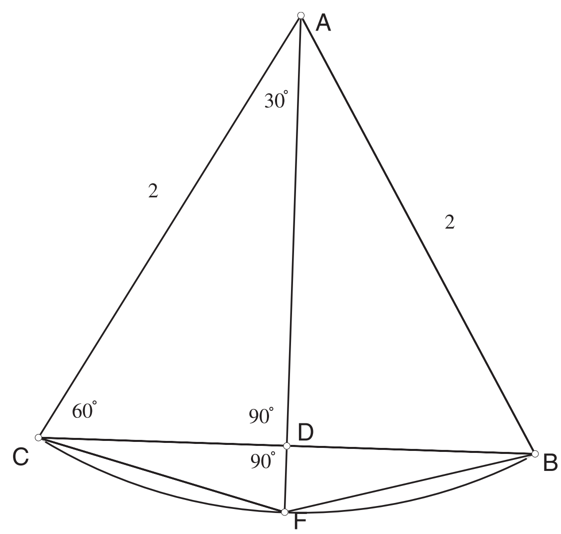
Scaffolding Questions:

Problems 1 and 2

- What is the measure of a central angle of a dodecagon?
- What must the measure of the arc be that is intercepted by the dodecagon's central angle?
- What segment can you extend to intersect the circle in order to create a 30° arc?

Problem 3

Note: If necessary have students copy the picture below.



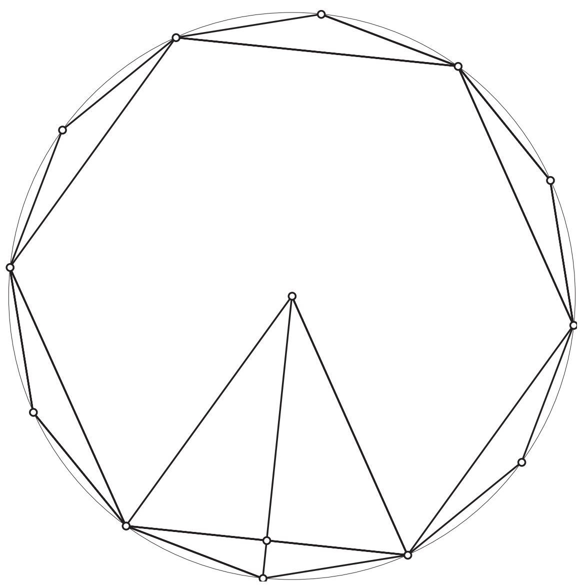
- What are the measurements of the segments?
- What is the length of segment $\overline{AF}$? What is the length of segment $\overline{AD}$?
- What is the length of segment $\overline{DF}$?

Note : If students use decimal approximations for their square roots, make sure they maintain about 4 decimal places of accuracy to ensure a good approximation of π at the final calculation.

Sample Solutions:

Problems 1 and 2

The hexagon was created by drawing a circle. With the radius of the circle as the compass width, place the compass point on the circle and strike an arc. Place the point on this mark, and strike another arc. The radius of the circle is the side of the hexagon. Connect these points to form the hexagon.



To double the sides of the hexagon, extend the hexagon's apothem until it intersects the circle. Then draw segments from the endpoints of the hexagon's sides to the point where the apothem intersects the circle. Every side of the hexagon results in 2 sides of the dodecagon. (Notice that this does **not** mean that the length of the dodecagon's sides is half the length of the hexagon's sides.)

Repeat this process as you go around the hexagon to create the inscribed dodecagon.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Texas Assessment of Knowledge and Skills:

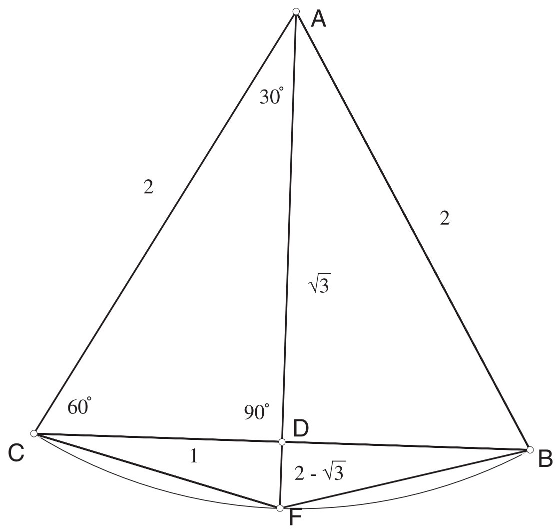
Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

Problem 3



CF, the length of the side of the dodecagon, needs to be found in order to complete the calculation for π.

In right $\triangle$ADC:

AC = 2; CD is one-half of AC, or 1 unit, and AD is $\sqrt{3}$.

The length of $\overline{AF}$ is 2, so DF = AF – AD or $2-\sqrt{3}$.

Use the Pythagorean theorem on right $\triangle$DCF.

Use $\sqrt{3} \approx 1.7321$

$$1^2 + (2 - 1.7321)^2 \approx CF^2$$

$$1 + (0.2679)^2 \approx CF^2$$

$$1 + 0.0718 \approx CF^2$$

$$\sqrt{1.0718} \approx CF$$

$$1.0353 \approx CF$$

The perimeter of the regular dodecagon is approximately 12(1.0353) or 12.4236 units.

So for the dodecagon, ratio $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{12.4236}{4} \approx 3.1059$.

Using exact values:

$1^2 + \left(2 - \sqrt{3}\right)^2 = CF^2$

$1 + 4 - 4\sqrt{3} + 3 = CF^2$

$8 - 4\sqrt{3} = CF^2$

$4(2 - \sqrt{3}) = CF^2$

$CF = \sqrt{4\left(2 - \sqrt{3}\right)} = 2\sqrt{\left(2 - \sqrt{3}\right)}$

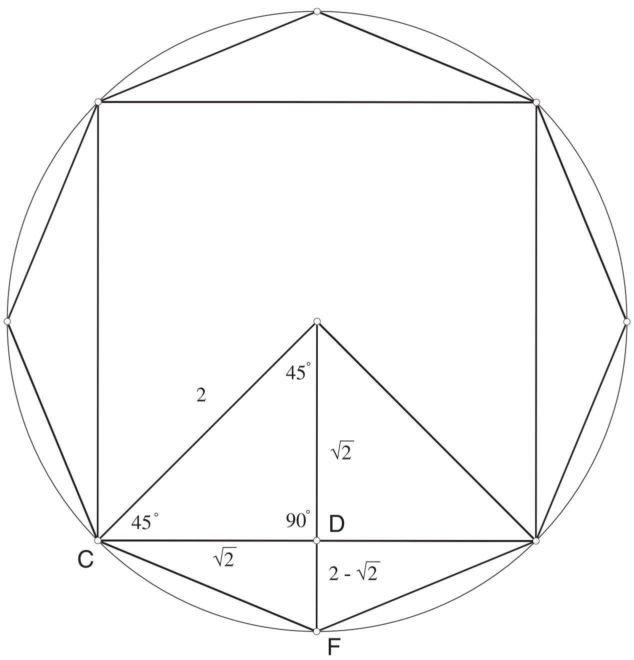
So for the dodecagon, ratio $\dfrac{\text{perimeter}}{\text{diameter}} \approx \dfrac{24\sqrt{2-\sqrt{3}}}{4} = 6\sqrt{2-\sqrt{3}} \approx 3.1058.$

Extension Problems:

- What two values should an estimate of π based on an octagon fall between?

