Chapter 3:
Properties and Relationships of Geometric Figures

Introduction

Chapter three's set of assessments requires students to analyze the properties and describe relationships in geometric figures, including parallel and perpendicular lines, circles and lines that intersect them, and polygons and their angles. The students will use explorations to formulate and test conjectures.

> Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, computation in problem solving contexts, language and communication, connections within and outside mathematics, and reasoning, as well as multiple representations, applications and modeling, and justification and proof. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999).

Angle Bisectors and Parallel Lines

Construct two parallel lines. Draw a transversal. Bisect two interior angles on the same side of the transversal.

Develop and write a conjecture about the intersection of the angle bisectors.

Materials:
Patty paper and one straightedge per student

Or geometry software

Connections to the Geometry TEKS:

(e.2) **Congruence and the geometry of size**. The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines;

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures.

Teacher Notes

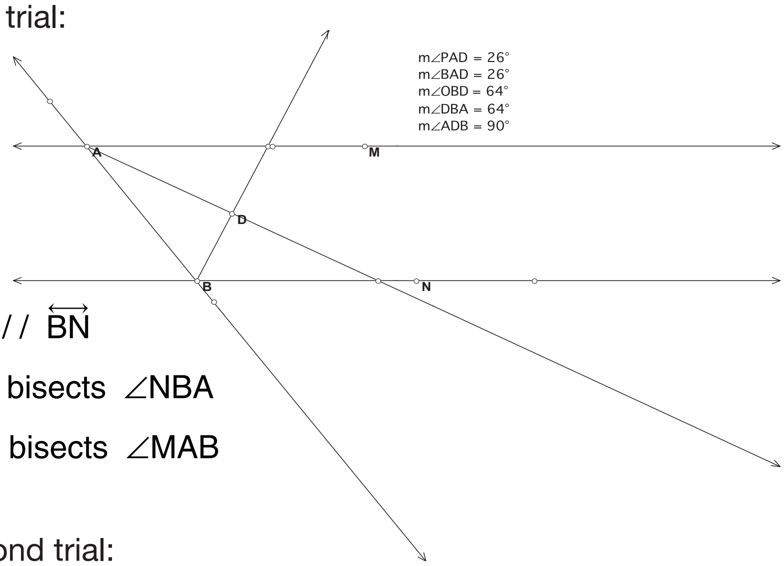
Scaffolding Questions:

- What does the given information imply about the angles in the figure?
- Which special properties of parallel lines can you use to describe the relationships between or among the angles?

Sample Solutions:

Construct the figure using geometry software. Measure the resulting angles.

First trial:

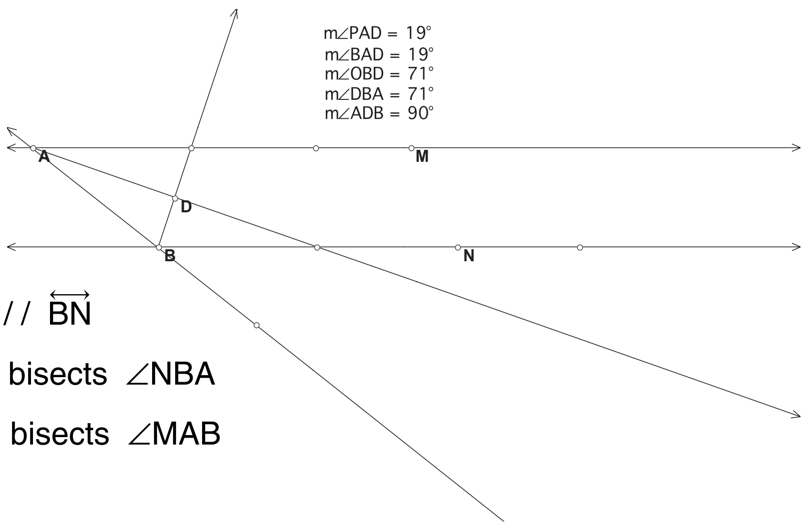


$\overleftrightarrow{AM}$ // $\overleftrightarrow{BN}$

$\overrightarrow{BD}$ bisects ∠NBA

$\overrightarrow{AD}$ bisects ∠MAB

Second trial:



$\overleftrightarrow{AM}$ // $\overleftrightarrow{BN}$

$\overrightarrow{BD}$ bisects ∠NBA

$\overrightarrow{AD}$ bisects ∠MAB

Conjecture: The two angle bisectors are perpendicular.

Extension Questions:

- Prove your conjecture using an axiomatic approach.

  $m\angle BAD = \frac{1}{2} m\angle BAM$ *because* $\overrightarrow{AD}$ *bisects* $\angle BAM$.

  $m\angle MBA = \frac{1}{2} m\angle ABN$ *because* $\overrightarrow{BD}$ *bisects* $\angle ABN$.

  The two interior angles on the same side of the transversal of two parallel lines are supplementary.

  Thus, $m\angle ABN + m\angle BAM = 180^\circ$

  $\frac{1}{2} m\angle ABN + \frac{1}{2} m\angle BAM = 90^\circ$

  By substitution:

  $m\angle MBA + m\angle BAD = 90^\circ$ *(Equation 1)*

  The sum of the angles of a triangle is 180°, *so in* ΔBDA

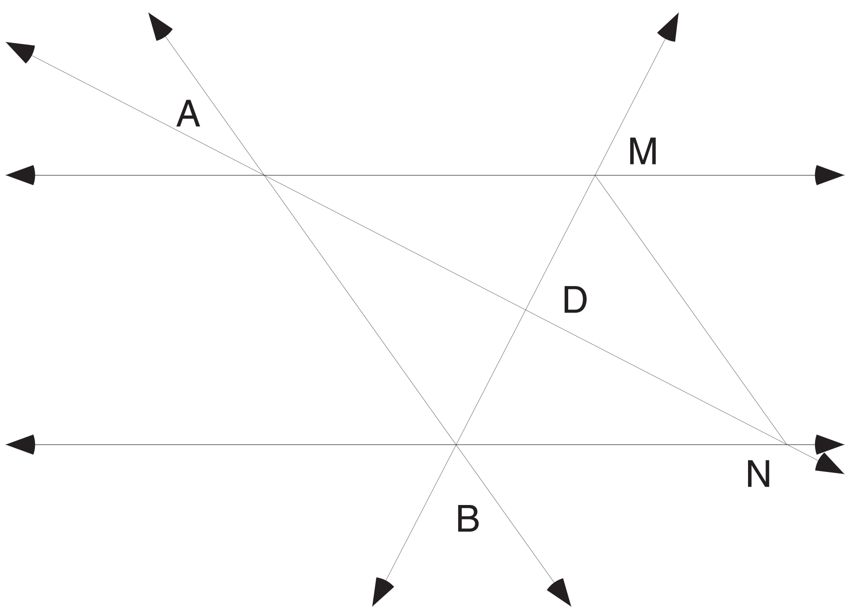
  $m\angle MBA + m\angle BAD + m\angle ADB = 180^\circ$. *(Equation 2)*

  Subtract Equation 1 from Equation 2.

  $m\angle ADB = 90^\circ$

Therefore $\angle ADB$ *is a right angle and the lines are perpendicular.*

- Connect the two points that are the intersection of the angle bisectors and the parallel lines. Describe the resulting triangles and the quadrilateral AMNB.



Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Triangles Tell All

The four triangles are congruent, and AMNB is a rhombus. The triangles AMD and ABD are congruent right triangles with common leg $\overline{AD}$ and congruent angles $(\angle DAB \cong \angle DAM)$.
AM = AB

Also, the triangles NBD and ABD are congruent right triangles with common leg $\overline{BD}$ and congruent angles $(\angle DBA \cong \angle DBN)$.
BN = AB

Therefore, AM = AB = BN.
Because AM is parallel to BN and AM = BN, the figure is a parallelogram.
AB = MN because they are opposite sides of the parallelogram.
MN = AM = AB = BN
The figure is a rhombus.

Another way to conclude that it is a rhombus is to realize that it is a parallelogram with perpendicular diagonals.

Student Work Sample

This student's work shows the use of a compass to construct the parallel lines and bisect the angles.

