Chapter 4:
*Area,
Perimeter,
and Volume*

Introduction

The problems in this chapter focus on applying the properties of triangles and polygons to compute area, perimeter, and volume.

> As presented in Grades K-8, the basic understandings of number, operation, and quantitative reasoning; patterns, relationships, and algebraic thinking; geometry; measurement; and probability and statistics are essential foundations for all work in high school mathematics. Students continue to build on this foundation as they expand their understanding through other mathematical experiences. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999).

Boxing Basketballs

1. A basketball (sphere) with a circumference of approximately 30 inches is packed in a box (cube) so that it touches each side of the interior of the box. Answer the following question, ignoring the thickness of the surface of the ball and the surface of the box.

   What is the volume of the wasted space in the box?

2. A box (cube) that has a side length of 9.5 inches is packed inside a ball (sphere) so that the corners of the box touch the interior of the sphere. Answer the following question, ignoring the thickness of the surface of the ball and the surface of the box.

   What is the volume of the wasted space in the ball?

Teacher Notes

Materials:
One pencil and graphing calculator per student

Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(d.1) **Dimensionality and the geometry of location**. The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

(e.1) **Congruence and the Geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

Scaffolding Questions:

- For the first situation how do the cube and the sphere come in contact with each other (sides or vertices)?
- Do the cube and the sphere have any common measurements?
- In the second problem how do the cube and the sphere come in contact with each other?
- What triangle is formed by the diagonal of the base of the cube and the edges of the base of the cube?
- What triangle is formed by the diagonal of the cube, the diagonal of the base, and one vertical edge of the cube?

Sample Solutions:

1. First, find the volume of the ball and the box. The diameter of the ball is the side length of the box. Use the circumference, *C*, to find the diameter, *d*.

$C = \pi d$

$30 = \pi d$

$d = 9.55$ in

The volume of the box, *V*, in terms of the length of the side, *s*, is found by using the formula

$$V = s^3.$$

$$V = 9.55^3$$

$$V = 870.98 \text{ in}^3$$

Find the volume of the ball.

$$V = \frac{4}{3}\pi r^3$$

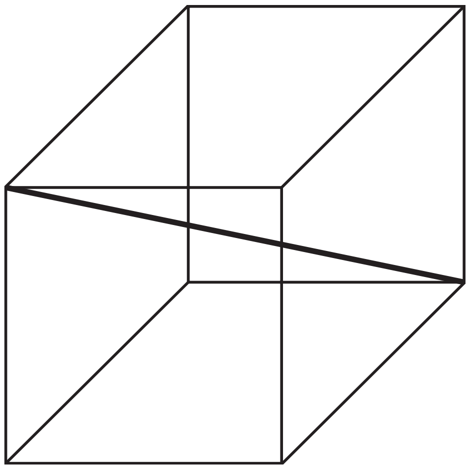
$$V = \frac{4}{3}\pi\left(\frac{9.55}{2}\right)^3$$

$$V = 456.05 \text{ in}^3$$

Subtract the volume of the ball from the volume of the box to find the volume of wasted space.

$870.98 \text{ in}^3 - 456.05 \text{ in}^3 = 414.93 \text{ in}^3$

2. If the cube is surrounded by the ball, the cube will touch the interior of the sphere at eight vertex points. To determine the diameter of the sphere we must find the diagonal of the cube. Since the diagonal of the cube passes through the center of the sphere and its endpoints touch the sphere, its length is the diameter of the sphere.



This can be found by using the Pythagorean Theorem or 45-45-90 special right triangles.

Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

III. Triangles

IV. Planar Figures: Stained Glass Circles

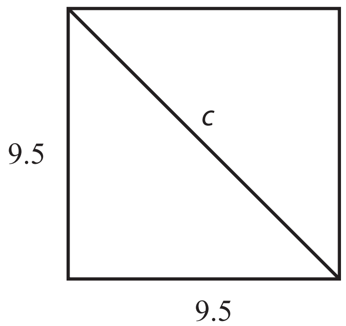
V. Solid Figures: Dwellings, Bayou City Dome

Teacher's Comments:

"I handed out a copy of the problem and we approached it first in a whole group discussion. I asked students to draw a picture and prompted them to determine what values were given for the critical attributes and how the parts were related."

"We reviewed formulas for the volume of spheres and rectangular prisms. We discussed the given scaffolding questions…. We need to work with 3-D models more often."

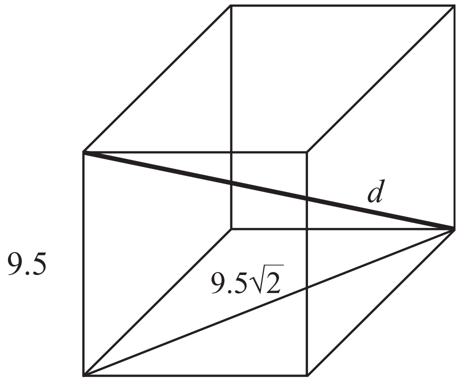
First find a diagonal of a side of the cube. The right triangle has sides measuring 9.5 inches.



$$9.5^2 + 9.5^2 = c^2$$

$$c = \sqrt{2\left(9.5^2\right)} = 9.5\sqrt{2}$$

Now use the Pythagorean Theorem again to find the diagonal of the cube. This diagonal is the hypoteneuse of a right triangle. One leg of the triangle is the side of the square, which measures 9.5 inches, and the other leg is the diagonal of the side face, $9.5\sqrt{2}$ inches.



$$9.5^2 + \left(9.5\sqrt{2}\right)^2 = d^2$$

$$d \approx 16.45 \text{ inches}$$

Now find the volume of the sphere and the cube.

Cube

$$V = s^3$$

$$V = 9.5^3$$

$$V = 857.375 \text{ in}^3 \cong 857.38 \text{ in}^3$$

Sphere

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi\left(\frac{16.45}{2}\right)^3$$

$$V = 2330.75 \text{ in}^3$$

The volume of the wasted space is the volume of the sphere minus the volume of the cube.