The value should be between the values for a hexagon and a dodecagon, or between 3.000 and 3.1058.

- Calculate an estimate for π based on an inscribed octagon.



Construct a square and extend the apothem to intersect the circle at point F. Connect this point to the adjacent vertices of the square. Repeat this process for each apothem. The segment $\overline{DF}$ measures $2 - \sqrt{2}$.

In right △CDF:

Using $\sqrt{2} \approx 1.4142$

$1.4142^2 + (2 - 1.4142)^2 \approx CF^2$

$2 + (0.5858)^2 \approx CF^2$

$2 + 0.3432 \approx CF^2$

$\sqrt{2.3432} \approx CF$

$1.5308 \approx CF$

So for the octagon, $\dfrac{perimeter}{diameter} \approx \dfrac{12.2464}{4} \approx 3.0616.$

This value is between the values for a hexagon and a dodecagon.

Using exact values:

$$\left(\sqrt{2}\right)^2 + \left(2 - \sqrt{2}\right)^2 = CF^2$$

$$2 + 4 - 4\sqrt{2} + 2 = CF^2$$

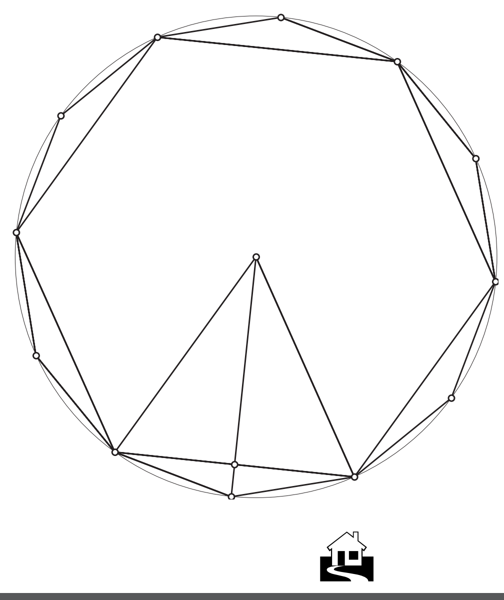
$$8 - 4\sqrt{2} = CF^2$$

$$4\left(2 - \sqrt{2}\right) = CF^2$$

$$CF = \sqrt{4\left(2 - \sqrt{2}\right)} = 2\sqrt{\left(2 - \sqrt{2}\right)}$$

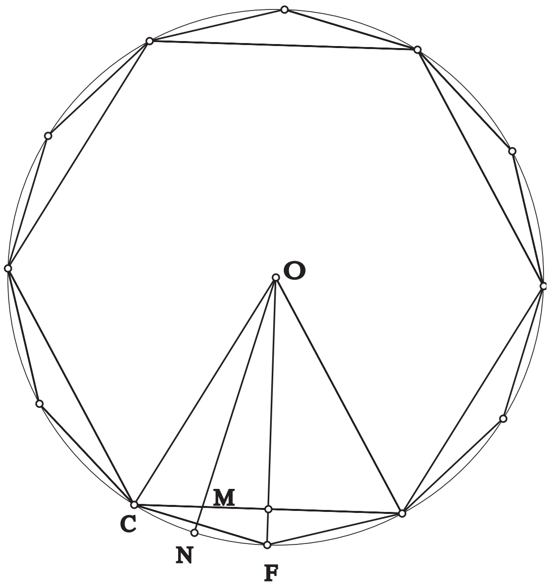
So for the octagon, ratio $\dfrac{perimeter}{diameter} \approx \dfrac{16\sqrt{2 - \sqrt{2}}}{4} = 4\sqrt{2 - \sqrt{2}} \approx 3.0615.$

- Double the sides of a dodecagon. Then calculate an estimate for π based on an inscribed 24-gon. (Circle with radius 2)



To double the sides of the hexagon, extend the hexagon's apothem until it intersects the circle. Then draw segments from the endpoints of the hexagon's sides to the point where the apothem intersects the circle. Every side of the hexagon results in two sides of the dodecagon. (Notice that this does **not** *mean that the length of the dodecagon's sides is half the length of the hexagon's sides.)*

Repeat this process as you go around the hexagon to create the inscribed dodecagon.



$CF = 1.0353$

$CM = \frac{1}{2}(1.0353)$

$CO = 2$

$OM^2 + (\frac{1}{2}(1.0353))^2 = 2^2$

$OM \approx 1.932$

$MN = 2 - OM \approx 0.06815$

$CN^2 = MN^2 + CM^2$

$CN^2 = (0.06815)^2 + (\frac{1}{2}(1.0353))^2$

$CN = 0.522$

$Perimeter = 24\ (0.522) = 12.528$

$\frac{perimeter}{diameter} \approx \frac{12.528}{4} \approx 3.132.$

Note: Archimedes extended his calculations to an estimate based on an inscribed 96-gon.

Mad as a Hatter or Hat as a Madder

> "Then you should say what you mean," the March Hare went on.
>
> "I do," Alice hastily replied, "at least—at least I mean what I say—that's the same thing, you know."
>
> "Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"
>
> Lewis Carroll, *Alice in Wonderland*

Problem Set 1

For each of the following conditional statements:

a) Determine if the original statement is true. Explain in writing the reasoning for each of your choices.

b) Write the converse of the statement.

c) Determine if the converse of the statement is true or false. Explain in writing the reasoning for each of your choices.

1. If a number is divisible by 4, then it is divisible by 2.
2. If it is raining in Las Vegas, then it is sunny in Nevada.
3. If it is sunny in Nevada, then it is sunny in Las Vegas.
4. If a number is greater than -500, then the number is greater than 500.
5. If a person is a teenager, then that person is between 13 and 19 years old.
6. If a musician plays classical guitar, then the musician is a girl.
7. If a girl plays classical guitar, then she is a musician.
8. If a person reads a lot, then that person is smart.
9. If a number is even, then it is divisible by 2.
10. If a car is an SUV, then it has four wheels.

11. If a number squared is greater than 1, then the number itself must be greater than 1.

12. If point B is the midpoint of $\overline{AC}$, then the distance from point A to point B is the same as the distance from point B to point C.

13. If a polygon has exactly 8 sides, then it is an octagon.

14. If a parallelogram has a right angle, then it is a square.

15. If a quadrilateral has exactly one pair of parallel sides, then it is a trapezoid.

Problem Set 2

Write: 1. a non-mathematical statement

and

2. a mathematical statement

that fulfill the following requirements.

1. A true conditional statement with a converse that is also true.
2. A true conditional statement with a converse that is false.
3. A false conditional statement with a converse that is true.
4. A false conditional statement with a converse that is also false.

Problem Set 3

1. Present your statements to fellow classmates. Be prepared to explain and justify why you consider your statements either true or false.
2. Write a few sentences discussing which types of statements were easiest to agree on and which types provoked the most disagreement and discussion.
3. Which statements in Problem Set 1 and Problem Set 2 would you consider to be definitions? Why?

Materials:
One pencil or pen for each student.

Connections to Geometry TEKS:

(b.1) **Geometric structure**. The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

(C) demonstrates what it means to prove mathematically that statements are true;

(b.4**) Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

Teacher Notes

Scaffolding Questions:

Problem Set 1

- In writing the converse of a conditional statement: "What is the hypothesis?" and "What is the conclusion?"
- Guide students as necessary toward composing correct grammatical sentences rather than simply mechanically switching the hypothesis with the conclusion.
- Guide students as necessary toward providing specific, concrete counterexamples in order to determine the converse or to establish that statements are false.

