Chapter 5:
Solids, and Nets

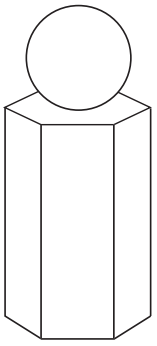
Introduction

The assessment problems in this chapter require students to analyze relationships between two- and three-dimensional objects and use these relationships to solve problems.

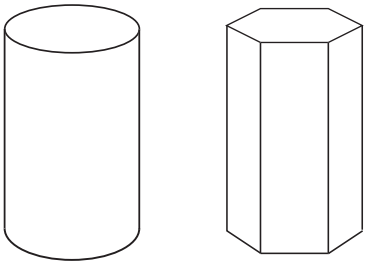
> Spatial reasoning plays a critical role in geometry; shapes and figures provide powerful ways to represent mathematical situations and to express generalizations about space and spatial relationships. Students use geometric thinking to understand mathematical concepts, and the relationships among them. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999).

Perfume Packaging

Ima Smelley would like to package her newest fragrance, *Persuasive*, in an eye-catching, yet cost-efficient box. The *Persuasive* perfume bottle is in the shape of a regular hexagonal prism 10 centimeters high. Each base edge of the prism is 3 centimeters. The cap is a sphere with a radius of 1.5 centimeters.



Ms. Smelley would like to compare the cost of the proposed box designs shown below. It is important that each bottle fit tightly in the box to avoid movement during shipping. The dimensions of the box must be a little bigger to allow for easy movement in getting the bottle in and out of the box. Thus, the box should only be 0.5 cm taller and 0.5 cm wider than the height and widest portion of the base of the bottle.



1. Determine the measurements needed for each dimension on the two boxes. Explain how you determined these dimensions.
2. Sketch the net for each package design, showing how you would make each box from a single sheet of cardboard. (Do not include the flaps needed to glue the box.)
3. Calculate the amount of cardboard used for each design. Which box will be more cost-efficient?

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

Teacher Notes

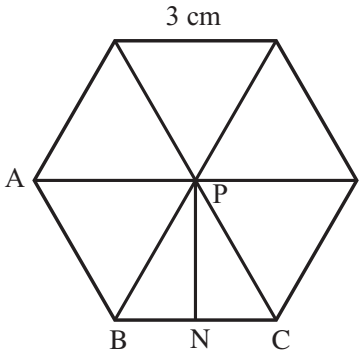
Scaffolding Questions:

- What is the total height of the perfume bottle and the cap?
- Describe the shape of the sides of each box.
- Describe the bases of each box.
- How will the dimensions of the perfume bottle compare to the dimensions of the box?

Sample Solutions:

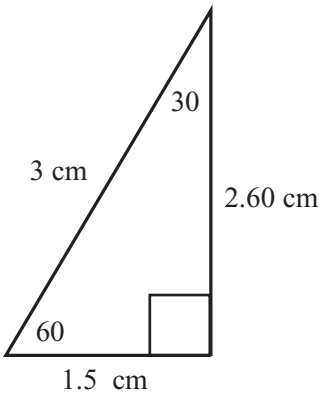
1. The total height of the perfume bottle and cap is 13 centimeters.

The base of the bottle is a regular hexagon with sides of 3 centimeters. A sketch of the base is shown below. The diagonals of the regular hexagon intersect in the center of the hexagon. The segment connecting the center of the regular hexagon and a vertex is the radius, $\overline{AP}$ or $\overline{BP}$, of the regular hexagon.

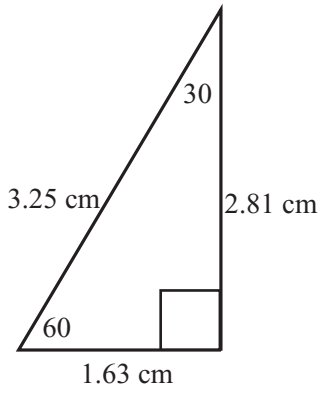


A regular hexagon has a radius equal to the length of the side of the regular hexagon. This can be established by drawing in the apothem of the hexagon and using a 30-60-90 triangle. The regular hexagon has an interior angle, $\angle ABC$, measuring 120 degrees. The radius of the hexagon, $\overline{BP}$, bisects this angle forming a 60-degree angle, $\angle CBP$. The apothem $\overline{PN}$ forms a 90-degree angle with the side of the hexagon. The apothem also bisects the side of the hexagon; therefore $\overline{BN}$ measures 1.5 centimeters. The remaining angle of the triangle, $\angle BNP$, measures 30 degrees.