2330.75 – 857.38 =1473.37

The volume of the wasted space is approximately 1473.37 in^3.

Extension Questions:

- If the circumference of the ball were doubled in problem 1, how would the wasted space be affected?

If the circumference is doubled, the diameter and the radius would also be doubled. The volume should be multiplied by two cubed or eight. This idea may be demonstrated by looking at the general formula.

The volume of the wasted space, in terms of the diameter for the original situation, is

$$(d)^3 - \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = (d)^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right]$$

If the diameter is doubled, the equation becomes

$$(2d)^3 - \frac{4}{3}\pi\left(\frac{2d}{2}\right)^3 = (2d)^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right] = 8(d)^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right]$$

The wasted space is multiplied by 8.

- If the length of the side of the box in problem 2 is multiplied by one-third, how will the diameter of the sphere be affected?

The diameter will also be multiplied by one-third. The diameter of the sphere is the diagonal of the cube. The diagonal of the cube is the hypotenuse of a right triangle. The right triangle is similar to the right triangle in the original cube. The corresponding sides of the two right triangles are proportional. Since the side of the new cube (also the leg of the right triangle) is one-third times the side of the original cube, the diagonal of the new cube (also the hypotenuse of the similar triangle) is one-third times the diagonal of the original cube.

Student Work Sample

The work on the next page was as an individual assessment following a unit on volumes. The student's work exemplifies many of the criteria from the solution guide. Note the following criteria:

- Shows an understanding of the relationships among elements.

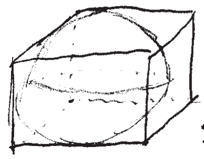
The student uses arrows in the first problem to show the circumference is used to determine the diameter which is equal to the length of the box's edges. In the second problem he shows how he used the edge of the box and the right triangles to find the radius of the sphere. He shows how he determines the answers in both problems by subtracting the two volumes, the volume of the box, and the volume of the sphere.

- States a clear and accurate solution using correct units.

The student not only uses the correct units "in^3" but indicates that it is the volume of the wasted space.

Boxing Basketballs

1



Circumference of a Basket Ball = $\underline{30 in}$

⇓

So the diameter of it is = $\underline{\frac{30}{\pi}}$ = the length of box's edges

⇓

the radius of it is = $\frac{30}{2\pi} = \underline{\frac{15}{\pi} in}$

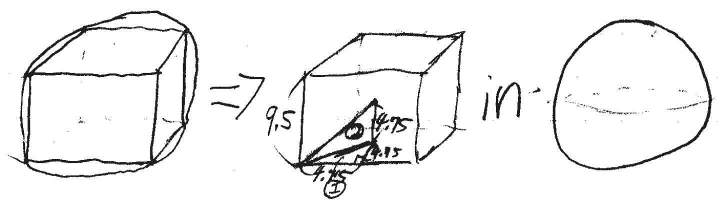
V of the box $= \left(\frac{30}{\pi}\right)^3$

V of the ball $= \left(\frac{15}{\pi}\right)^3 \pi \times \frac{4}{3}$

V of the wasted space $= \left(\frac{30}{\pi}\right)^3 - \left(\frac{15}{\pi}\right)^3 \pi \times \frac{4}{3}$

$= \boxed{44.85 in^3}$

2



To find the radius of the sphere (look at the diagram):

① Use special triangle: $\underline{4.75\sqrt{2}}$

⇓

then ② use pythagorean therom: $\underline{\sqrt{4.75^2+(4.75\sqrt{2})^2}}$

⇓

This is the radius of the sphere

V of the box $= 9.5^3$

V of the sphere $= \left(\boxed{\sqrt{4.75^2+(4.75\sqrt{2})^2}}\right)^3 3.14 \times \frac{4}{3}$

V of the wastech space $= \left(\boxed{\sqrt{4.75^2+(4.75\sqrt{2})^2}}\right)^3 \times 3.14 \times \frac{4}{3} - 9.5^3$

$= \boxed{1474.10 in^3}$

Flower

A landscape company has produced the garden design below (see figure 1). The petals are constructed by drawing circles from the vertex to the center of a regular hexagon with a radius of 20 feet (see figure 2).

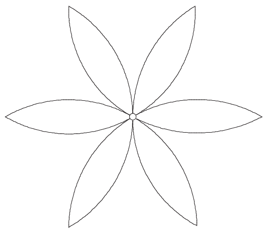


figure 1

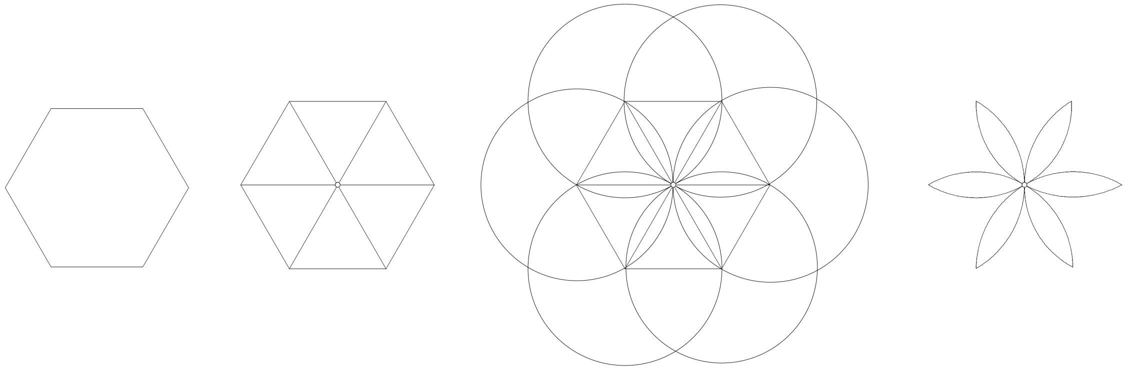


figure 2

The company must know the area of the petals in order to determine how many flowers to purchase for planting. Find the area of the petals. Give answers correct to the nearest hundredth.