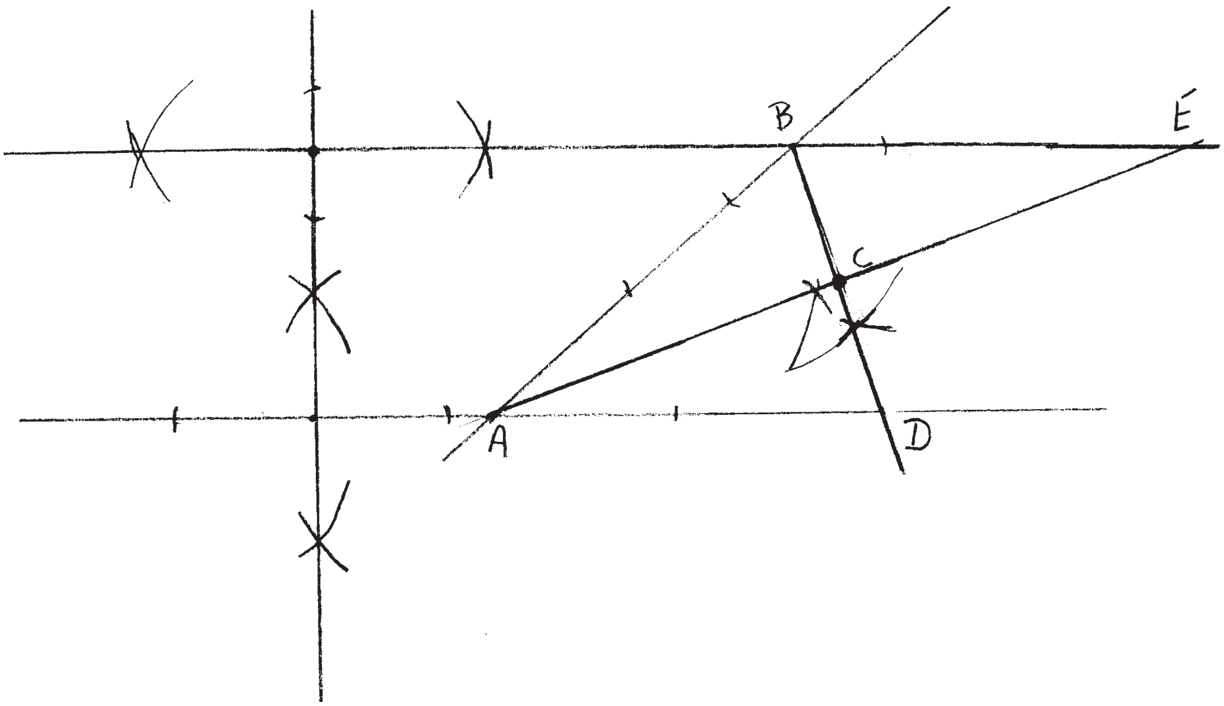
His work exemplifies many of the criteria on the solution guide, especially the following:

- Identifies the important elements of the problem.

  The student understands what the problem requires. He understands the lines referenced in the problem--the angle bisectors and writes his conjecture about these bisectors.

- States a clear and accurate solution using correct units.

This problem requires a construction and a conjecture. The student's construction is accurate, and his conjecture is clearly stated. Note the problem does not require that the student justify his conjecture, but he writes an explanation of how he came to his conjecture. The solution does not require units, but the student used the correct units in his justification.



$\angle BAD + \angle ABE = 180^\circ$

$\frac{1}{2}\angle BAD + \frac{1}{2}\angle ABE = 90^\circ$

$-\quad \frac{1}{2}\angle BAD + \frac{1}{2}\angle ABE + \angle C = 180^\circ$

$-\angle C = -90^\circ$

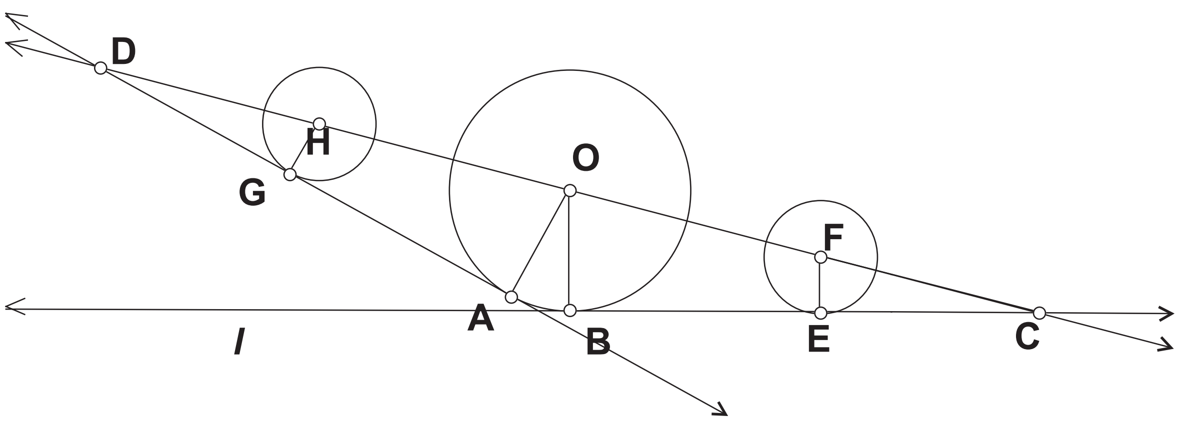
$\angle C = 90^\circ$

When a transversal intersects a pair of parallel lines, and two of the interior angles on the same side are bisected, the bisectors are perpendicular

Circles and Tangents

Maren constructed the figure below by going through the following process:

- Draw a line, *l*.
- Construct a circle with center O tangent to the line at point B.
- Construct a smaller circle with center F that is tangent to line *l* at point E. Draw $\overleftrightarrow{OF}$.
- Construct another circle on the other side of circle H, such that the new circle is congruent to circle F and has center H on $\overleftrightarrow{OF}$, such that HO = FO.
- Construct two angles congruent to angle COB, one with center O and another with center H as shown in the diagram.
- The intersection point with circle O and the angle ray is point A and the intersection point of the angle ray and circle H is G.
- Draw the line $\overleftrightarrow{GA}$.



Develop and write a conjecture about the relationships among the angles and the triangles in the figure and between the circles and the line $\overleftrightarrow{GA}$.

Teacher Notes

Materials:
One straightedge and compass per student

Or geometry software

Connections to the Geometry TEKS:

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them;

(f) **Similarity and the geometry of shape.** The student applies the concepts of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures.

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Scaffolding Questions:

If the student uses constructions to analyze the problem, the following questions may be asked.

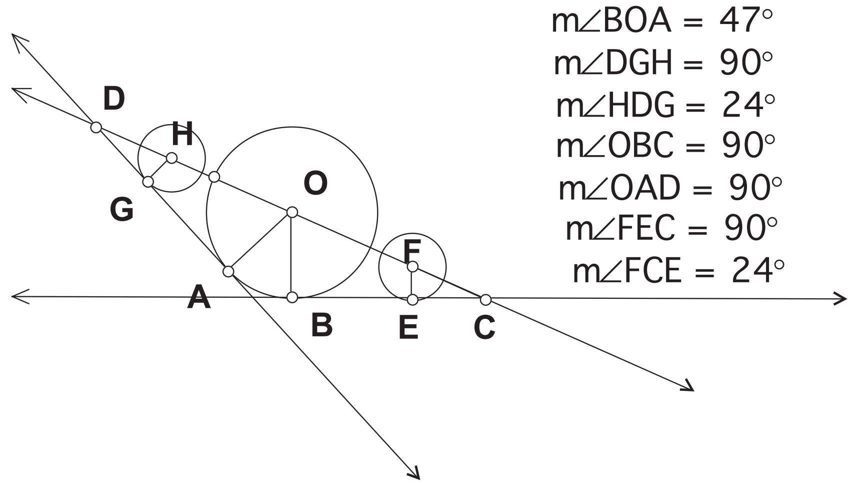
- What is the definition of the tangent to a circle?
- Given a line how can you construct a circle tangent to the line?
- What is the relationship between $\overline{GH}$ and $\overline{OA}$?
- Which segments in the figure must be congruent?
- Describe the relationship between $\angle DHG$ and $\angle CFE$.
- Explain what you know about the relationship between ΔDGH and ΔCEF.
- What is special about $\angle DGH$?
- What does it tell you about the figure?

If the student does not have access to geometry software, the problem may be approached analytically.