  For the converse of the statement in Problem Set 1, question 1: "Can you think of a number that is divisible by 2, but that is not even?"

  In Problem Set 1, question 1: student response: "If it is divisible by two, then a number is divisible by 4." Guiding question: "Can you rewrite your sentence so that the subject (a number) is part of the hypothesis?"

  For Problem Set 1, question 10: students will justify that the converse of this statement is false with a general statement like: "Not all cars are SUVs." An appropriate question would then be: " Can you think of a specific car that has four wheels but is not an SUV?"

Problem Sets 1, 2, and 3

- Student examples often provoke debate among classmates about the relative merit and/or truth of counterexamples. They might not always agree that a statement is a valid counterexample.
- Students are asked to address this issue in Problem Set 3. Guide students as necessary toward making the distinction between non-mathematical statements, whose truth can depend on people's viewpoints or opinions, and mathematical statements, whose truth depends on logical reasoning.

Sample Solution:

Problem Set 1

1. **True**. Since 4 is divisible by 2, then numbers divisible by 4 must also be divisible by 2.

   Converse: If a number is divisible by 2, then it is divisible by 4.

   False: Numbers such as 2, 6,and 10 are divisible by 2, but not by 4.

2. **False**. If it is raining in Las Vegas, then it is not sunny throughout the entire state of Nevada.

   Converse: If it is sunny in Nevada, then it is raining in Las Vegas.

   False: Las Vegas would be sunny if it were sunny in Nevada; therefore it can't be raining in Las Vegas.

3. **True**. Since Las Vegas is part of the entire sunshine-drenched state of Nevada.

   Converse: If it is sunny is Las Vegas, then it is sunny in Nevada.

   False. It could be raining in Reno, Nevada while it is sunny in Las Vegas.

4. **False**. 300 is greater than -500 but not greater than 500.

   Converse: If a number is greater than 500, then it is greater than -500.

   True. All numbers greater than 500 are positive, and by definition all positive numbers must be greater than negative numbers like -500.

5. **True**. This is the definition of a teenager.

   Converse: If a person is between 13 and 19 years old, then that person is a teenager.

   True: This is the definition of a teenager.

6. **False**. For example, Segovia was a famous male classical guitarist.

   Converse: If a musician is a girl, then the musician plays classical guitar.

Texas Assessment of Knowledge and Skills:

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Bayou City Geometry

False: The Dixie Chicks do not play classical guitar, but they are girl musicians. (This is, of course, open to debate. Like many non-mathematical statements, we can't always be perfectly sure if they are true or false.)

7. **True**. Classical guitar players are all musicians.

   Converse: If a girl is a musician, then she plays classical guitar.

   False. For example, Gloria Estafan, who is a percussionist.

8. **False**. Students can argue that there are many ways to have intelligence which have nothing to do with reading.

   Converse: If a person is smart, then they read a lot.

   False: For the same reason as above. (This is another example of a non-mathematical statement whose truth is open to debate and interpretation.)

9. **True**. There are no even numbers that are not divisible by 2.

   Converse: If a number is divisible by 2, then it is even.

   True. There are no numbers that are even that are not divisible by 2. Or, True by definition of an even number.

10. **True**. Since all cars have 4 wheels and an SUV is a type of car.

    Converse: If a car has 4 wheels, then it is an SUV.

    False. A Volkswagen Beetle is not an SUV and it is a car with 4 wheels.

11. **False**. -2.

    Converse. If a number is greater than 1, then the number squared is greater than 1.

    True. Squaring a number greater than 1 makes the number even larger.

12. **True**. If B is the midpoint of $\overline{AC}$, then by definition, AB = BC.

    Converse: If the distance from point A to point B is the same as the distance from point B to point C, then B is the midpoint of $\overline{AC}$.

    False. $\overline{AB}$ and $\overline{BC}$ can be the legs of an isosceles triangle.

13. **True**. This is the definition of an octagon.

    Converse: If a polygon is an octagon, then it has eight sides.

    True. By definition, an eight-sided polygon is an octagon.

14. **False**. It could be a rectangle.

    Converse: If a parallelogram is a square, then it has a right angle.

True: All squares have right angles by definition.

15. **True**. This is the definition of a trapezoid.

    Converse: If a quadrilateral is a trapezoid, then it has exactly one pair of parallel sides.

    True. Definition of a trapezoid.

Problem Set 2

Answers will vary. Check Student Work.

1. Question 5, Problem Set 1 is a true conditional with a true converse.
2. Question 1, Problem Set 1 is a true conditional with a false converse.
3. Question 14, Problem Set 1 is a false conditional with a true converse.
4. Question 2, Problem Set 2 is a false conditional with a false converse.

Problem Set 3

1. and 2. Answers will vary. Students should be able to articulate that, in general, it's easier to decide the truth/falseness of mathematical statements. The terms in mathematical statements are precisely defined, and their truth depends on logic. In contrast, the truth/falseness of non-mathematical statements often depends on people's opinions and personal understanding of what words mean.

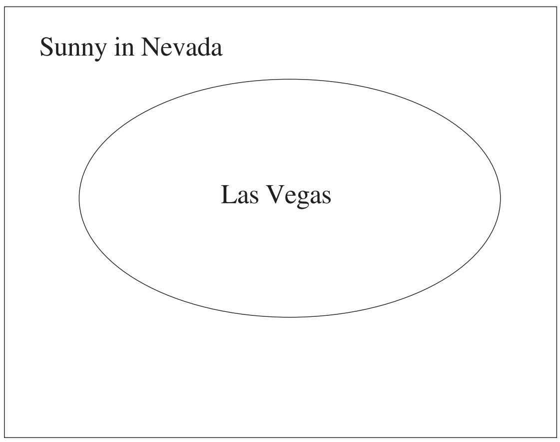
3. The statements that make good definitions are the conditional statements that have true converses.

Extension Questions:

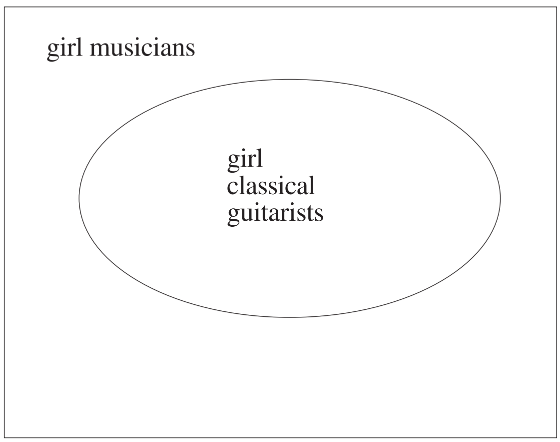
- Refer to questions 3, 7, and 10 in Problem Set 1. For each of these questions, represent the conditional statement in an Euler Diagram.

  Note: Euler (pronounced "oiler") diagrams are often called Venn Diagrams.