Using this information, it can be determined that the hypotenuse of the triangle is 3 centimeters. The hypotenuse of the triangle is also the radius of the hexagon. The following figure shows the measurements in the 30-60-90 triangle. Measurements are rounded to the nearest hundredth.



The required 0.5 cm allowance for packaging space must be considered. This changes the length of the diagonal of the hexagon to 6.5 cm. The diagonal is made up of 2 radii (shown as the hypotenuse in the triangle below). Therefore the actual measurement needed for the packaging must be adjusted on the triangle as follows:



The widest dimension of the base will be 6.5 centimeters. This dimension will be the same for the hexagon and the circular base.

The shorter distance across the hexagon (2 apothems, end to end) will be approximately 5.62 cm. (This amount is rounded.)

The measurement of the distance needed to go around the bottle will be the perimeter of the hexagon and the circumference of the circular base of the cylinder.

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(B) uses nets to represent and construct three-dimensional objects; and

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

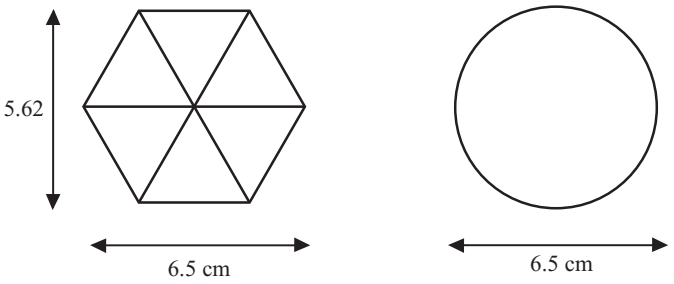
(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(D) analyzes the characteristics of three-dimensional figures and their component parts.

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.



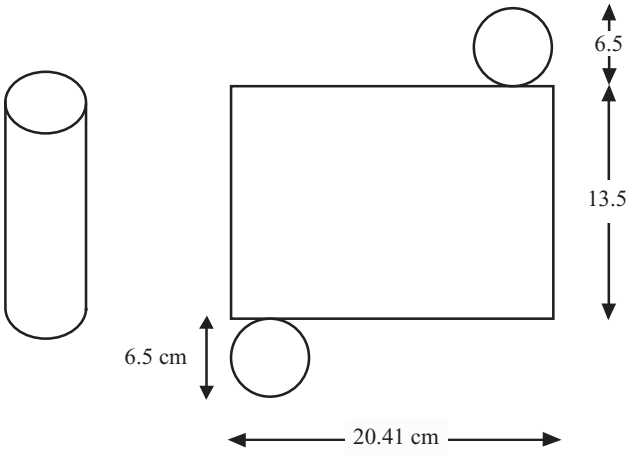
The perimeter of the regular hexagon is calculated by multiplying the length of one side by six. P = 3.25 cm • 6 = 19.5 cm.

The circumference of the circular base is calculated using $C = 2\pi r$. The radius of the circle is 3.25 cm. $C = 2\pi (3.25\text{cm}) = 6.5\pi$ cm. Using 3.14 for π, the length of the circumference is:

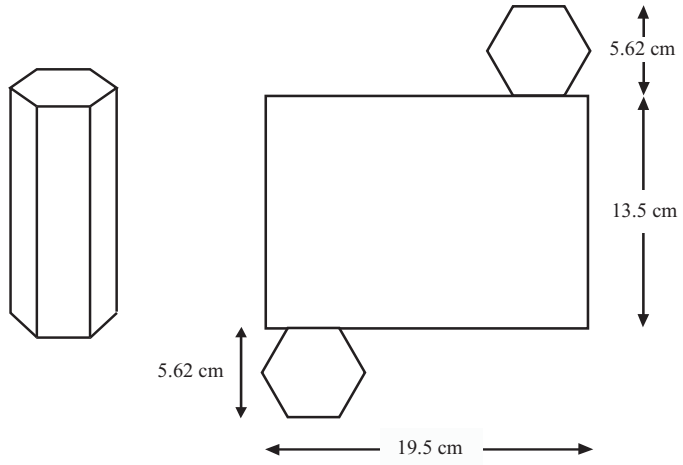
C = 6.5(3.14) cm = 20.41 cm.

2. The sketch of the nets for the boxes is shown below along with the surface area of each net. The 0.5 cm allowance for space is included in the dimensions. This space allows for the bottle to slide in and out of the box with ease, yet provides minimum movement during shipping. Values are rounded to the nearest hundredth of a centimeter.

Cylindrical Box:



Hexagonal Box:



3. The net of the cylindrical box consists of 2 congruent circles and a rectangle.

The area of each circle is $A = \pi r^2$.
$A = \pi (3.25 \text{ cm})^2$
$A = (3.14)(10.56 \text{ cm})$
$A = 33.16 \text{ cm}^2$

The area of the rectangle is $A = bh$.
$A = 20.41 \text{ cm} \bullet 13.5 \text{ cm}$
$A = 275.54 \text{ cm}^2$

The total surface area of the cylinder is found by finding the sum of the areas of the circles and the rectangle. Surface Area = $33.16 \text{ cm}^2 + 33.16 \text{ cm}^2 + 275.54 \text{ cm}^2 = 341.86 \text{ cm}^2$

The net of the hexagonal box is comprised of 2 regular hexagons and a rectangle.

The area of the rectangle is $A = bh$.
$A = 19.5 \text{ cm} \bullet 13.5\text{cm}$
$A = 263.25 \text{ cm}^2$

The area of the regular hexagon is $A = (ap) / 2$.
We know that $a = 2.81$ cm and $p = 6(3.25 \text{ cm}) = 19.5$ cm
$A = (2.81 \text{ cm} \bullet 19.5 \text{ cm}) / 2$
$A = (54.8 \text{ cm}^2) / 2$
$A = 27.4 \text{ cm}^2$

The total surface area of the hexagonal box is
$A = 263.25 \text{ cm}^2 + 27.4 \text{ cm}^2 + 27.4 \text{ cm}^2$
$A = 318.05 \text{ cm}^2$

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Dwellings, Bayou City Dome

Teacher's Comment:

" The students were hesitant to get started. No one wanted to give a wrong answer to the whole class. We share in class in groups, but not often as a class looking at each group's work. I will do more of this to encourage participation and understanding of different approaches. I will give more problems to be worked this way. I need to start this kind of problem at the beginning of the school year."

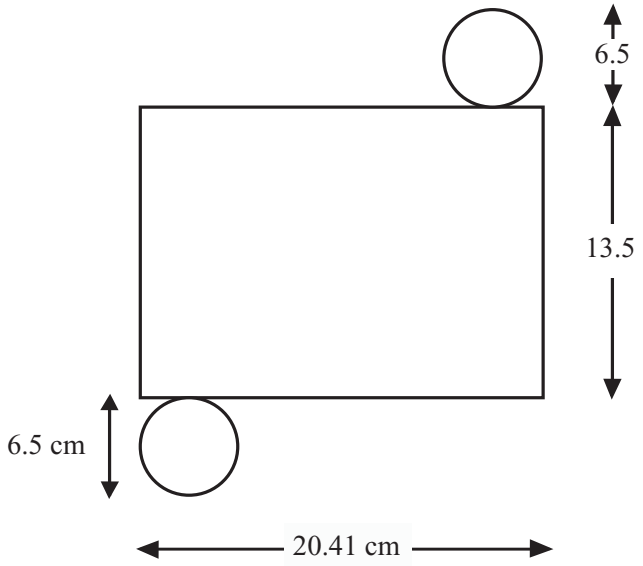
The hexagonal box has a smaller total surface area of 318.05 cm^2, while the cylindrical box has a total surface area of 341.86 cm^2. Ms. Ima Smelley should select the hexagonal box for the perfume because it is more cost-efficient.