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(B) finds areas of sectors and arc lengths of circles using proportional reasoning; and

(C) develops, extends, and uses the Pythagorean Theorem.

Teacher Notes

Scaffolding Questions:

- Are all of the petals the same size?
- What is the combination of figures that form a petal?
- How can the measure of the arc be determined?

Sample Solutions:

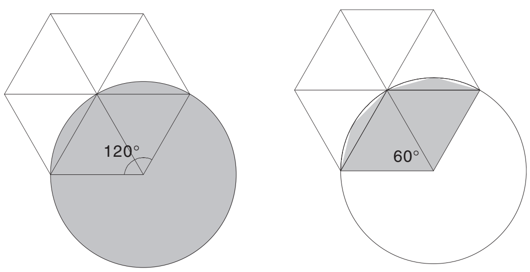
The hexagon eases the solution in that it can be subdivided into 6 equilateral triangles.

First, find the area of one of the circles.

$$A = \pi r^2$$

$$A = \pi(20)^2 = 1256.64 \text{ ft}^2$$

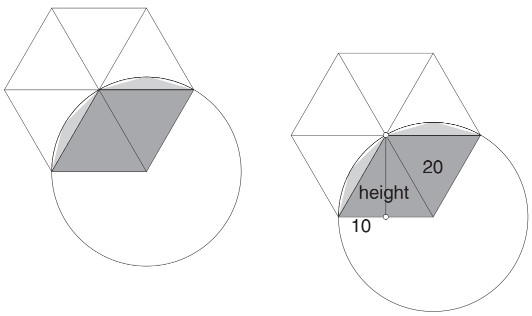
Each angle of the hexagon is 120°. Find the area of the sector of the circle.



The hexagon is composed of six equilateral triangles. The area of half the sector shown is $\frac{60}{360}$ or $\frac{1}{6}$ of the area of the circle. Thus, to determine the area of the sector multiply the area of the circle by one-sixth.

$$A = \frac{1}{6}(1256.64) = 209.44 \text{ ft}^2$$

In order to find the area of the segments of the circle that form a petal, the area of the triangle must be subtracted from the area of the one-sixth sector of the circle.



Determine the height of the triangle.

$$h^2 + 10^2 = 20^2$$
$$h^2 = 300$$
$$h = 10\sqrt{3} \approx 17.32 \text{ ft}$$

The area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(20)(17.32) = 173.2 \text{ ft.}$$

The area of one-half of a petal is the area of the sector minus the area of the triangle.

$209.44 - 173.2 = 36.24 \text{ ft}^2$.

The area of one petal is $2(36.24) \text{ ft}^2 = 72.48 \text{ ft}^2$
Multiplying by 6 yields the total area of the petals.
Total area = $6(72.48) = 434.88 \text{ ft}^2$

Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

Teacher's Comment:

"Stress to students that they must write explanations of steps in words and that they must explain how they found anything that was not given."

Extension Questions:

- If the length of the hexagon side is doubled, how is the area affected?

If the length is doubled, then the area is multiplied by a factor of two squared.

- Is it possible to use other regular polygons to create flowers using the same process?

What makes this activity possible is the fact that the side of the hexagon becomes the radius of the circle. The hexagon is the only one for which this relationship is true.

Student Work Sample

The next page shows the work of an individual student. The teacher required her students to give a verbal description of the process used to solve the problem. The second page shows the students computations. The work exemplifies the following criteria:

- Communicates clear, detailed, and organized solution strategy.

The student provides a step by step description of the steps. For each step he gives a reason for each step. "you know that the hexagon is regular, which means that all of the sides are equal…. If you draw a circle around a regular hexagon…"

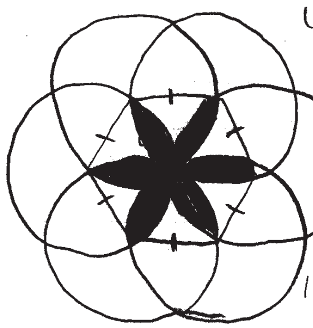
- Demonstrated geometric concepts, process, and skills.

He used the Pythagorean Theorem. He showed how to get the area of the hexagon and the area of a circle. He showed how to find the area of the petal region by subtracting the area of the hexagon from the area of the circle.

Flower

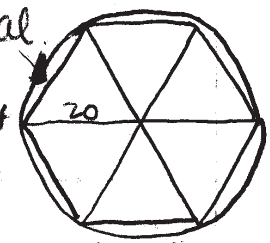
a landscaping company has a flower design. The petals are constructed by drawing circles from the vertices to the center of a regular hexagon, with a radius of 20 feet. The company must know the area of the petals so they will know how many flowers to buy. Find the area of the petals.

To start off the problem, you know that the hexagon is regular, which means that all of the sides are equal.



you also know that if you draw a circle around a regular hexagon, to where it touches each of the 6 corners, the space outside of the hexagon is equal to 1/2 of a petal.

1/2 of a single petal



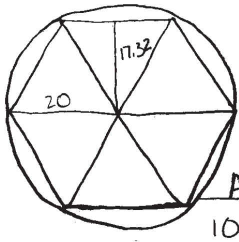
you find the area of the hexagon, and then subtract that from the area of the circle that you drew.

$(10(\sqrt{300})/2) \cdot 12 = 1039.23$. Then the circle $\pi 20^2$ so subtract, $\pi 20^2 - (10(\sqrt{300})/2) \cdot 12$ and you get 217.41. you know thats 1/2 of all of the petals, so you multiply that by 2 to get 434.81 ft^2.

→

This problem used the pythagorean therom, so that you could figure out the area of the hexagon. Also, you used πr^2 to find the area of the circle. you then used subtraction and addition to find the total area of the petals.