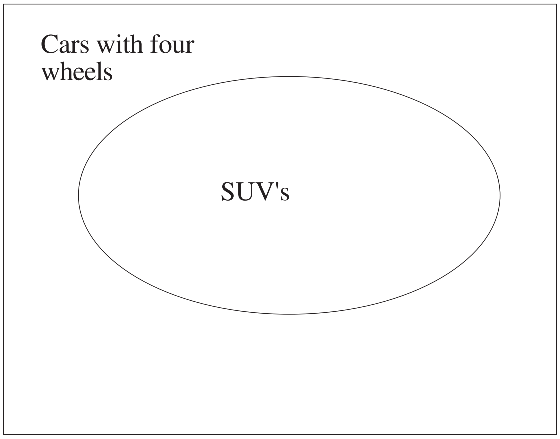
Problem 3



Problem 7



Problem 10



- Write definitions of the following terms as true conditional statements with true converses. Then, rewrite the definitions as biconditional statements.

  a) complementary angles

  b) isosceles triangle

  c) polygon

  a) If two angles are complementary, then the sum of their measures is 90°. Two angles are complementary if and only if the sum of their measures is 90°.

  b) If a triangle is isosceles, then it has at least 2 sides that are congruent. A triangle is isosceles if and only if it has at least two sides that are congruent.

  c) If a shape is a polygon, then it is a plane figure formed from 3 or more line segments, such that each segment intersects exactly 2 other segments, one at each endpoint, and no 2 segments with a common endpoint are collinear.

  A shape is a polygon if and only if it is a plane figure formed from 3 or more line segments, such that each segment intersects exactly 2 other segments, one at each endpoint, and no two segments with a common endpoint are collinear.

- Create and name your own object. It does not have to be mathematical, but it should be something you can draw. Then, write the definition of your object as a biconditional statement.

  Answers will vary. Check student work.

Going the Distance in Taxicab Land Assessment

Write a paragraph comparing and contrasting the characteristics of the geometric objects you studied in the Going the Distance in Taxicab Land lesson as they are represented on the traditional coordinate grid and as they are represented on a taxicab coordinate grid.

1. Be sure to include all of the following objects in your analysis:

    line segments
    circles
    perpendicular bisectors

2. Discuss whether you think the definitions or characteristics of these objects are valid and useful for both geometric systems.

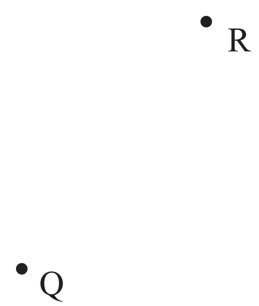
3. Can you think of situations that might be better represented in Taxicab geometry than in traditional geometry?

Going the Distance in Taxicab Land Lesson:
Describing and Analyzing Objects in Two Different Geometric Systems.

Activity 1:

Draw accurate sketches to represent the following definitions, theorems, or postulates, and answer the questions that follow.

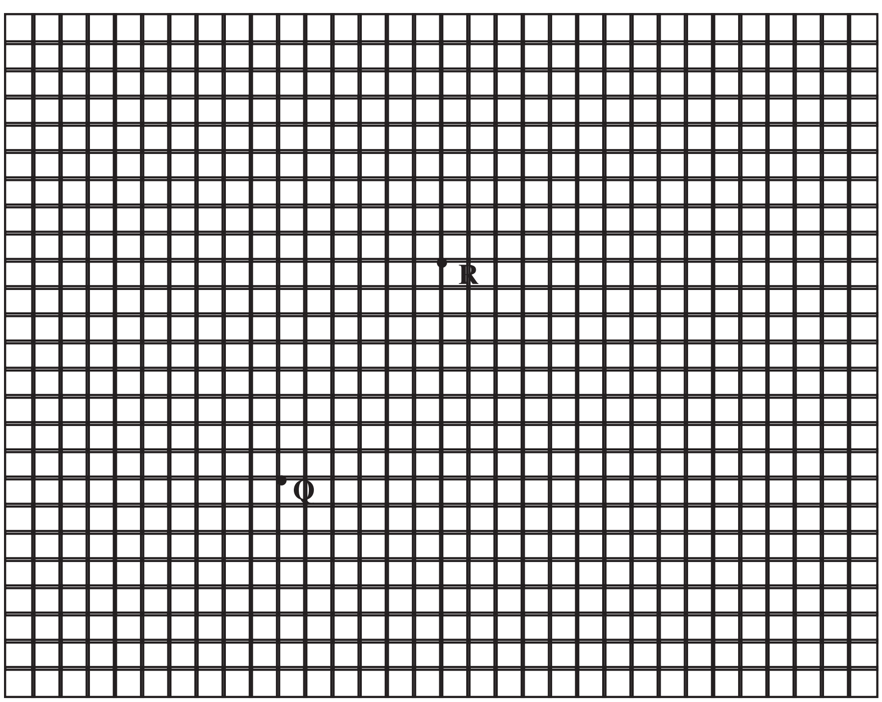
1. **Two points determine a line segment.**



   a) How many line segments can be drawn between the two given points?

   b) The line segment drawn between the two points represents the ________________________ distance between the two points.

2. Make the same sketch on a coordinate grid.



a) How many line segments can be drawn between the two points?

b) The line segment drawn between the two points represents the

__________________________ distance between the two points.

c) Does placing the geometric object on the coordinate grid change its characteristics? Explain your answer.

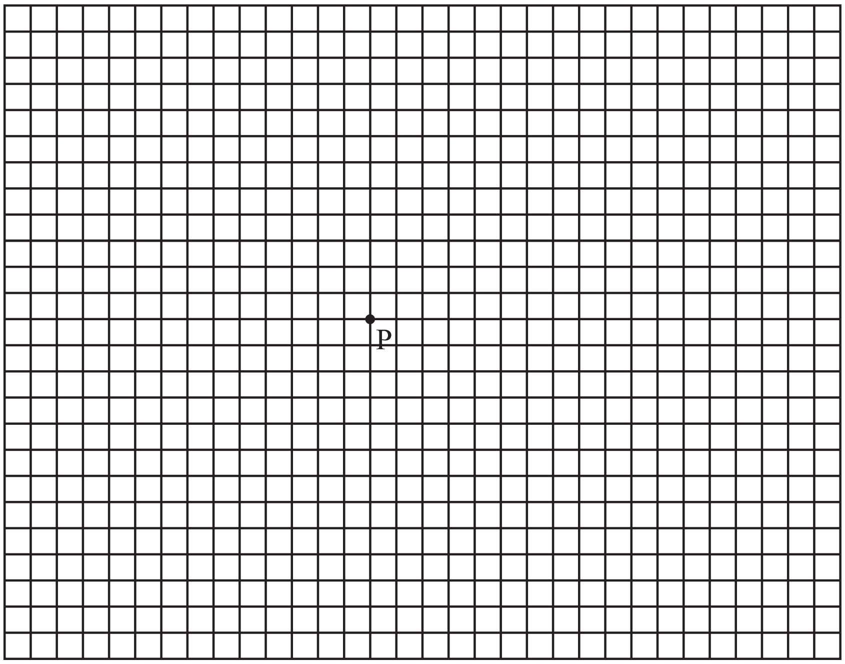
3. **A circle is the set of points in a plane that are the same distance from a given point in the plane.**

Sketch a circle with center at point P and a radius of 6 cm.

• P

a) How many different circles can you draw with this center and radius?

4. Make the same sketch on a coordinate grid. (For convenience, make the radius of the circle 6 grid units.)

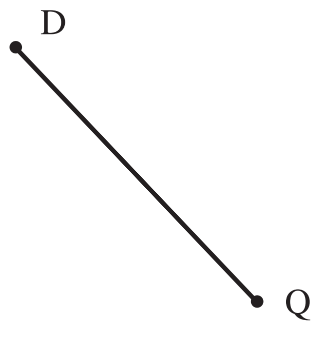


a) How many different circles can you draw with this center and radius?

b) Does placing the geometric object on the coordinate grid change its characteristics? Why?