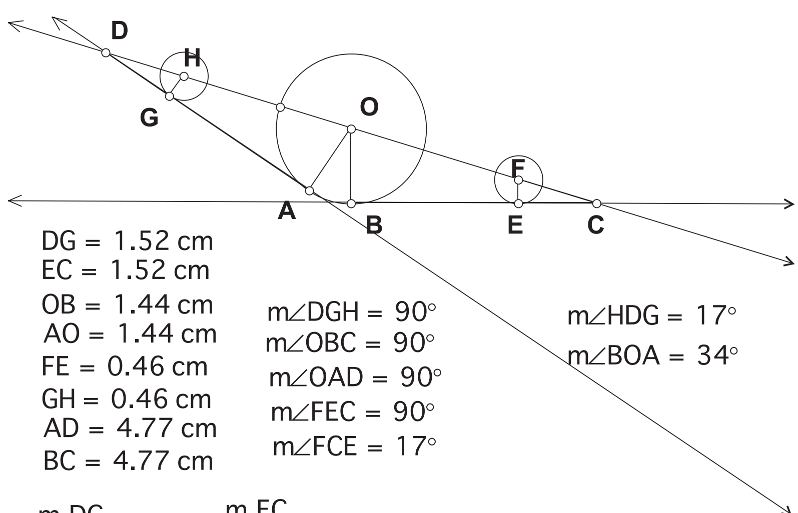
- What does the given information imply about the triangles in the figure?
- Which special properties of right triangles can you use to describe the relationships between or among the segments in the figure?
- What is the relationship between a central angle and its intercepted arc?

Sample Solutions:

Construct the figure using geometry software.



Try a second example. Measure the angles and the ratios of the corresponding sides of triangles.



$\frac{m\ DG}{m\ AD} = 0.32$ $\frac{m\ EC}{m\ BC} = 0.32$

$\frac{m\ FE}{m\ OB} = 0.32$ $\frac{m\ GH}{m\ AO} = 0.32$

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

Teacher's Comment:

"It's hard for some teachers to let students struggle before answering questions for the students. We as teachers need to allow them to struggle and ask their teammates before answering questions. This allows them to work on teamwork skills needed for the work place in their future."

Possible Conjectures:

$\overleftrightarrow{GA}$ is tangent to circles O and H.

The angle BOA is equal to twice the measure of the angle C or angle GDH.

$\Delta CFE \cong \Delta DHG$

$\Delta CBO \cong \Delta DAO$

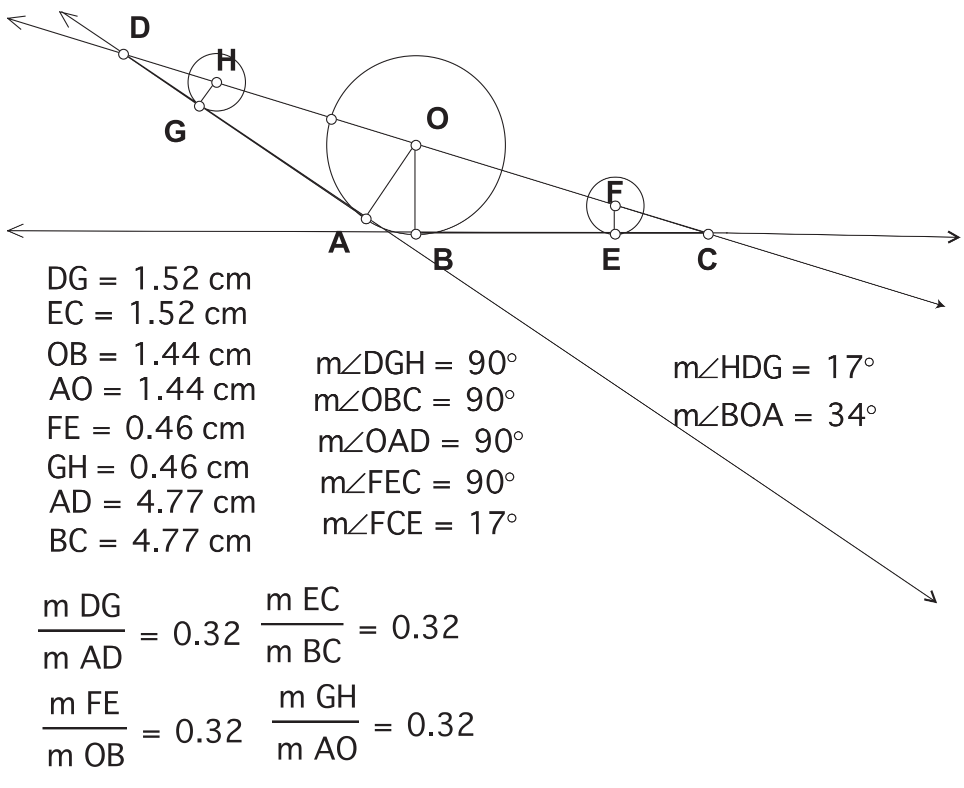
$\Delta CBO \approx \Delta CEF$

$\Delta DAO \approx \Delta DGH$

Extension Questions:

- How would your conjecture change if the circles were tangent or intersecting?

The relationships are the same no matter what the position of the circles.



- Prove that the measure of <AOB is twice the measure of <OCB.

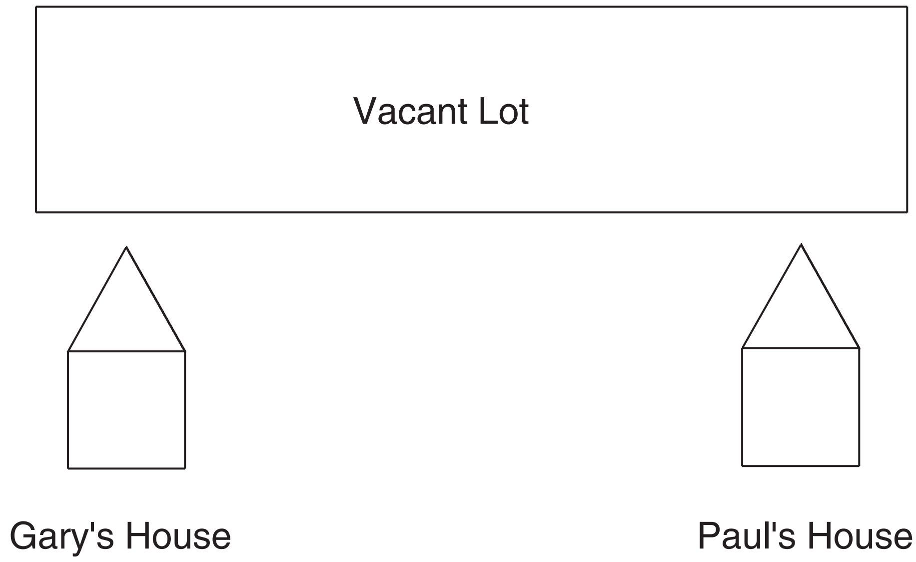
The proof will depend upon the theorems that have previously been developed in the geometry classroom. One possible proof follows:

$m\angle POA \cong m\angle BOT$	*Given (constructed to be congruent.)*
$m\angle \widehat{TOP} = 180^\circ$	*PT is a diameter of the circle,*
$m\angle POA + m\angle AOB + m\angle BOT = m\angle TOP$	*An angle is the sum of its parts.*
$m\angle BOT + m\angle AOB + m\angle BOT = 180^\circ$	*Substitution*
$\overleftrightarrow{BC}$ *is tangent to circle O*	*Given*
$\overline{OB} \perp \overleftrightarrow{BC}$	*Definition of tangent*
$m\angle OBC = 90^\circ$	*Definition of perpendicular*
$m\angle BOT + m\angle ECF = 90^\circ$	*Sum of the acute angles of a triangle is 90degrees.*
$2(m\angle BOT + m\angle ECF) = 180^\circ$	*Multiplication*
$m\angle BOT + m\angle AOB + m\angle BOT = 2(m\angle BOT) + 2(m\angle ECF)$	*Substitution*
$m\angle AOB = 2(m\angle ECF)$	*Subtraction*

The Clubhouse

Gary and Paul want to build a clubhouse on the lot next door to their houses. To be fair they decide it has to be the same distance from both of their houses. Describe all the possible locations of the clubhouse so that it is the same distance from both of their houses and in the vacant lot. Justify your answer.

Include a discussion of the properties of the lines and figures you have created. Can you generalize your conjecture?



Teacher Notes

Materials:
Tracing paper and one straightedge per student

Connections to Geometry TEKS:

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines;

(f) **Similarity and the geometry of shape**. The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures.

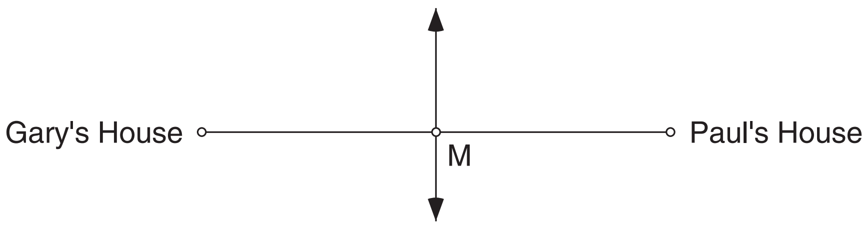
Scaffolding Questions:

- What are some strategies that you could use to solve this problem?
- Is there more than one location in the vacant lot that they could place the clubhouse?