Work



Height of 1 hexagon

$10^2 + x^2 = 20^2$

$x^2 = 300$

$x = \sqrt{300}$ or 17.32

Area of all Hexagons

$10(\sqrt{300}) = 173.21 / 2 = 86.60(12) = 1039.23$

Area of circle

$\pi 20^2 = 1256.64$

$$\begin{array}{r} 1256.64 \\ -\ 1039.23 \\ \hline 217.41 \\ \times\ \ 2 \\ \hline 434.81 \end{array}$$

Area of flower → 434.81

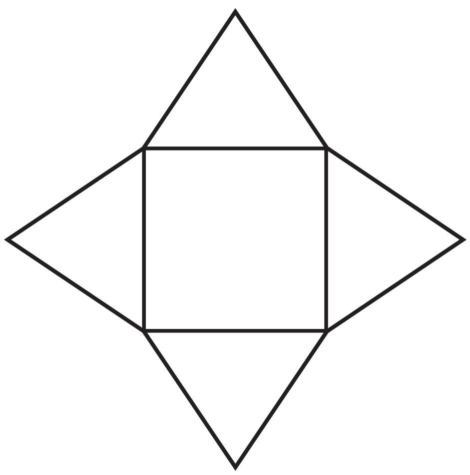
Area of flower petals: 434.81 ft^2

Great Pyramid

About 2,550 B.C., King Khufu, the second pharaoh of the fourth dynasty, commissioned the building of his tomb at Giza. This monument was built of solid stone and was completed in just under 30 years. It presides over the plateau of Giza on the outskirts of Cairo, and is the last survivor of the Seven Wonders of the World. At 481 feet high, the Great Pyramid stood as the tallest structure in the world for more than 4,000 years. The base of the Great Pyramid was a square with each side measuring 756 feet.

1. If the average stone used in the construction was a cube measuring 3.4 feet on a side, approximately how many stones were used to build this solid monument? Justify your answer.

2. If you could unfold the pyramid like the figure below, how many acres of land would it cover? (1 acre = 43,560 square feet)



Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(B) uses nets to represent and construct three-dimensional objects

(e.1) **Congruence and the geometry of size**. The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(D) analyzes the characteristics of three-dimensional figures and their component parts.

Teacher Notes

Scaffolding Questions:

- What shape is the solid figure? How do you know?
- What formula do you need in order to find the number of stones?
- Explain what measurements you will use in order to determine the volume of the pyramid.
- What are the component parts of the net of the pyramid?
- Describe how you could find the surface area of the pyramid.

Sample Solutions:

1. Find the volume of a stone and the pyramid.

Stone

$V = s^3$

$V = 3.4^3$

$V = 39.304 \text{ ft}^3$

Pyramid

$V = \frac{1}{3}Bh$

$V = \frac{1}{3}(756)^2(481)$

$V = 91{,}636{,}272 \text{ ft}^3$

$$\text{Number of stones} = \frac{V_{\text{pyramid}}}{V_{\text{stone}}} = \frac{91636272}{39.304} \approx 2{,}331{,}474.456$$

If whole number of stones are required, the number of stones would be 2,332,475.

2. Find the area of component parts.

Square

$A = s^2$

$A = 756^2$

$A = 571536 \text{ ft}^2$

Triangle

Find the slant height of the pyramid; this is the hypotenuse of the cross section of the pyramid.

$a^2 + b^2 = c^2$

$481^2 + \left(\frac{1}{2}756\right)^2 = (\text{slant height})^2$

$\sqrt{374245} = \text{slant height}$

$611.76 \text{ ft} \approx \text{slant height}$

Now find the area of a side.

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2} \bullet 756 \bullet 611.76$$

$$A \approx 231245.28 \text{ ft}^2$$

The four sides of the pyramid are congruent. The total surface area of the pyramid is the area of the square base plus the area of the four sides.

$$\text{Total area} \approx 571536 + 4(231245.28)$$
$$\approx 1496517.12 \text{ ft}^2$$

Convert to acres: $1496517.12 \text{ ft}^2 \left(\frac{1 \text{ acre}}{43560 \text{ ft}^2} \right) \approx 34 \text{ acres}$

Extension Questions:

- If all the dimensions of the pyramid were multiplied by 4, how would the volume of the pyramid be changed?

If a side is multiplied by a factor of 4, the volume is multiplied by a factor of 4^3.

- If you want to have the net cover twice the surface area, how would the sides of the pyramid need to be changed?

If a side is multiplied by a factor of k, the area is multiplied by a factor of k^2*. If the area is multiplied by a factor of h, the side is multiplied by a factor of* $\sqrt{h}$.

Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connections to High School Geometry: Supporting TEKS and TAKS Institute:

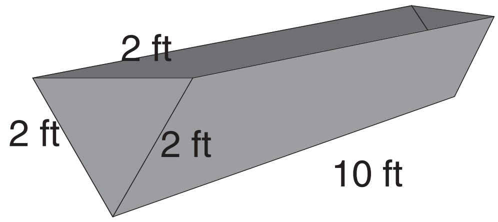
III. Pythagorean Theorem: A Patty Paper Proof

V. Solid Figures: Dwellings

Walter and Juanita's Water Troughs

You have been hired as chief mathematician by a company named Walter and Juanita's Water Troughs. This company builds water troughs for various agricultural uses. The company has one design (see figure 1). Your job is to perform mathematical analysis for the owners.

figure 1



1. A customer would like to know what the depth of the water is (in inches) if the trough only has 32 gallons in it.

2. The interior of the troughs must be coated with a sealant in order to hold water. One container of sealant covers 400 square feet. Will one container of sealant be enough to seal ten troughs? Why or why not?