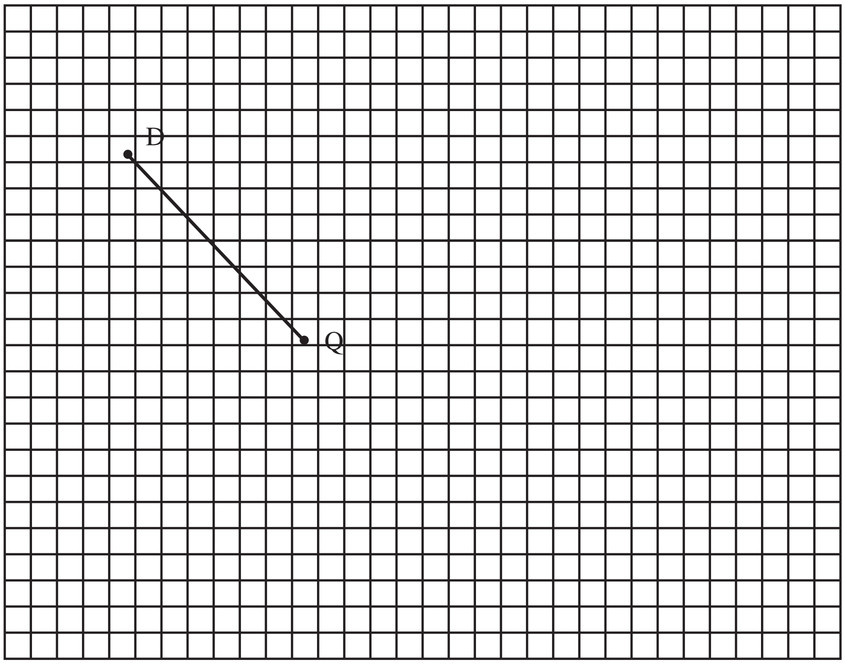
c) Does changing the location of the geometric object on the coordinate grid change its characteristics?

5. **A point is on a segment's perpendicular bisector if and only if it is the same distance from each of the segment's endpoints.** (Represent all the points that satisfy this requirement.)



a) How many different perpendicular bisectors is it possible to draw for the given line segment?

6. Make the same sketch on a coordinate grid.



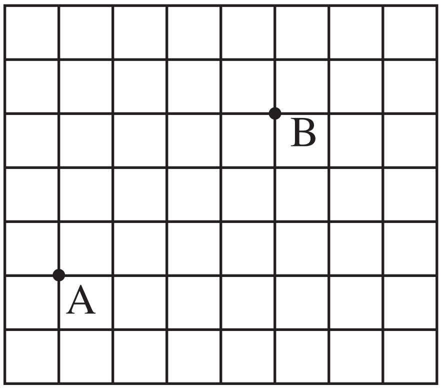
a) How many different perpendicular bisectors is it possible to draw for the given line segment?

b) Does placing the geometric object on the coordinate grid change its characteristics? Why or why not?

c) Describe how changing the position of the geometric object on the coordinate grid changes its characteristics.

Activity 2:

1. Determine the distance between points A and B on the coordinate grid below.



2. Imagine you are a taxicab driver and the coordinate grid above represents the grid of city streets you can travel on. Determine the **taxidistance** from point A to point B in Taxicab Land.

3. In what ways do you think the coordinate grid in Taxicab Land is different from the traditional coordinate grid you worked with in Activity 1?

Activity 3:

You have seen that representing a geometric object on a traditional coordinate grid does not change any of its characteristics. Neither does repositioning the object on the grid.

Your task is to analyze what happens when geometric objects are placed on a taxicab grid which has the following characteristics:

Points on a taxicab grid can only be located at the intersections of horizontal and vertical lines.

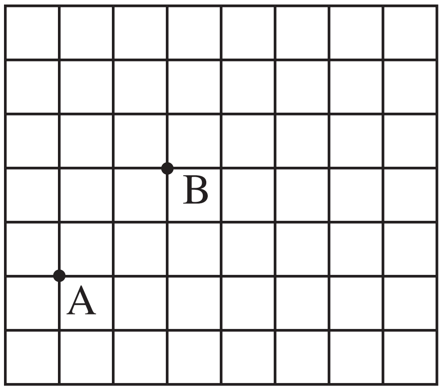
One unit will be one grid unit.

The numerical coordinates of points in **taxicab geometry** must therefore always be integers.

The **taxidistance** between 2 points is the smallest number of grid units that an imaginary taxi must travel to get from one point to the other. In Activity 2, the **taxidistance** between point A and point B is 7.

Draw accurate sketches on a **taxicab geometry** coordinate grid to represent the following definitions, theorems, or postulates, and answer the questions that follow.

1. **Two points determine a line segment.**

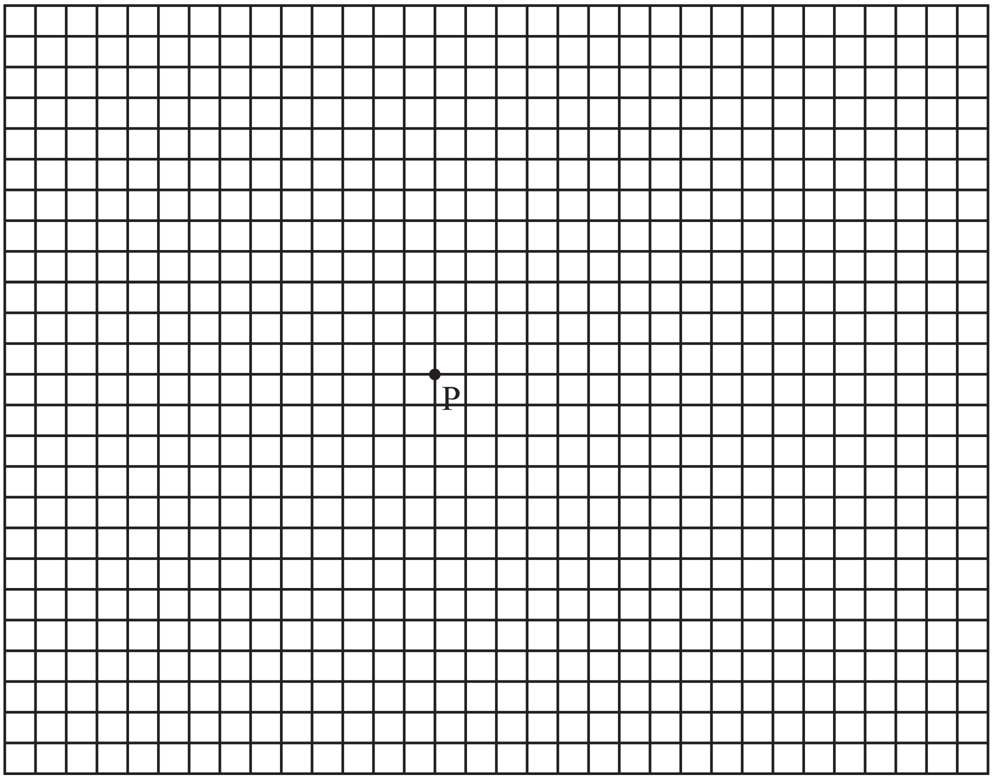


a) Is this the only line segment that can be drawn between the two given points? Explain.

b) The line segment(s) drawn between the two points represents the ___________________________________ taxidistance between the two points.

c) Does changing the position of the geometric object on the taxicab grid change its characteristics? Explain.