Extension Questions:

- Suppose the packaging manufacturer used sheets of cardboard 1 meter long and 1 meter wide. How many of each type of packaging could be made from a single sheet? Explain your answer in detail.

The area of the cardboard sheet is 1 square meter or 10000 square centimeters. You may not, however, divide this total by the total surface area of the box because the shapes, when placed on the sheet, will not use all of the space.

The cylindrical shape with tangent circle bases:



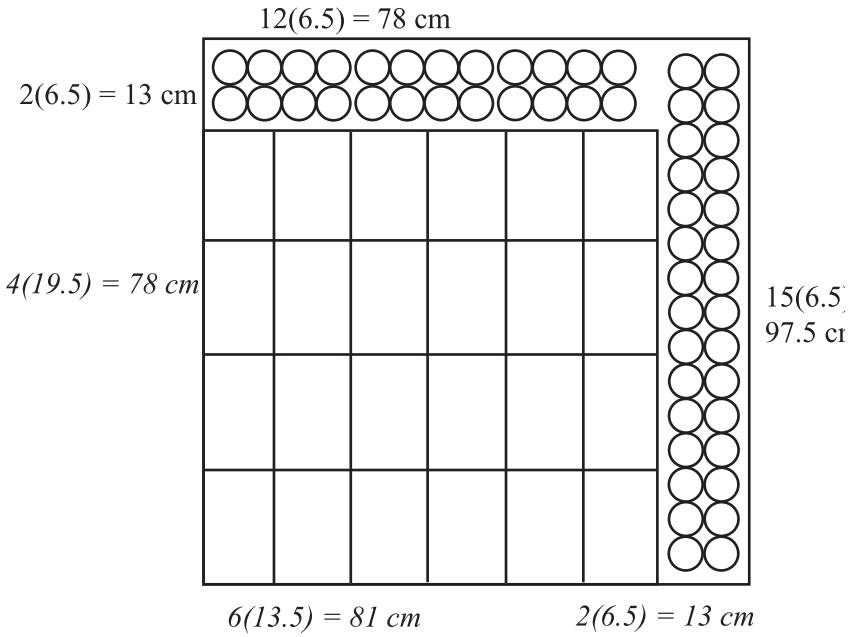
The total width is 20.41 cm, and the length is 6.5 + 6.5 + 13.5 or 26.5 cm. If it is assumed that the shape must be placed with the tangent circles, the question is, "How many rectangles of this size will fit in a rectangle 100 cm by 100 cm?"

$100 \div 26.5 \approx 3.77$

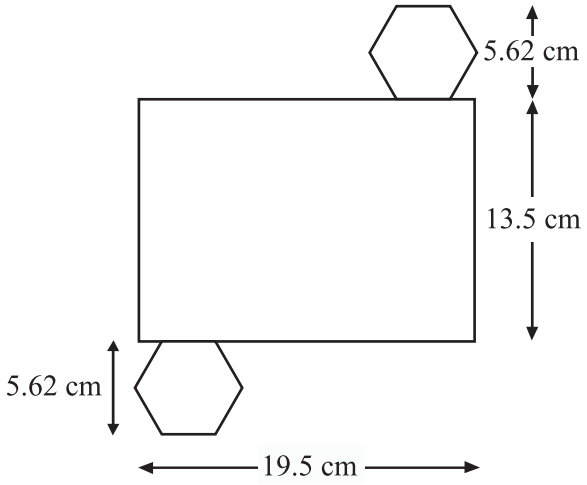
$100 \div 20.41 \approx 4.9$

An array of three rectangles by four rectangles would fit on the sheet. At least 12 shapes would fit on the sheet.