3. Walter and Juanita would like to explore some minor modifications of their original design. They would like to know which change will produce a water trough that would hold more water—adding one foot to the length of the trough, making it 11 feet long, or adding three inches to each side of the triangular bases, making them 2 feet 3 inches on each side (see figures 2 and 3). Justify your answer.

figure 2

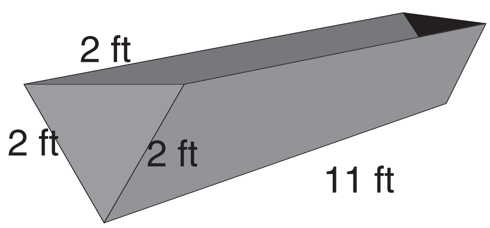
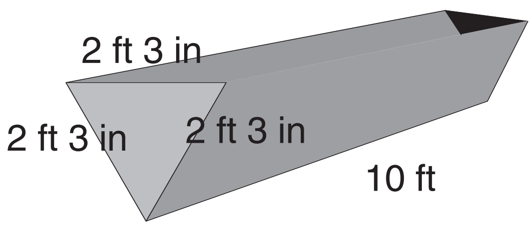


figure 3



Teacher Notes

Materials:
One calculator and straightedge per student

Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem.

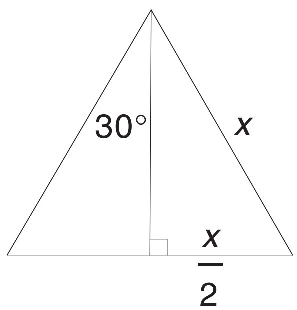
(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

Scaffolding Questions:

- What important components of the trough are needed to solve the problem?
- What is your prediction for which trough will hold the most water in number 3? Justify your answer.

Sample Solutions:

1. To solve this problem the area of the triangular base must be found in terms of an unknown side. If water is poured into the trough, the height of the water is a function of the side of the triangle. Consider the end of the trough that is an equilateral triangle.



Using 30-60-90 special right triangle properties or the Pythagorean Theorem the altitude is $\frac{x\sqrt{3}}{2}$ ft.

Area of the base can be found by:

$$\text{Area} = \frac{1}{2}(x)\left(\frac{x\sqrt{3}}{2}\right)$$

$$\text{Area} = \frac{x^2\sqrt{3}}{4}\ \text{ft}^2$$

The desired volume is given in gallons and must be converted to cubic feet because the measurements are given in feet.

$$32\ \text{gal.} \bullet \frac{1\ \text{ft}^3}{7.48\ \text{gal}} \approx 4.28\ \text{ft}^3$$

The volume of the tank is the area of the triangular base times the length of the tank.

$$\left(\frac{x^2\sqrt{3}}{4}\right)10 = 4.28 \text{ ft}^3$$

$$\left(x^2\sqrt{3}\right)10 \approx 17.12$$

$$x \approx \pm\sqrt{0.988} \approx 0.944 \text{ ft}$$

The side length is 0.994 ft.

The depth of the water is the altitude of the base.

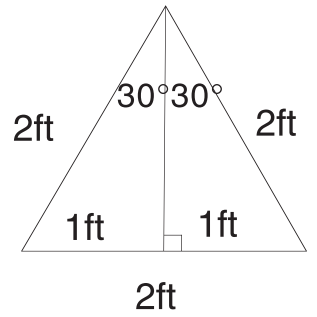
$$\text{altitude} = \frac{0.994\sqrt{3}}{2}$$

$$\text{altitude} \approx 0.861 \text{ ft}$$

$$\text{depth of water} \approx 10.3 \text{ in}$$

2. Find the interior surface area of one trough and multiply by 10.

   Figure 2: First, find the area of the base (the ends of the trough). In order to do this, find the altitude of the triangle. The base is an equilateral triangle, so the altitude bisects the side and the intercepted angle. The angles of an equilateral triangle are all 60º.



Use the Pythagorean Theorem to find the length of the altitude.

$$a^2 + b^2 = c^2$$

$$a^2 + 1^2 = 2^2$$

$$a^2 + 1 = 4$$

$$a^2 = 3$$

$$a = \pm\sqrt{3}$$

Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connections to High School Geometry: Supporting TEKS and TAKS Institute:

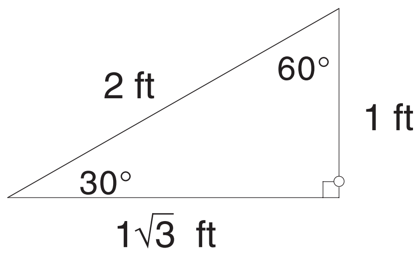
III. Pythagorean Theorem: A Patty Paper Proof

V. Solid Figures: Dwellings

The altitude must be positive.

$a = \sqrt{3}$

Another approach to determining the altitude is to use 30-60-90 special right triangle properties. The hypotenuse is twice the shortest side. The side opposite the 60-degree angle is the shorter leg times $\sqrt{3}$.



The altitude is $1\sqrt{3} \approx 1.73$ ft.

The area of the base may be found using the altitude and the base of the triangle.

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(2)\left(\sqrt{3}\right) = \sqrt{3} \approx 1.73 \text{ ft}^2$$

The surface area is the sum of the areas of the 2 ends and the 2 sides.

$$2\sqrt{3} + 2(20) \approx 43.64 \text{ ft}^2$$

The surface are of ten troughs is 10 times the surface area of one trough.

$$(10)43.46 = 434.6 \text{ ft}^2$$

Since the gallon of paint covers 400 ft^2, there will not be enough sealant to seal ten troughs.

3. The volume of a right prism is the area of the base times the height.

   The area of the base for the first case was found in problem 2.