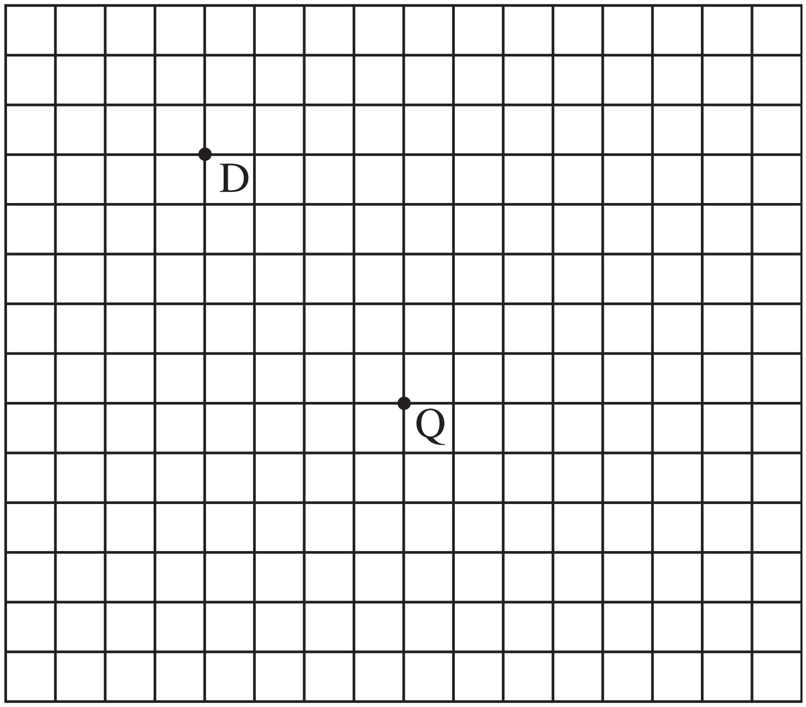
2. **A circle is the set of points in a plane that are the same distance from a given point in the plane.** (Sketch a circle with a radius of 6.)



a) How many different circles can you draw with this center and radius? Explain your answer.

b) Explain how changing the position of the geometric object on the taxicab grid changes its characteristics.

3. **A point is on a segment's perpendicular bisector if and only if it is the same distance from each of the segment's endpoints.** (Represent all the points that satisfy this requirement.)



a) Is the set of points you represented in your drawing the only perpendicular bisector for the two given endpoints? Explain.

b) Does changing the position of the geometric object on the coordinate grid change its characteristics? Explain.

Teacher Notes

Materials:
One compass and ruler per student.

Connections to Geometry TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(C) compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(C) develops and uses formulas including distance and midpoint.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures

The lesson is intended to allow students to develop "an awareness of the structure of a mathematical system," (See TEK b.1 (A)). If students are already familiar with taxicab geometry, the assessment may be used to evaluate their understanding of this mathematical system.

Scaffolding Questions:

Assessment

Encourage students to review all the information in the lessons. They may want to organize the information in a chart form before they write their summary paragraph.

Activity 1:

1.

- What does the word "determine" mean in the sentence: "Two points determine a line segment"?

2.

- Is the coordinate grid line segment drawn between point Q and R unique? Does it represent the shortest distance between those two points?

4. Students may notice that point P does not lie at a grid intersection.
   - Is it possible to draw a circle with a radius of 6 with point P positioned this way?
   - After students draw the initial circle, they may position point P at a grid point and redraw the circle.

6. Students may notice that the segment does not have endpoints that lie on grid intersections.
   - Is it possible to draw the perpendicular bisector with the points in their present position?
   - Students may reposition the segment so that its endpoints lie on grid intersections and then complete the sketch.

Activity 2:

1.

- Is it possible to find the distance between A and B by counting along the grid lines?
- What formula could we use to determine the distance between A and B?

2.

- In Taxicab Land is it possible to find the distance between A and B by counting along the grid lines?
- Is there more than one way to count out the distance?
- Which way (or ways) of counting out the distance do you think is correct?

3.

- Where can points be located on a traditional coordinate grid?
- Where can points be located on a taxicab coordinate grid?
- What does this tell us about the numerical coordinates of the points on each type of coordinate grid?
- How is the distance between two points on a taxicab coordinate grid different from the distance on a traditional coordinate grid?

Activity 3:

1. Students may notice that there are many ways to draw a minimum distance pathway between the two points.

- Do all these segments represent the shortest taxidistance between the two points?

2.

- Do the points you drew satisfy the definition of a circle?
- Should you connect the points by drawing along the grid lines so that it looks more like an enclosed shape?

about the properties of parallel and perpendicular lines;

(C) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

Texas Assessment of Knowledge and Skills:

Objective 7. The student will demonstrate an understanding of two and three dimensional representations of geometric relationships and shapes.

Connection to High School Geometry: Supporting the TEKS and TAKS Institute:

I. Structure: Bayou City Geometry

Student's Response to Assessment Question 2:

"In normal Geometry, a line is the shortest distance between two points. In Taxicab it is not. In normal Geometry circles have points in which the shortest distance from the center to each point is equal. In taxicab geometry, the circles are different. They are not curved. In normal geometry, all segments have a perpendicular bisector. In taxicab geometry only segments with an even taxicab distance have perpendicular bisectors. The definitions for geometric shapes are only useful in normal geometry, not taxicab. You would use taxicab geometry if you wanted to find out how far a car travel to get from one street corner to another."

3.

- Do you think there will be any points that are the same distance from both endpoints?

Sample Solution:

Assessment

Traditional Coordinate System

Line Segment:	Uniquely determined by two points. Represents shortest distance between two points.
Circle:	Uniquely determined by given center and radius. Characteristics do not change when repositioned.
Perpendicular Bisector:	Uniquely determined by endpoints of segment. Characteristics do not change when repositioned.

Taxicab Coordinate System

Line segment:	Not necessarily uniquely determined by two points. Represents shortest distance between two points.
Circle:	Uniquely determined by center and radius. Characteristics do not change when repositioned.
Perpendicular Bisector:	May not exist at all. If it does exist, however, then it is uniquely determined by endpoints of segment. Characteristics change when repositioned.

Students should realize that the definitions and characteristics may be more useful and familiar in the traditional system, but they are equally valid in both systems.

Any sort of problem where one needs to find locations along a grid of city streets might be more successfully represented using taxicab geometry.

Activity 1:

1. a) One

   b) The line segment drawn between the two points represents the shortest distance between the two points.

2. a) One

   b) The line segment drawn between the two points represents the shortest distance between the two points.

   c) No. There is one and only one line segment between the two points, and it represents the shortest distance between them.

3. Draw circle with a compass.

   a) Only one circle may be drawn with a given radius.

4. Circle can be drawn on coordinate grid with a radius of 6 grid units by using a compass.

   If students subsequently reposition point P on a grid point, then they can sketch the circle by counting grid units, make a ruler in grid units, etc.

   a) One

   b) No. There is only one circle with the given radius and center.

   c) No. The coordinate name of the point changes but the properties of the circle stay the same.

5. Compass and straightedge construction is the most accurate way to do this.

   a) One

6. Compass and straightedge construction is still an appropriate way to do this. If students reposition points D and Q to the nearest grid intersections, then the 10 unit segment's midpoint will also fall on a grid intersection.

   a) One

   b) No. There is only one set of points that satisfy the requirement of being the same distance from the endpoints of segment $\overline{DQ}$.