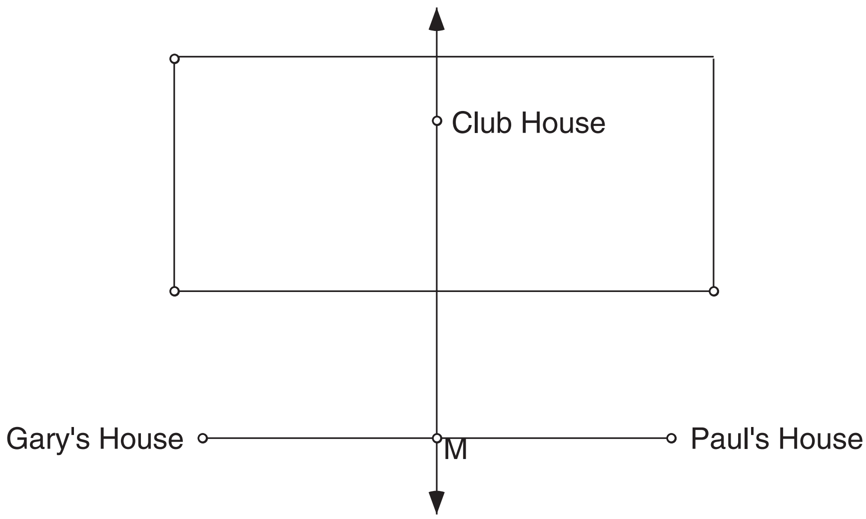
Sample Solution:

Draw a segment connecting the two houses.

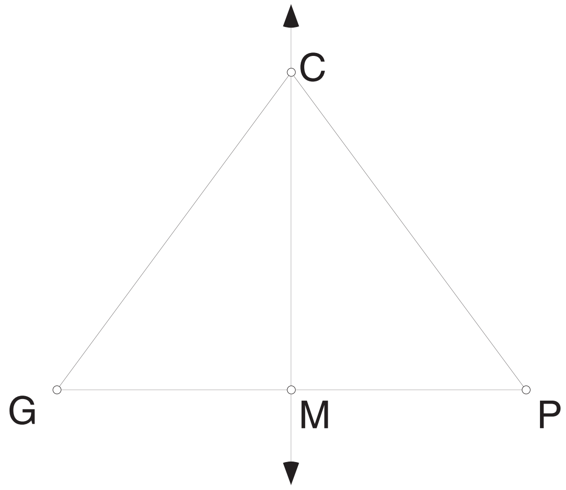
Construct the perpendicular bisector of the connecting segment by reflecting it onto itself.



Place the clubhouse anywhere on this perpendicular bisector and within the lot.



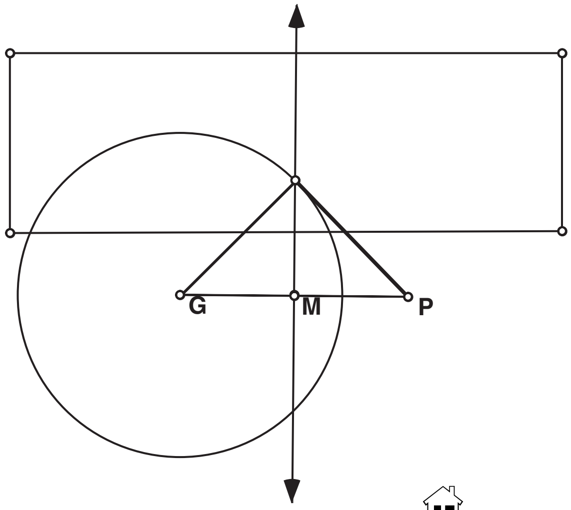
Draw segments connecting their houses to the clubhouse.



Because $\overleftrightarrow{CM}$ is the perpendicular bisector of $\overline{GP}$, $\overline{GM} \cong \overline{MP}$ and ΔGMC and ΔPMC are right triangles with common leg $\overleftrightarrow{CM}$. The two right triangles are congruent. Therefore, $\overline{GC} \cong \overline{PC}$. This could also be demonstrated by reflecting the segment connecting Gary's house to the clubhouse onto the segment connecting Paul's house to the clubhouse by folding along the perpendicular bisector. The segments are congruent so any point on the perpendicular bisector is equidistant from the endpoints of the segment it bisects.

Extension Questions:

- Suppose you have selected the position for the clubhouse. Are there other points in the vacant lot that is the same distance away from Gary's house as the distance between Gary's house and the clubhouse?



Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

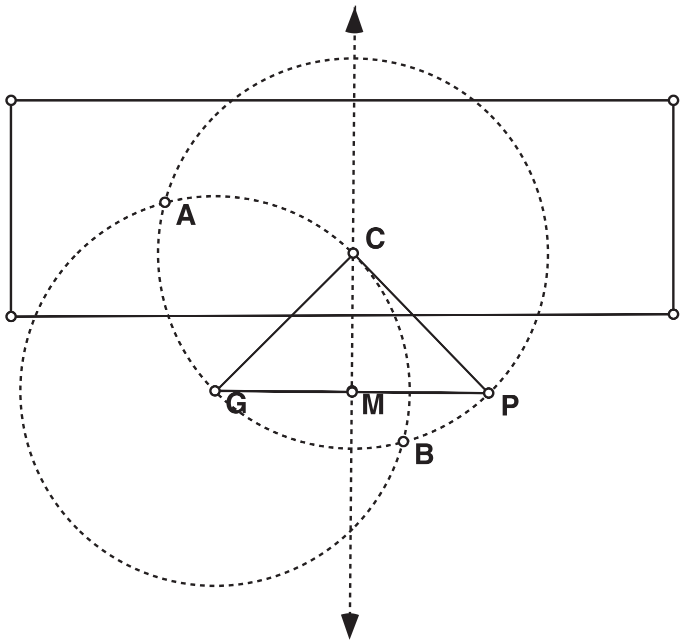
Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Triangles Tell All

Any point on the circle with center G and radius $\overline{GC}$ that is in the vacant lot will be the same distance from Gary's house as the clubhouse is from Gary's house.

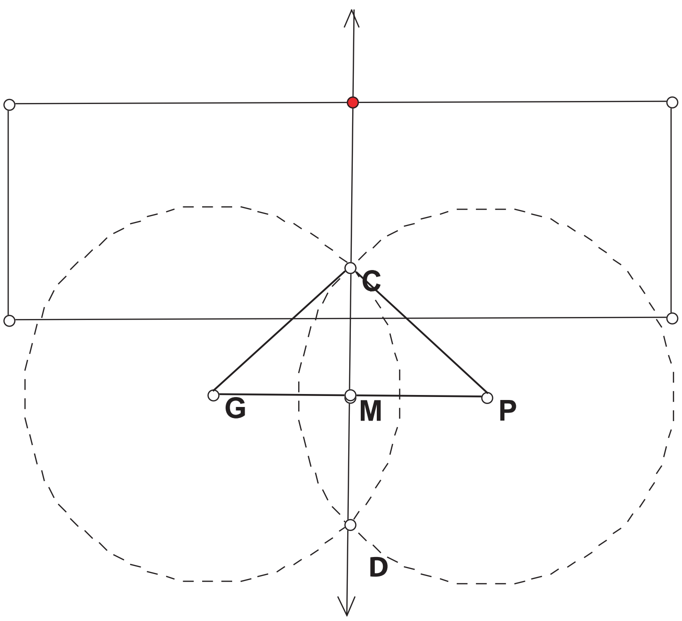
- Describe the location of the points such that the distance from the points to the clubhouse is the same as the distance from Gary's house to the clubhouse.

Using C as a center point and GC as the radius construct a circle. The intersections of this circle and circle G are the points, A and B, that are the same distance from G and from C. One of these points could be in the vacant lot.

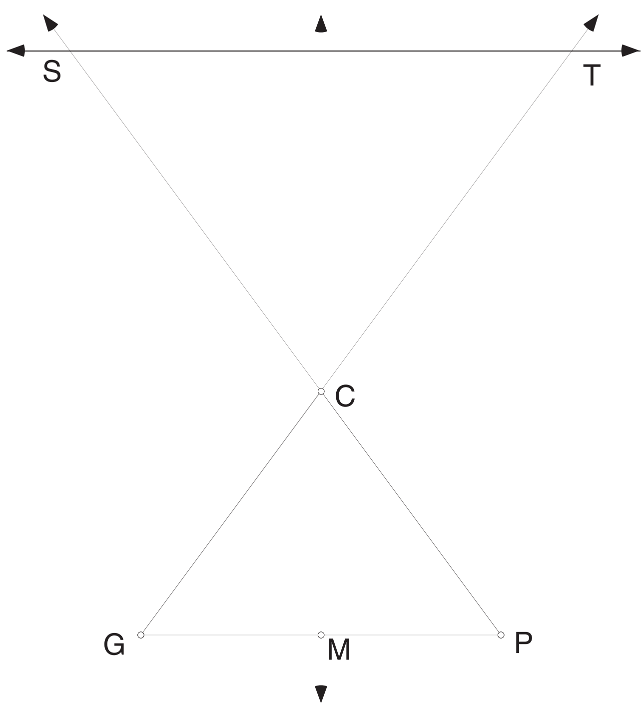


- Is there a point that is the same distance away from G and P as the clubhouse?