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(2)\left(\sqrt{3}\right) = \sqrt{3} \approx 1.73 \text{ ft}^2$$

$$V = Bh = \sqrt{3}(11) \approx 1.73(11) \approx 19.05 \text{ ft}^3$$

For the second case, find the altitude of the right triangle base with side of 2.25 feet and the shorter leg, one-half of 2.25 or 1.125. Use the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$
$$a^2 + (1.125)^2 = (2.25)^2$$
$$a^2 \approx 5.0625 - 1.266$$
$$a \approx \sqrt{3.8}$$
$$a \approx 1.95 \text{ ft}$$

Find the area of the base.

$$A = \frac{1}{2}bh$$
$$A \approx \frac{1}{2}(2.25)(1.95)$$
$$A \approx 2.19 \text{ ft}^2$$

Find the volume.

$$V = Bh$$
$$V \approx (2.19)(10)$$
$$V \approx 21.9 \text{ ft}^3$$

Adding 3 inches to each side of the base will produce a greater increase in volume than adding a foot to the distance between the bases.

Extension Questions:

- A new tank is designed in the shape of a hemisphere (half of a sphere) to hold the same volume of water as the tank in figure one. What is the depth of the water if the tank is full?

The area of the base (the ends of the trough) is the same as the area for figure 2. The area is $\sqrt{3} \approx 1.73\ ft^3$*. The volume of a right prism is the area of the base times the height.*

$V = Bh = \sqrt{3}(10) \approx 17.3\ ft^3$.

Volume of a sphere = $\frac{4}{3}\pi r^3$*, but we only want* $\frac{1}{2}$ *of this amount.*

$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$$17.3 = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

$$17.3 = \frac{2}{3}\pi r^3$$

$$\frac{3}{2} \bullet 17.3 = \frac{3}{2} \bullet \frac{2}{3}\pi r^3$$

$$\frac{25.95}{\pi} = \frac{\pi}{\pi} r^3$$

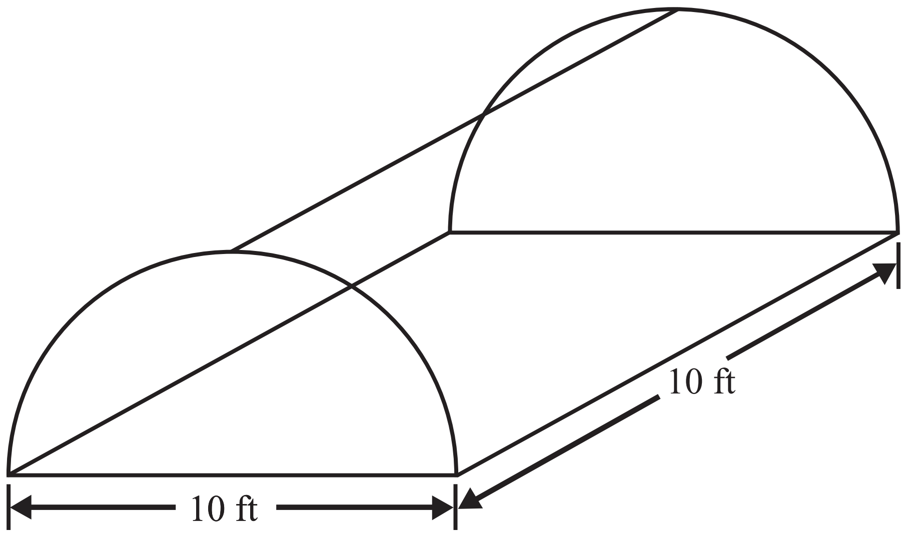
$$\sqrt[3]{8.26} = \sqrt[3]{r^3}$$

$$2.02 \approx r$$

The depth is approximately 2.02 feet.

Greenhouse

The backyard greenhouse in the figure below uses plastic tubing for framing and plastic sheeting for wall covering. The end walls are semicircles, and the greenhouse is built to the dimensions in the figure below. All walls and the floor are covered. The door is formed by cutting a slit in one of the end walls.



1. How many square feet of plastic sheeting will it take to cover top, sides, and floor of the greenhouse?
2. What is the volume of the greenhouse?

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds area of regular polygons and composite figures

(B) finds areas of sectors and arc lengths of circles using proportional reasoning; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

Teacher Notes

Scaffolding Questions:

- Describe the shape of the greenhouse.
- What dimensions do you need to determine the surface area?

Sample Solutions:

1. The ends of the greenhouse are semicircles that have a diameter of ten. Area of each end:

$A = \pi r^2$

$A = \pi(5)^2 \approx 78.54$

The area of one-half of the circle is $A \approx 78.54 \bullet \frac{1}{2} \approx 39.27 \text{ ft}^2$

Area of side:

Since the greenhouse is half of a cylinder, we can unwrap the sides, find the area of the rectangle, and divide by 2. One dimension of the rectangle is the circumference, and the other is 10.

$C = \pi d$

$C = \pi 10$

$C \approx 31.4$ ft

Find the area.

Surface area of the cylinder $\approx 31.4 \bullet 10 \approx 314 \text{ ft}^2$

Surface area of half of the cylinder $\approx 314 \text{ ft}^2 \bullet \frac{1}{2}$

$\approx 157 \text{ ft}^2$

The floor is the area of the square.

$A = 10 \bullet 10$

$A = 100 \text{ ft}^2$

Total square feet of plastic needed is

$39.27 + 39.27 + 157 + 100 = 335.54 \text{ ft}^2$.

2. The volume of the greenhouse is the area of the base multiplied by the distance between the bases. The base is a semicircle with a diameter of 10.

$$A = \frac{1}{2}\pi r^2$$ because we only need half of the circle.

$$A = \frac{1}{2}\pi(5)^2$$

$$A \approx 39.27 \text{ ft}^2$$

The volume of the greenhouse may be computed using the formula.

$$V = Bh$$

$$V = 39.27 \bullet 10$$

$$V \approx 392.7 \text{ ft}^3$$

Extension Question:

- What would the dimensions of a new square floor greenhouse have to be in order to double the volume of the greenhouse in this problem?

  Let the side length be x. The radius of the semicircular ends would be expressed as $\frac{x}{2}$.

  The area of the base is one-half of the area of a circle with radius $\frac{x}{2}$.

  $$A = \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2$$

  $$A = \frac{1}{8}\pi x^2$$

  The volume of the greenhouse is the area of the base (the area of the semicircle) times the length, x.

Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Dwellings

The rule for the volume as a function of the side is:

$$V = \frac{1}{8}\pi x^2 \bullet x$$

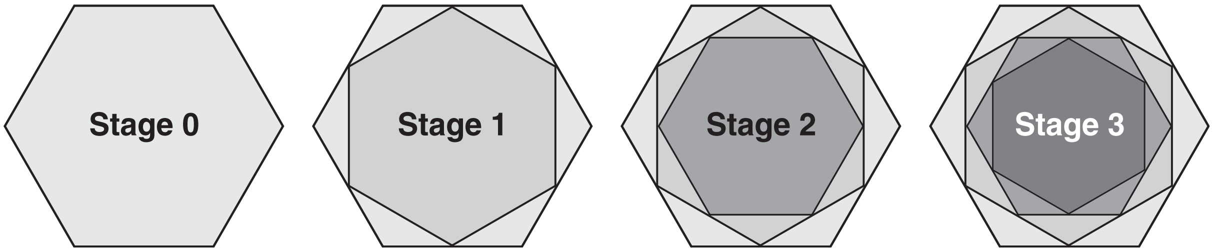
The volume must be double the original volume, or two times 392.7 ft^3. The volume must be 785.4 ft^3.

$$785.4 = \frac{1}{8}\pi x^3$$
$$x \approx 12.6 \text{ ft}$$

The length of the side of the base must be about 12.6 feet.

Nesting Hexagons

The first four stages of nested hexagons are shown below. Connecting the midpoints of the sides of the previous stage creates each successive stage. The side of the hexagon in Stage 0 measures 3 units.



1. Create a function rule to find the area of the innermost hexagon of any stage.
2. Find the area of the innermost hexagon in stage 10.
3. What is the domain of your function rule?
4. What is the range of your function rule?

Materials:
One graphing calculator and straightedge per student

Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem.

Teacher Notes

Note: It may prove beneficial to create a geometry software sketch of "Nesting Hexagons" once students have performed the calculations for Stages 1 and 2. Students can explore whether the size of the hexagon has any impact on the relationship between the stages. They can also use the measurements to help determine a function rule for this situation.

It may also prove beneficial to model the organization of data in a table such as the one in the possible solution strategies. Encourage students to round area values to the nearest hundredth.

Scaffolding Questions:

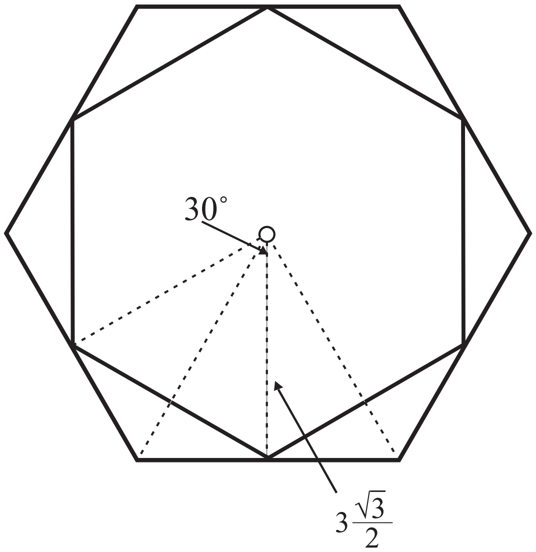
If students have created a table, the following questions may be asked:

- Is there a common difference between the terms in the area column of the table?
- Is there a common second difference between the terms in the table?
- Is there a common ratio between the terms in the table?
- How is each stage related to the previous stage?

Sample Solutions:

1. The perimeter is 18. Use $A = \frac{1}{2}ap$ to find the area of the regular hexagon. Each interior angle of a regular hexagon is 120°. The segments from each vertex to the center bisect the interior angles forming 6 equilateral triangles. Therefore the radius and the sides of the hexagon have the same measure.

   Using 30-60-90 right triangle properties, the length of the apothem is found by dividing the hypotenuese, or in this case the radius, by 2 and multiplying by $\sqrt{3}$. The length of the apothem for stage 0 is $3\frac{\sqrt{3}}{2}$, and the length of the radius is 3. The perimeter, p, is 6 times 3 units or 18 units.



$$A = \frac{1}{2}ap$$

$$A = \frac{1}{2}\left(3\frac{\sqrt{3}}{2}\right)(18)$$

$$A \approx \frac{1}{2} \bullet 3 \bullet 18 \bullet \frac{\sqrt{3}}{2}$$

$$A = 27\frac{\sqrt{3}}{2} \quad \text{or}$$

$$A \approx 23.38 \text{ square units}$$

Stage	Apothem	Radius	Side length	Perimeter	Area	Area Rounded to nearest hundredth
0	$3\frac{\sqrt{3}}{2}$	3	3	18	$27\frac{\sqrt{3}}{2}$	23.38

The apothem of the stage 0 hexagon is now the radius of the stage 1 hexagon. The length of the side of the stage 1 hexagon is $3\frac{\sqrt{3}}{2}$.

Using 30-60-90 right triangle properties, the length of the side is $3\frac{\sqrt{3}}{2}$, and the length of the apothem is $3\frac{\sqrt{3}}{2} \bullet \frac{\sqrt{3}}{2}$.

Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

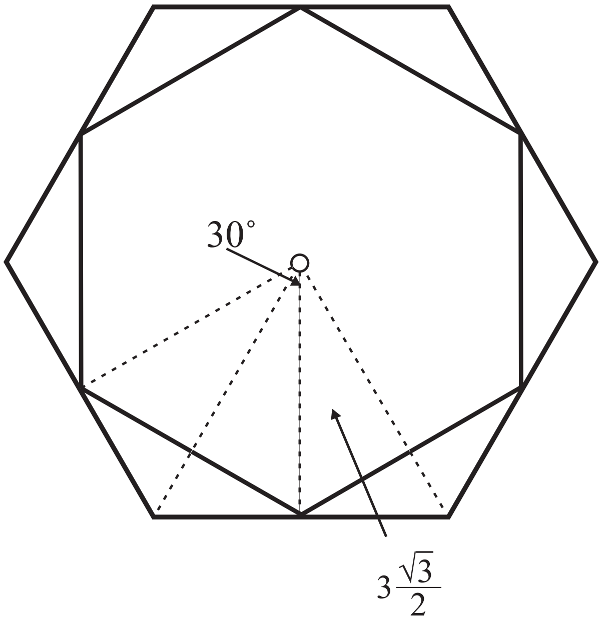
Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connections to High School Geometry: Supporting TEKS and TAKS:

II. Transformations: Fractals

IV. Planar Figures: Stained Glass Circles



$$A = \frac{1}{2}ap$$

$$A = \frac{1}{2}\left(3\frac{\sqrt{3}}{2} \bullet \frac{\sqrt{3}}{2}\right)\left(18\frac{\sqrt{3}}{2}\right)$$

$$A = \frac{1}{2} \bullet 3 \bullet 18 \bullet \frac{\sqrt{3}}{2} \bullet \frac{\sqrt{3}}{2} \bullet \frac{\sqrt{3}}{2}$$

$$A = 27\frac{\sqrt{3}}{2} \bullet \frac{3}{4} \approx 17.54 \text{ square units}$$

Stage	Apothem	Radius	Side length	Perimeter	Area	Area Rounded to nearest hundredth
0	$3\frac{\sqrt{3}}{2}$	3	3	18	$27\frac{\sqrt{3}}{2}$	23.38
1	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} * \frac{3}{4}$	17.54

A pattern emerges when the rest of the table is completed. In each new stage the side length, radius, apothem, and perimeter change by a factor of $\frac{\sqrt{3}}{2}$ times the amount in the previous stage. The area in the new stage changes by a factor of $\frac{3}{4}$ times the amount in the previous stage.

Stage	Apothem	Radius	Side Length	Perimeter	Area	Area Rounded to nearest hundredth
0	$3\frac{\sqrt{3}}{2}$	3	3	18	$27\frac{\sqrt{3}}{2}$	23.38
1	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} * \frac{3}{4}$	17.54
2	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4}$	13.15
3	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4} * \frac{3}{4}$	9.86

A recursive process of repeated multiplication can be used to generate the function rule for the area.

Stage	Area
0	$27\frac{\sqrt{3}}{2}$
1	$27\frac{\sqrt{3}}{2} * \frac{3}{4}$
2	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4}$
3	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4} * \frac{3}{4}$
4	$27\frac{\sqrt{3}}{2}\left(\frac{3}{4}\right)^4$
x	$27\frac{\sqrt{3}}{2}\left(\frac{3}{4}\right)^x$

Another approach is to look for patterns in the area values themselves. It will facilitate the development of a pattern if the values are rounded to the nearest hundredth. If the difference between each term is found, no pattern emerges. If the difference between each of those differences is found, no pattern emerges. However, if the ratios of the terms are found, a common ratio emerges. This means that each successive term can be found by multiplying the previous term by this common ratio. This leads to the generation of the exponential rule for this situation.

Stage	Process	Area
0	23.38 or $23.38 = 23.38\,(0.75)^0$	23.38
1	23.38 (0.75) or $23.38\,(0.75) = 23.38\,(0.75)^1$	17.54
2	17.54 (0.75) or $23.38\,(0.75)(0.75) = 23.38\,(0.75)^2$	13.15
3	13.15 (0.75) or $23.38(0.75)(0.75)(0.75) = 23.38\,(0.75)^3$	9.86
Description	The area of the current stage hexagon is the area of the stage 0 hexagon multiplied by 0.75 raised to the current stage number.	
x	$23.38\,(0.75)^x$	A

1st differences	2nd differences	common ratio
5.84	1.45	$\frac{17.54}{23.38} \approx 0.75$
4.39	1.1	$\frac{13.15}{17.54} \approx 0.75$
3.29		$\frac{9.86}{13.15} \approx 0.75$

$$A = 27\frac{\sqrt{3}}{2}\left(\frac{3}{4}\right)^x$$

2. $A = 27\frac{\sqrt{3}}{2}\left(\frac{3}{4}\right)^{10} \approx 1.32$

3. Possible notation for the domain of the function might include one of the following: all real numbers $-\infty < x < +\infty$ or $(-\infty, +\infty)$.

   The domain for the problem situation would be non-negative integers.

4. Possible notation for the range of the function might include one of the following: real numbers greater than $0 < y < +\infty$ or $(0, +\infty)$.

   The range for the problem situation is more complicated. The range is the specific set of numbers corresponding to the domain values for the problem situation. For example, $\left\{\frac{27\sqrt{3}}{2}, \frac{81\sqrt{3}}{8}, \frac{243\sqrt{3}}{32}, \ldots\right\}$.

Extension Questions:

- What is the relationship between the ratio of side lengths $\frac{\text{stage}(n)}{\text{stage}(n-1)}$ and the ratio of perimeters? How is this number related to the ratio of areas?

  The ratio of side lengths and the ratios of perimeters are both $\frac{\sqrt{3}}{2}$. *This ratio squared is the ratio of the areas.*

- What is the largest possible area in this problem? Justify your answer.

  The largest possible area is $27\frac{\sqrt{3}}{2}$ *or 23.38.*

  The area of each successive stage is $\frac{3}{4}$ *times the area of the hexagon of the previous stage so the area is always smaller than the area of the hexagon of stage 0.*

- What is the smallest possible area in this problem? Justify your answer.

  There will never be a smallest area. Any area thought to be the smallest area can be multiplied by $\frac{3}{4}$ *according to our rule in order to find a new stage with a smaller area. The area will approach 0 as the stage number gets very large but will never actually be 0.*