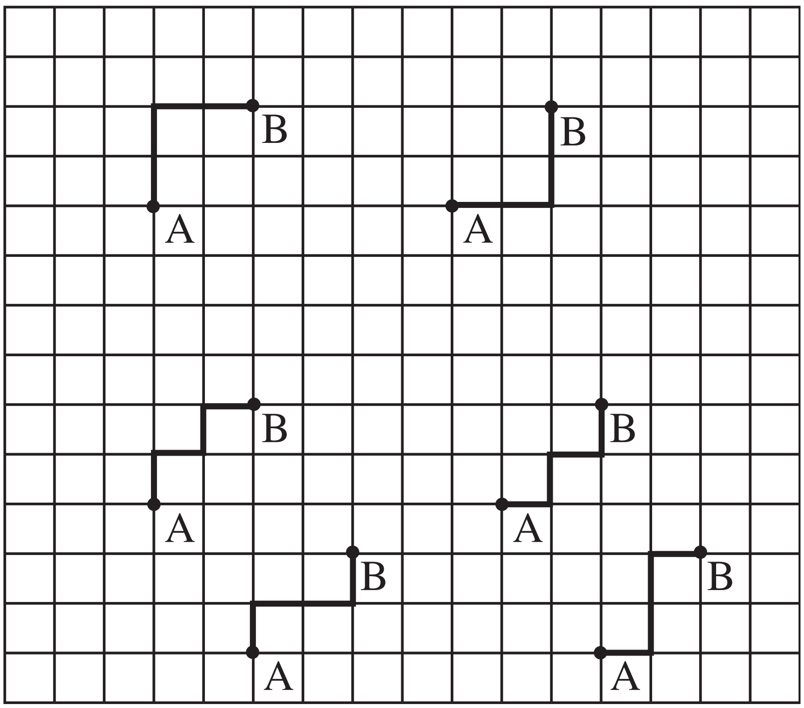
   c) No

Activity 2:

1. $\overline{AB}$ is the hypoteneuse of a right triangle with sides of length 3 units and 4 units. By the Pythagorean Theorem the length of $\overline{AB}$ is 5 units.
2. The taxi driver must travel on the streets so he would travel 3 blocks vertically and 4 blocks horizontally. He would travel a total distance of 7 blocks.
3. Points on a taxicab grid can only be located at grid intersections. This makes their numerical coordinates integers. The taxidistance between two points must always be an integer.

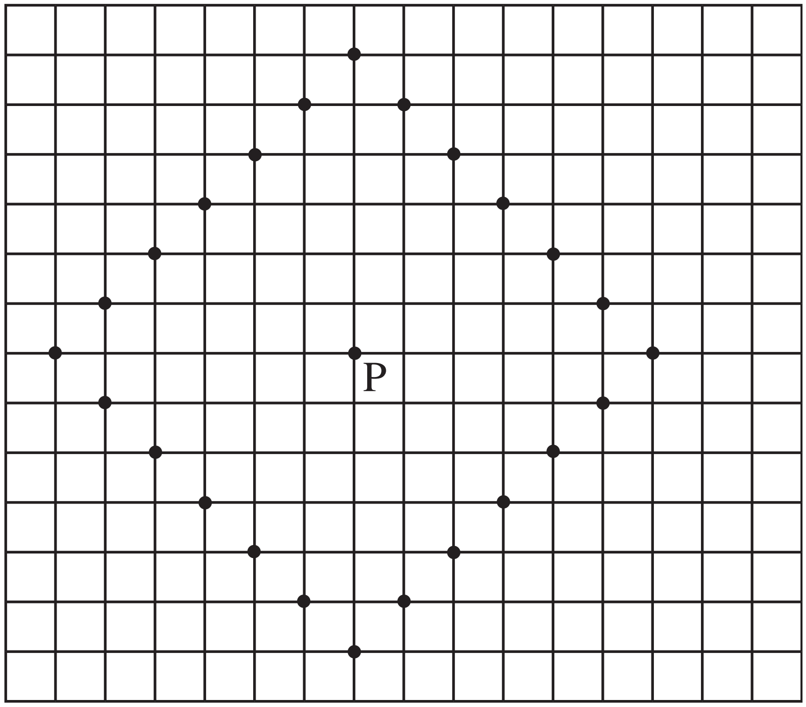
Activity 3:

1. a) There are six minimum distance pathways that can be drawn between the two points.



b) The line segment(s) drawn between the two points represent the <u>shortest</u> taxidistance between the two points.

c) Yes. Students should realize that if they position the two points along either a horizontal or a vertical grid line, then there will only be one minimum distance segment between them.

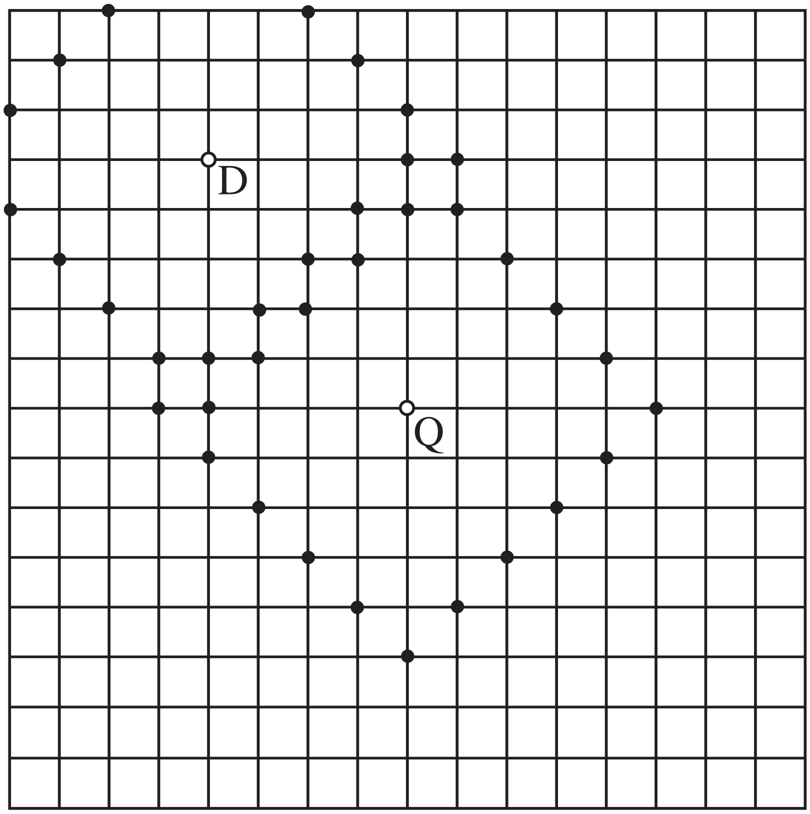


2. a) There is only one set of points that are all 6 units from P. They are points that lie on "street corners," such that the sum of the horizontal and vertical distances is 6 units.

   b) The coordinates of the points would change, but the shape of the figure would remain the same.

3. a) There are no points that satisfy the requirement of being the same distance from points D and Q. The distance from point D to point Q is 9 units.