There is another point, D, that is on the intersection of the two circles with centers at G and P, but it would be in the opposite direction as C and would not be in the vacant lot.



- Suppose that a street runs parallel to the segment connecting Gary and Paul's houses, and it is on the other side of the vacant lot. Gary and Paul both walk from their own house through the clubhouse to the street as shown in the diagram. Tell me anything you can about the resulting figures.



Possible answers:

If $\overleftrightarrow{ST} \parallel \overleftrightarrow{GP}$, then the alternate interior angles are congruent.

$\angle S \cong \angle P$ *and* $\angle T \cong \angle G$

Vertical angles are congruent, $\angle SCT \cong \angle GCP$.

Two similar triangles are formed because three angles of one triangle are congruent to three angles of the other triangle.

$\Delta SCT \approx \Delta PCG$

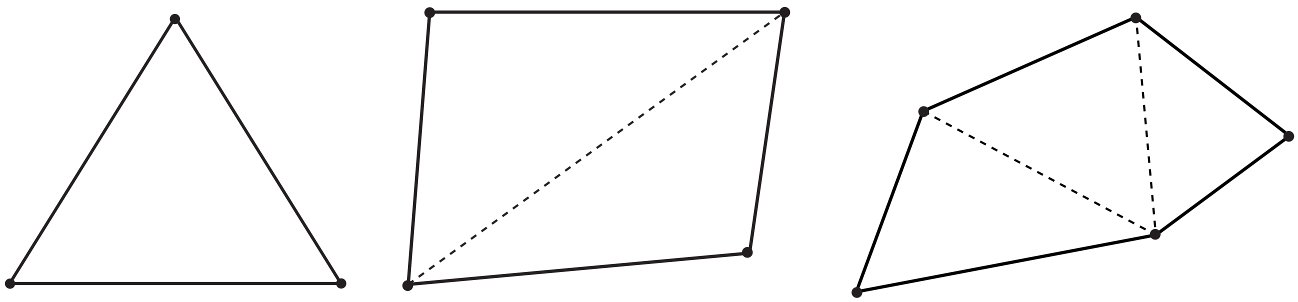
Because CG = CP, then triangle GCP is isosceles.

Since $\Delta SCT \approx \Delta PCG$, *triangle SCT is also isosceles and CS = CT.*

The distance Gary would walk would be equal to the distance Paul would walk.

Diagonals and Polygons

Write a conjecture about the relationship between the number of non-intersecting diagonals and the sum of the interior angles in a convex polygon. Justify your conjecture. Represent this function using symbols and a graph.



Polygon Name	Number of Sides	Number of Diagonals From One Vertex	Process to Find the Sum of the Interior Angles	Sum of the Interior Angles
		0		
		1		
		2		
		3		
		4		
		10		
		n		

Teacher Notes

Materials:
One straightedge, protractor, and graphing calculator per student

Connections to Geometry TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

Scaffolding Questions:

- How many triangles do the diagonals form?
- What do you know about the sum of the interior angles of a triangle?
- What do you already know about the sum of the interior angles of a polygon?
- How do the number of sides, the number of triangles and the number of diagonals from one vertex relate to the sum of the interior angles?

Sample Solutions:

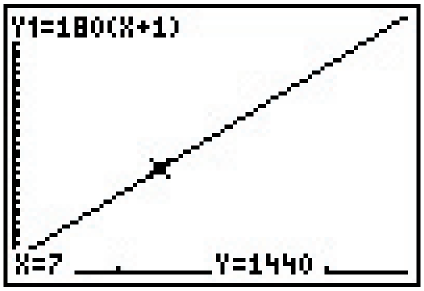
Draw the convex polygons; draw the diagonals. The only diagonals that satisfy the requirement to be non-intersecting are the ones drawn from one vertex. If another diagonal is drawn from a different vertex, it will intersect one of the diagonals from the first vertex. Count the number of triangles. There seems to be one more triangle than there are diagonals. To find the sum of their interior angles multiply the number of triangles by 180 degrees.

Complete the table.

Polygon Name	Number of Sides	Number of Diagonals From One Vertex	Process to Find the Sum of the Interior Angles	Sum Interior Angles
Triangle	3	0	180 or 180(0+1)	180
Quadrilateral	4	1	180(2) or 180(1+1)	360
Pentagon	5	2	180(3) or 180(2+1)	540
Hexagon	6	3	180(4) or 180(3+1)	720
Heptagon	7	4	180(5) or 180(4+1)	900
		10	180(10+1) or 180(11)	1980
n-gon		*n*	180(n+1)	

The number of triangles is 1 more than the number of diagonals. The sum of the interior angles of a convex polygon is 180 times 1 more than the number of non-intersecting diagonals. If *s* represents the sum of the angles of the polygon, and *n* represents the number of non-intersecting diagonals, then

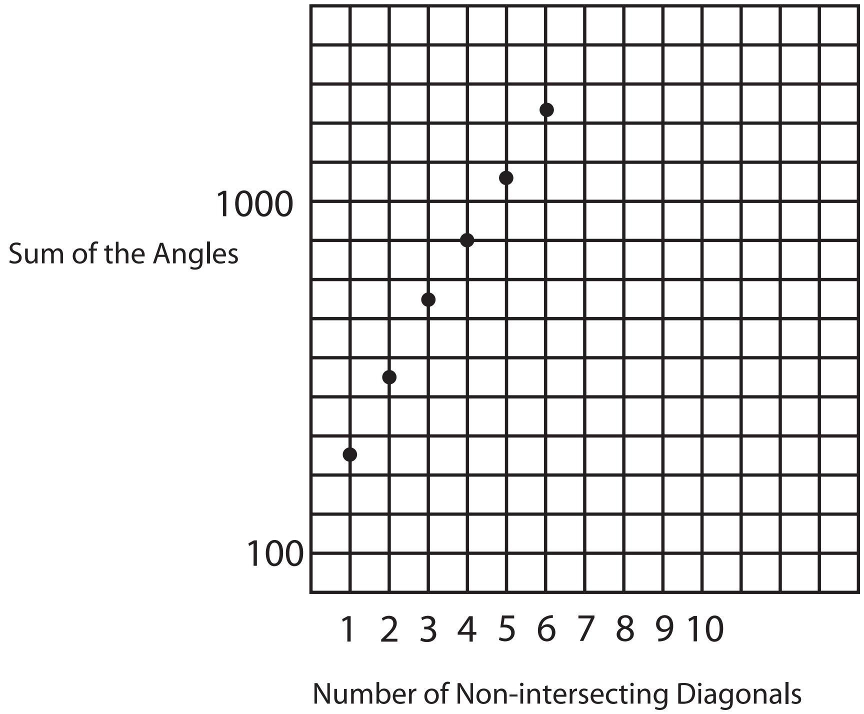
$s = 180(n + 1)$.



Extension Questions:

- What are the restrictions on the domain of the function?

The domain values must be the counting numbers. The graph should be a set of points.



Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Additional objectives are addressed by the Extension Questions that follow the sample solution:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

- How does the number of diagonals from one vertex relate to the number of sides?

There are 3 fewer diagonals than sides. If n is the number of diagonals, the number of sides is expressed as n + 3.

- Will there be a polygon such that the sum of the angles of the polygon is 2000 degrees? Explain why or why not.

The table shows that the polygon with 10 diagonals has a sum of angles of 1980 degrees and the polygon with 11 diagonals has a sum of angles of 2160 degrees. Since the number of diagonals must be a whole number there is no polygon with an angle sum of 2000 degrees.

X	Y_1	
8	1620	
9	1800	
10	1980	
11	2160	
12	2340	
13	2520	
14	2700	

X=10

- Use your table or graph to determine the sum of the angles of a polygon with 20 diagonals.

  The sum of the angles of a polygon with 20 diagonals is 3780 degrees.