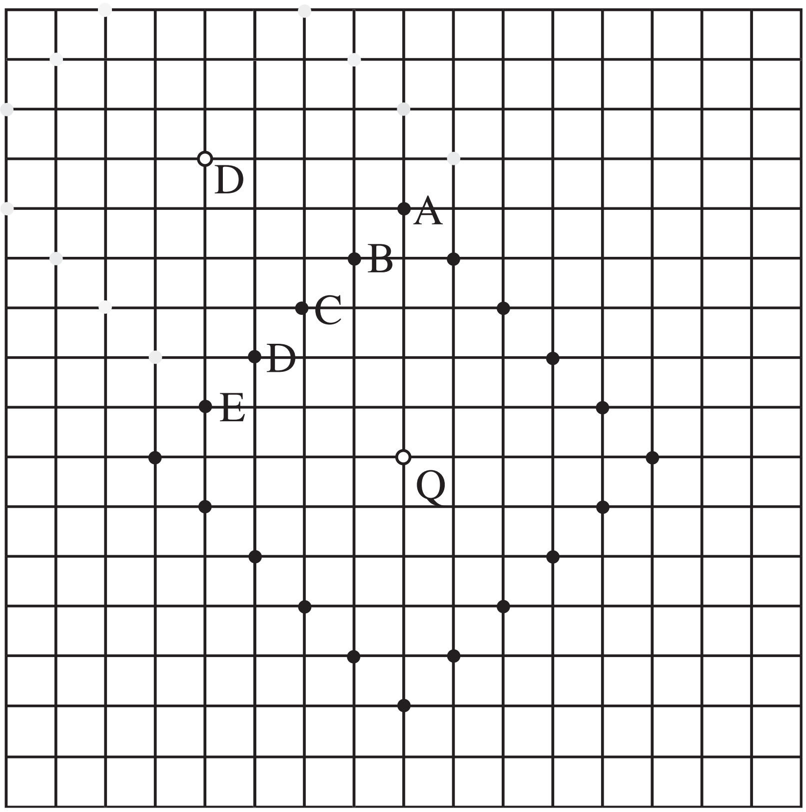
   The graph shows the set of points that are 5 units from D and the set of shaded points that are 5 units from Q. There are no common points. The two taxicab circles will not intersect.



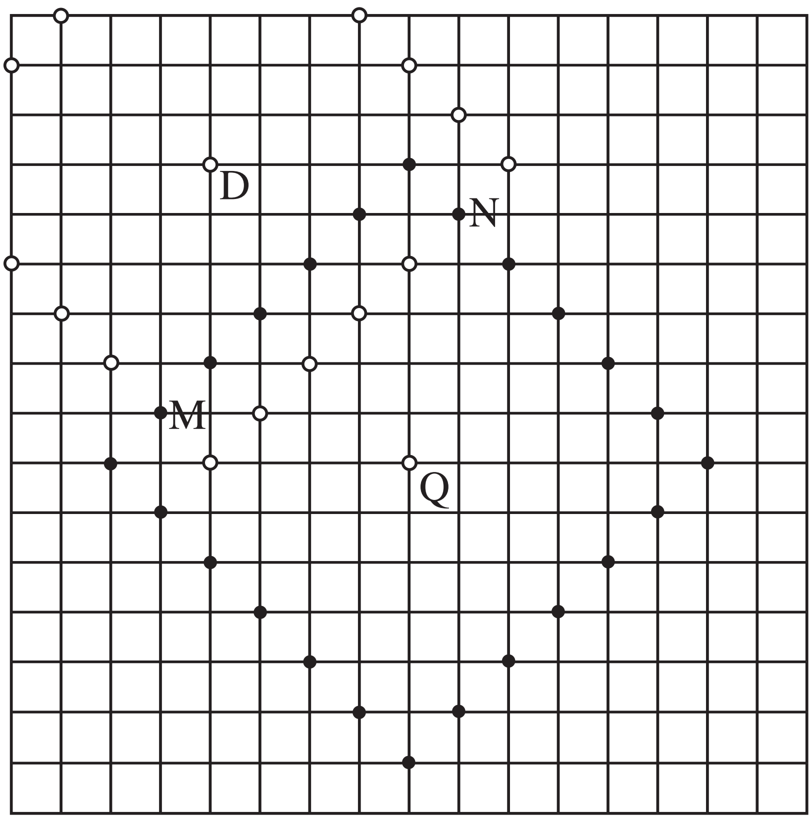
b) Students may experiment and discover that if the taxidistance between the two points is odd, then there will be no perpendicular bisector.

If, however, the taxidistance between the two points is even, there will be a perpendicular bisector, and it can take on a variety of configurations, depending on how the points are positioned.

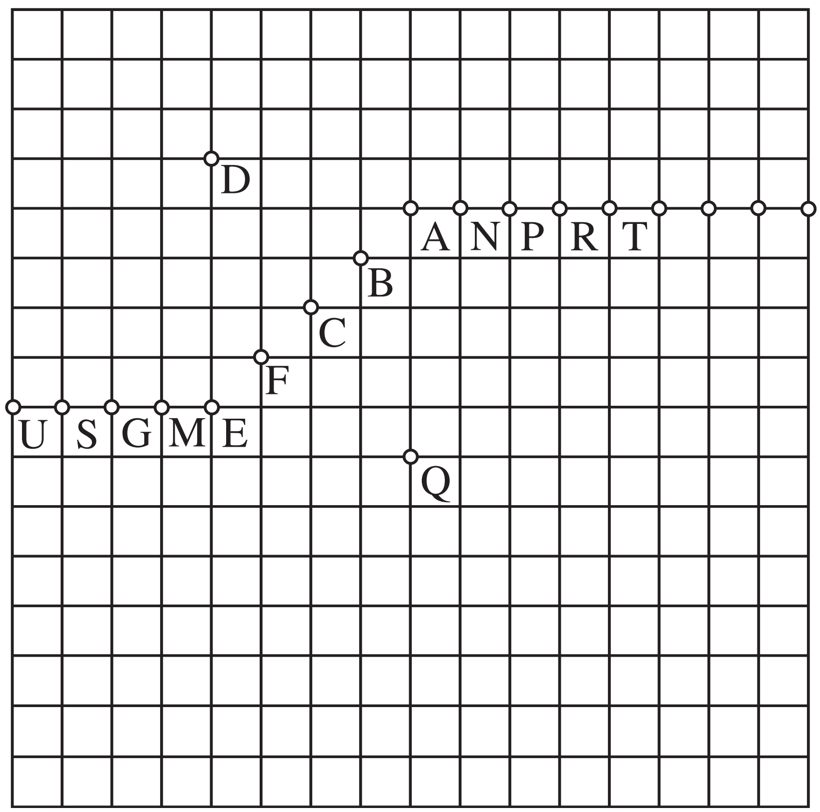
D and Q in this graph are 10 units apart. The next graph shows the two circles that have a radius of 5 units and centers D and Q. The points that the two circles have in common (A, B, C, D, and E) are all 5 units from both D and Q. Then A, B, C, D, and E are points on the perpendicular bisector of D and Q.



The next graph shows the two circles that have a radius of 6 units and centers D and Q. The points that the two circles have in common, M and N, are 6 units from both D and Q.



This next graph shows the points A, B, C, D, and E , 5 units from D and Q, M and N (six units from D and Q), P and Q (seven units from D and Q), R and S (eight units from D and Q), T and U (nine units from D and Q). This collection of points is a part of the set of points equidistant from D and Q.



Extension Questions:

- For each pair of points given, decide if it is possible to draw a perpendicular bisector in a taxicab coordinate system.

(1, 3) and (5, 3)	*yes*
(-2, 0) and (-6, -4)	*yes*
(5, -2) and (2, 1)	*yes*
(-3, 3) and (2, 1)	*no*
(5, -2) and (-3, 3)	*no*
(5, 3) and (5, -1)	*yes*

- Write a conjecture about what must be true in order for it to be possible to draw a perpendicular bisector in a taxicab coordinate system.

  The taxidistance between the two endpoints must be an even number.

- Experiment with circles of varying radii on a taxicab coordinate system. Write a conjecture about the value of π in taxicab geometry.

  The value of π in taxicab geometry is 4.