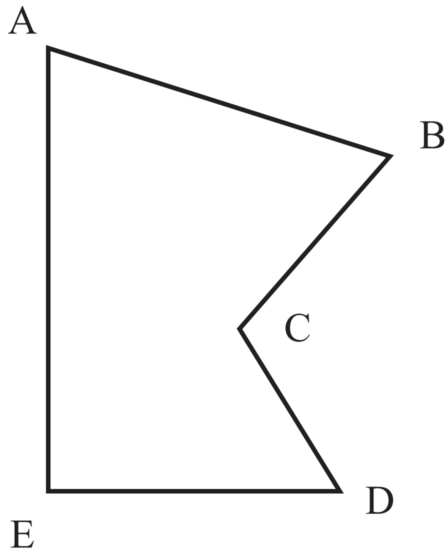
X	Y_1	
15	2880	
16	3060	
17	3240	
18	3420	
19	3600	
20	3780	
21	3960	

X=20

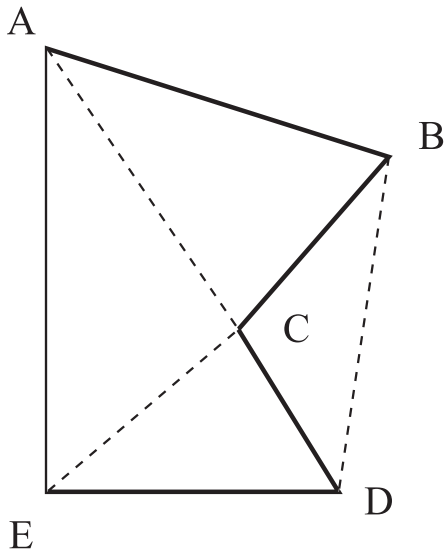
- Does the relationship hold true if the polygons are concave?

The relationship does not hold true for concave polygons. See the example below.

This is a concave polygon. A diagonal is a segment that has endpoints that are non-adjacent vertices of the polygon.



Draw non-intersecting diagonals.



If the formula is applied using the number 3, the sum of the interior angles is (3+1)180 =4(180) or 720 degrees. The sum of the interior angles of this polygon is not 720 degrees. To determine the sum, note that the quadrilateral is composed of three triangles, ΔACE, ΔACB, and ΔECD. The sum of the angles of the 3 triangles is 3(180) or 540 degrees. The formula does not work for this concave polygon.

Student Work Sample

This student completed the given table and then wrote the explanation and constructed the graph on the next page.

- Evaluates reasonableness or significance of the solution in the context of the problem.

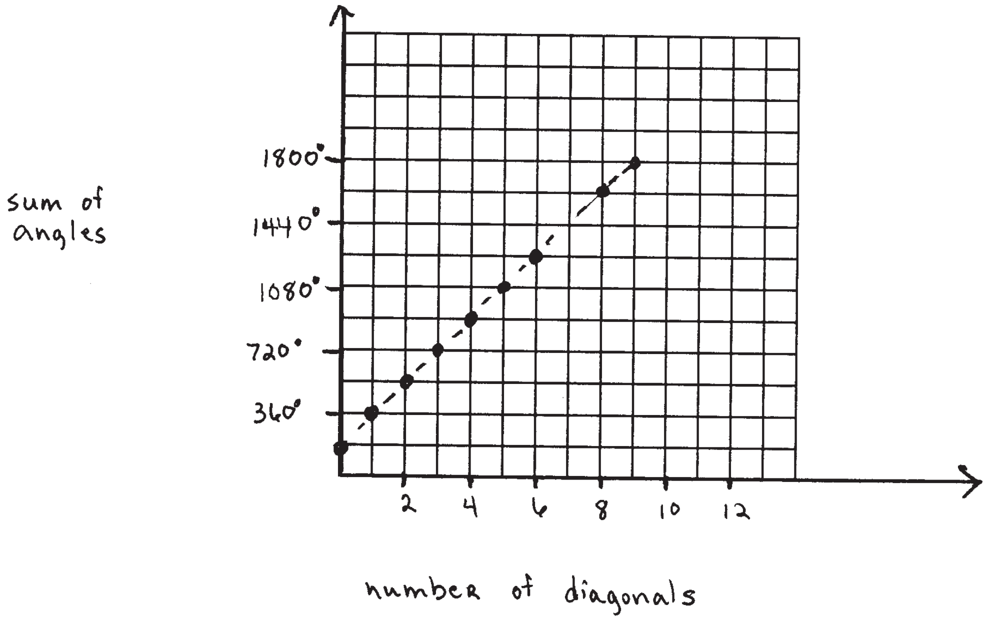
The graph is an accurate representation of the situation. The student plots points on the graph and shows a dashed line to indicate that the whole line does not represent the situation. (The number of diagonals must be a whole number.)

- Uses appropriate terminology and notation.

In his conjecture, the student identifies the variables, describes the relationships using an accurate verbal description of the relationship between the variables.

Diagonals and Polygons

A polygon with n diagonals from each vertex can be split into n + 1 triangles. Since triangles have an interior angle sum of 180°, and the triangles interior ∠s make up the larger polygons interior ∠s, the sum of the interior angles is 180° times the number of triangles formed by the diagonals of one vertex, or (the number of diagonals from 1 vertex + 1) times 180°.



The Most Juice

You are in charge of buying containers for a new juice product called Super-Size Juice. Your company wants a container that provides the greatest volume for the given parameters.

- A six-inch straw must touch every point on the base with at least one inch remaining outside of the container.
- The base of the container must either be a square, 4 inches by 4 inches, or a circle that would be inscribed in that square.
- The hole that the straw is inserted into in the top of the container must be exactly one inch from the side of the cylinder or from both sides of the prism.

You have two containers to choose from—a rectangular prism or a right cylinder. Which one would you choose? Justify your answer.

Materials:
One straightedge and one graphing calculator per student

Connections to Geometry TEKS:

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

Teacher Notes

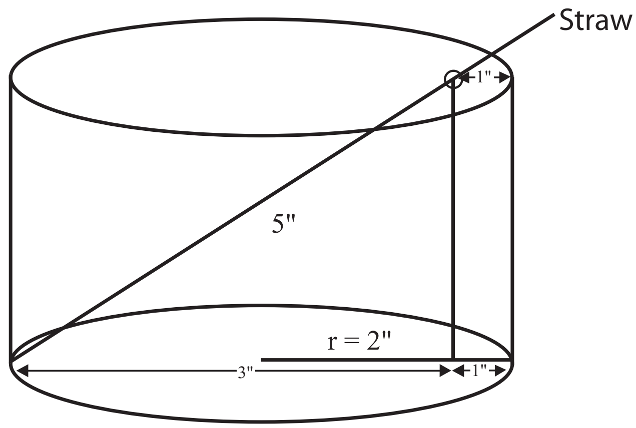
Scaffolding Questions:

- Can you place the hole for the straw in more than one place?
- Does one inch from the side mean one inch from every side?
- What would pictures of the various containers and hole positions look like?

Sample Solutions:

For the cylinder:

If the base of the figure is a circle inscribed in a square of sides 4 inches, then the diameter of the inscribed circle is 4 inches. The cylinder has a radius of 2 inches.



If the hole is 1 inch from the edge, a segment from the hole through the center to the other side of the cylinder is 4 – 1 or 3 inches. One inch of the straw is outside of the can, so 5 inches are inside the can. When meeting the parameters for the straw, a right triangle with a leg of 3 inches and a hypotenuse of 5 inches is formed. This is a Pythagorean triple, so the other leg must be 4 inches. This makes the minimum height for the cylinder 4 inches.

The volume is:

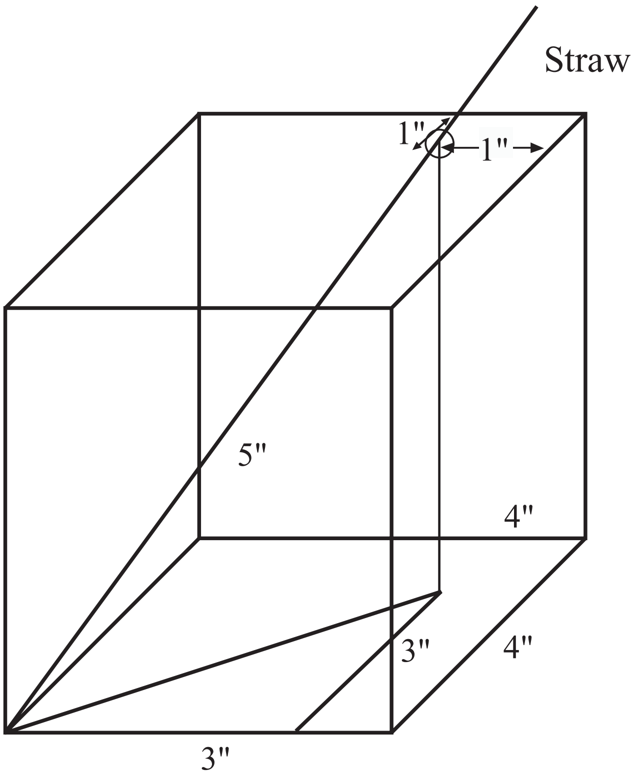
$V = Bh$

$V = \pi r^2 h$

$V = 3.14 \times 2^2 \times 4$

$V = 50.24 \text{ in}^3$

For the rectangular prism with a square base:



The triangle is a right triangle with legs of measure 3 inches. The hypoteneuse is $3\sqrt{2}$. Use the Pythagorean Theorem to determine the height of the box.

$\left(3\sqrt{2}\right)^2 + h^2 = 5^2$

$18 + h^2 = 25$

$h^2 = 7$

$h \approx 2.65$ in.

(e.2) **Congruence and the geometry of size**. The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(C) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them; and

(D) analyzes the characteristics of three-dimensional figures and their component parts.

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of concepts and uses of measurement and similarity.

Additional objectives are addressed by the Extension Questions that follow the sample solution:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Texas Assessment of Knowledge and Skills: (cont.)

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Now find the volume of this box.

$V = Bh$

$V = s^2 h$

$V \approx 4^2 \times 2.65$

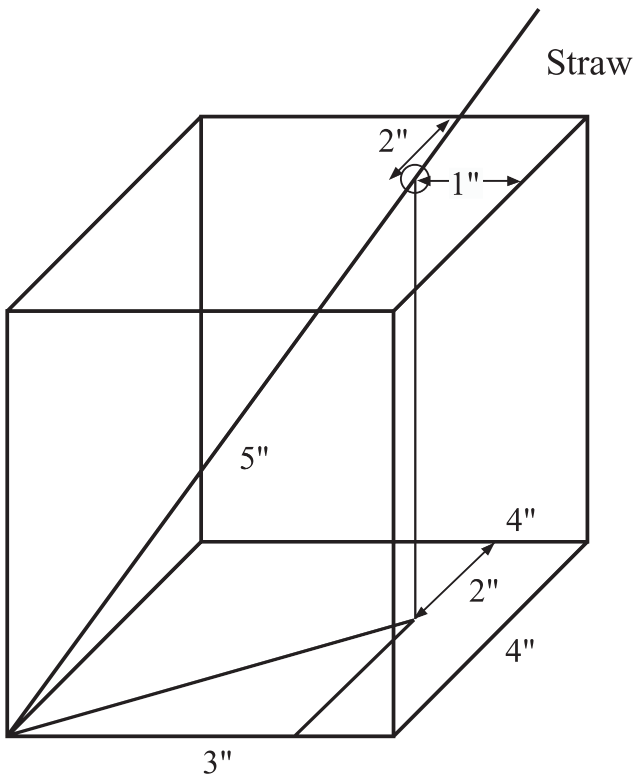
$V \approx 42.4 \text{ in}^3$

The volume of the cylinder is approximately 50.24 in^3; the cylinder has the larger volume and is a better choice than the rectangular prism.

Extension Questions:

- Suppose that the straw must be 1 inch from one edge and 2 inches from the other edge. How does the volume of this box compare to the other two choices?

The second yields the largest volume and places the hole 1" from one side in the center of the box.



We must use the Pythagorean Theorem to find the base leg of our right triangle.

$$a^2 + b^2 = c^2$$
$$\sqrt{a^2 + b^2} = c$$
$$\sqrt{2^2 + 3^2} = c$$
$$\sqrt{13} = c$$
$$c = \sqrt{13} \ in$$

Now use the Pythagorean Theorem again to find the height of the box.

$$a^2 + b^2 = c^2$$
$$\left(\sqrt{13}\right)^2 + b^2 = 5^2$$
$$13 + b^2 = 25$$
$$b^2 = 12$$
$$b \approx 3.46 \, in$$

Now find the volume of this box.

$$V = Bh$$
$$V = s^2 h$$
$$V \approx 4^2 \times 3.46$$
$$V \approx 55.36 \ in^3$$

This box is the best choice because it has the greatest volume.

- Suppose that the length of the straw and the diameter of the base of the cylinder are doubled. How will the volume be affected?

Usually if the dimensions of a solid are doubled, the volume is affected by a factor of 2 cubed. However, there are other factors to consider. The diameter of the base is 8 inches. The cylinder has a radius of 4 inches.

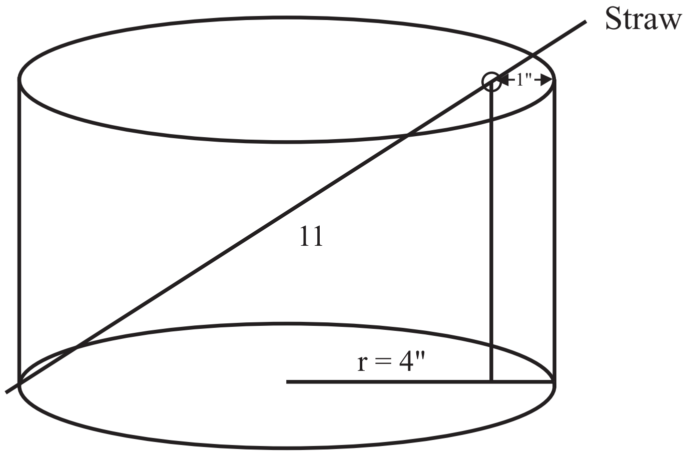
Connection to High School Geometry: Supporting TEKS and TAKS Institute:

III. Triangles: Distance in Space

V. Solid Figure: Dorothy You are Not in Bayou City Anymore

Teacher's Comment:

"The students needed clarification on the straw placement. I asked them about drinking out of a cup with a straw and how the straw reaches inside the cup to the corner."



If the hole is 1 inch from the edge, a segment from the hole through the center to the other side of the cylinder is 8 – 1 or 7 inches. The part of the straw that is inside the can is 12 – 1 or 11 inches.

The height is found using the Pythagorean Theorem.

$$11^2 = 7^2 + h^2$$
$$h^2 = 121 - 49 = 72$$
$$h = \sqrt{72} \approx 8.48$$

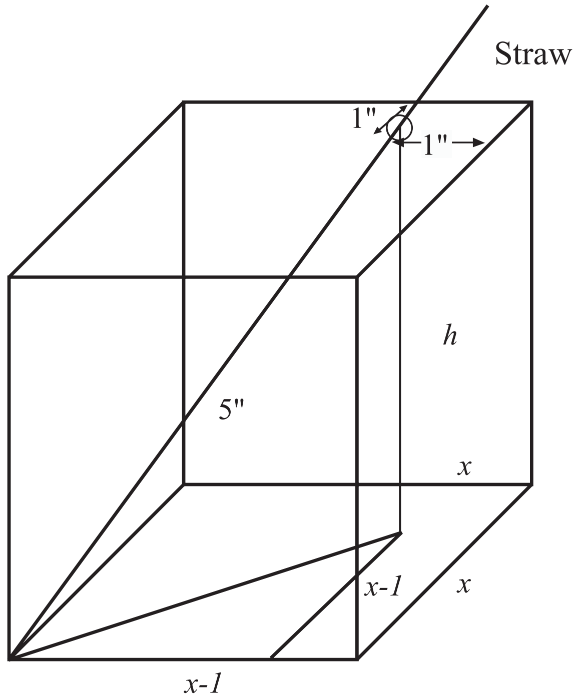
The volume is:

$$V = Bh$$
$$V = \pi r^2 h$$
$$V \approx 3.14 \times 4^2 \times 8.48$$
$$V \approx 426.04 \text{ in}^3$$

This number is more than 8 times 50.24 cubic inches. Not all of the dimensions were changed by a multiple of 2, so the theory of changing the volume by a factor of 2 cubed does not apply. Since the height of the new cylinder is 2.12 times the height of the original cylinder, the volume of the new cylinder is 2(2)(2.12) times the volume of the original cylinder.

- Suppose that the base of the rectangular prism could be any square that had dimension 4 inches or less. If the straw hole is one inch from each side of the base, what can you tell about the variables and the volume?

Let the side of the square base be x inches. The length of one side of the right triangle shown on the base is represented by (x-1) inches.



The hypoteneuse of the right triangle is (x-1) $\sqrt{2}$ *.*

The relationship between the side, x, and the height, h, is $5^2 = \left[(x-1)\sqrt{2}\right]^2 + h^2$.

The equation may be solved for h.

$$h^2 = 5^2 - \left[(x-1)\sqrt{2}\right]^2$$

$$h = \pm\sqrt{\left(5^2 - \left[(x-1)\sqrt{2}\right]^2\right)}$$

$$h = \pm\sqrt{\left(25 - 2(x-1)^2\right)}$$

Because the height cannot be negative, $h = +\sqrt{\left(25 - 2(x-1)^2\right)}$.

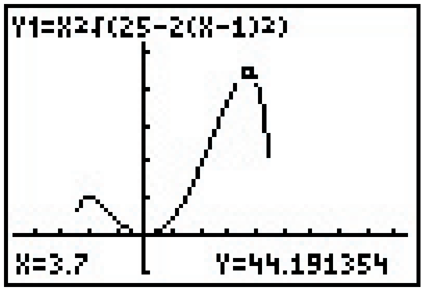
The volume of the box is

$$V = x^2 h$$

$$V = x^2\sqrt{\left(25 - 2(x-1)^2\right)}.$$

It is possible to graph this function to realize that the maximum volume occurs when x is about 3.7 inches. Under the given restrictions that x be less than or equal to 4, a base of about 3.7 inches gives the greatest volume.

```
WINDOW
 Xmin=-4.7
 Xmax=9.4
 Xscl=1
 Ymin=-10
 Ymax=60
 Yscl=10
 Xres=1
```



X	Y1	
3.5	43.31	
3.6	43.911	
3.7	44.191	
3.8	44.083	
3.9	43.502	
4	42.332	
4.1	40.414	

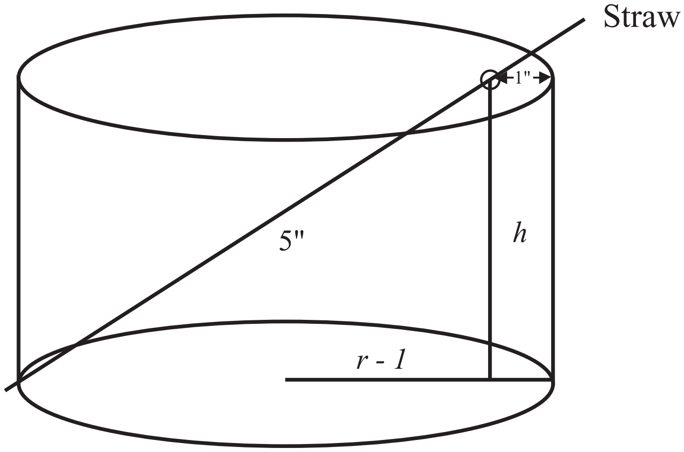
X=3.7

Making the increments on the table smaller would show that the value is between 3.72 and 3.73 inches.

X	Y1	
3.7	44.191	
3.71	44.199	
3.72	44.203	
3.73	44.203	
3.74	44.199	
3.75	44.191	
3.76	44.178	

X=3.72

- If the base may be a circle with a diameter that is less than or equal to 4, what can you conclude about the volume?



Let the radius of the base be represented by r and let the height of the cylinder be represented by h.

The right triangle with hypoteneuse 5 inches has legs represented by 2r-1 and h. Apply the Pythagorean Theorem.

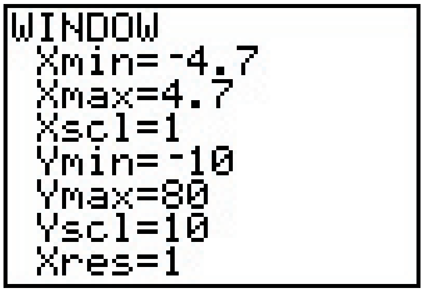
$$h^2 + (2r-1)^2 = 5^2$$
$$h^2 = 5^2 - (2r-1)^2$$
$$h = \pm\sqrt{\left(5^2 - (2r-1)^2\right)}$$
$$h = \pm\sqrt{\left(25 - (2r-1)^2\right)}$$

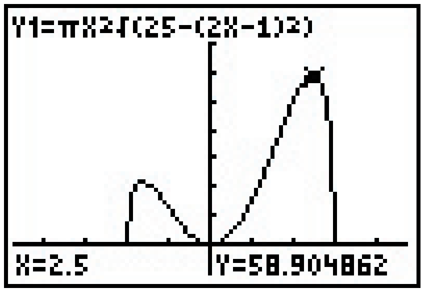
The height must be positive. $h = \sqrt{\left(25 - (2r-1)^2\right)}$

The volume of the cylinder is

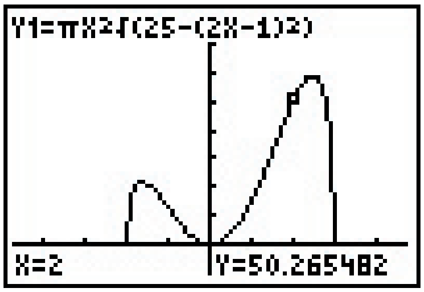
$$V = \pi r^2 h$$
$$V = \pi r^2 \sqrt{\left(25 - (2r-1)^2\right)}.$$

The graph or table of this function illustrates that the maximum volume occurs at a radius of 2.5 inches. However, because of the given restriction that the diameter is less than or equal to 4 , the radius must be less than or equal to 2 inches. The maximum volume occurs when the radius is 2 inches.





The maximum volume is approximately 50.27 in^3.



Student Work Sample

The work on the next two pages was created by a group of students.

The solution illustrates many of the solution guide criteria. For example,

- Shows an understanding of the relationships among elements.

The diagrams and explanations demonstrate the understanding of the relationships between the length and position of the straw and the radius length. These measurements are used to determine the height of the cylinder. Their work shows that they know what measurements they must determine to find the volume of the cylinders.

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

The Pythagorean Theorem was used to compute the heights of the cylinder and the prism.

- Communicates clear, detailed, and organized solution strategy.

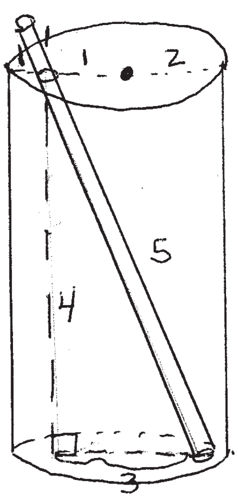
The students described what they were doing to solve the problem. The mathematical processes are detailed. They presented the equations in organized and detailed manner. There is a clear, step-by-step process showing the formulas that are used. They wrote detailed statements to explain their work.

The Most Juice

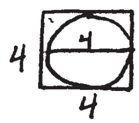
1. First, we are given that we are looking for a juice container that holds the most juice. We know the container has to have a six-in. straw that can touch every point on the base with one inch remaining outside. Also, the hole that contains the straw must be 1 inch from the side of the container. Also, the base must be a square with dimensions of 4 x 4 or a circle that would be inscribed in that square. Finally, the containers could either be a rt. cylinder or a rectangular prism.

So we came up with these diagrams.

$3^2 + x^2 = 5^2$
$9 + x^2 = 25$
$x^2 = 16$
$x = 4$



2. For a circle to be inscribed in a square its diameter must be the side of a square

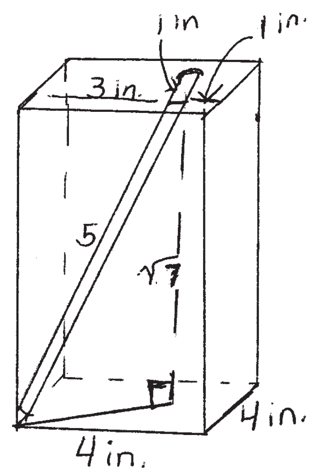


the radius is also 2

$3^2 + 3^2 = x^2$
$x = \sqrt{18}$

We use the Pythagorean Theorem to find the height.

$\sqrt{18}^2 + x^2 = 5^2$
$\sqrt{18}^2 + x^2 = 25$
$x^2 = 7$
$x = \sqrt{7}$



3. Now, since the straw has to have at least 1 inch of it above the surface we know that the farthest reach the straw has in the cylinder must be 5 inches. Therefore we get a triangle with 3,4,5 in which four is the hight.

3. cont... Now, since the straw has to have at least 1 inch of it above the surface we know that the farthest reach the straw has in the cylinder must be 5 inches. Therefore we get a right triangle with a height of $\sqrt{7}$

4. Then we find the volume of each container.

Volume of cylinder

$V = bh$
$V = \pi r^2 h$
$V = \pi 2^2 (4)$
$V = 16\pi$
$V = 50.2 \text{ in}^3$

Volume of Rectangular Prism

$V = bh$
$V = 4^2(\sqrt{7})$
$V = 16(\sqrt{7})$
$V = 42.3 \text{in}^3$

5. In conclusion, we have concluded that the rt. cylinder would hold the most juice for the given specifications because it has more volume than the rectangular prism.