If the circles may be separated from the rectangle, more of the shapes could be cut out of the cardboard. One possible arrangement is shown below.



24 rectangles and 54 circles could be arranged on the square.



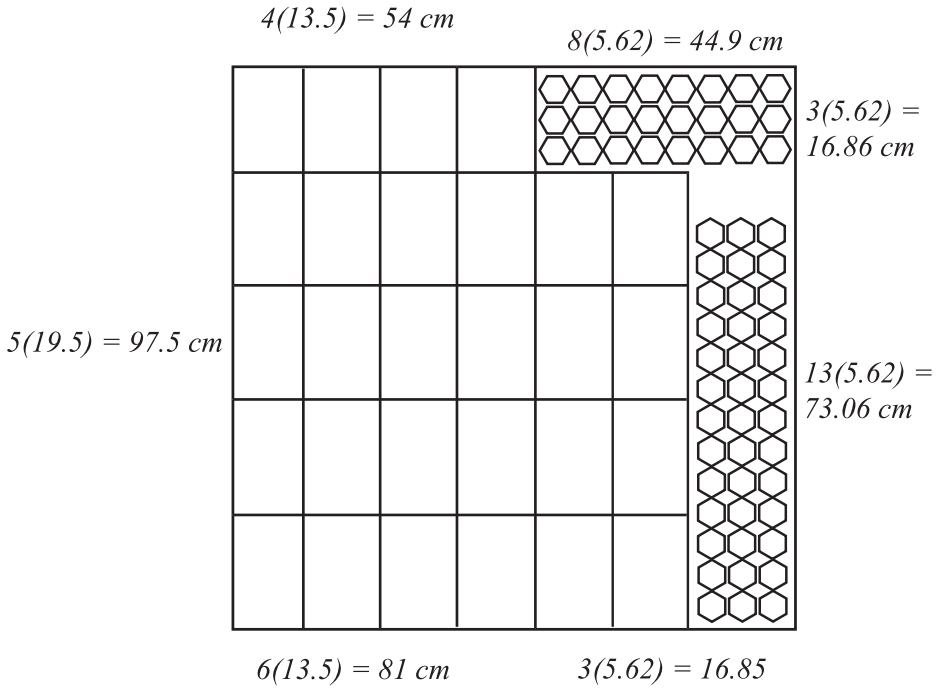
The total width is 19.5 cm, and the length is 5.62 + 5.62 + 13.5 or 24.74 cm. The question is, "How many rectangles of this size will fit in a rectangle 100 cm by 100 cm?"

$100 \div 19.5 \approx 5.128$

$100 \div 24.74 \approx 4.04$

An array of five shapes by four rectangles would fit on the sheet. At least 20 shapes would fit on the sheet.

If the hexagons are not connected to the rectangles, more could be positioned on the cardboard. One possible arrangement is demonstrated in the diagram below.



Rectangles:

4(5) = 20
2(4) = 8
28 rectangles

Hexagons:

13(3) = 39
8(3) = 24
63 hexagons

Thus, 28 sides and 56 tops and bottoms will fit in the space.

Student Work

A group of four students created a poster of their solution to this problem. The work has been copied on the next three pages.

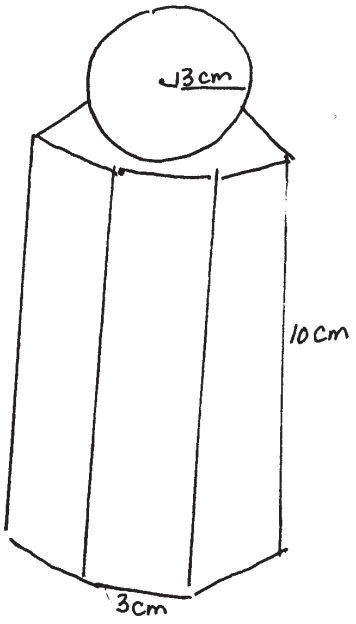
This work exemplifies all of the solution guide criteria. For example:

- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

The students' diagrams of the solids and nets are correct. They have labeled the measurements of the perfume bottle, the containers and the nets.

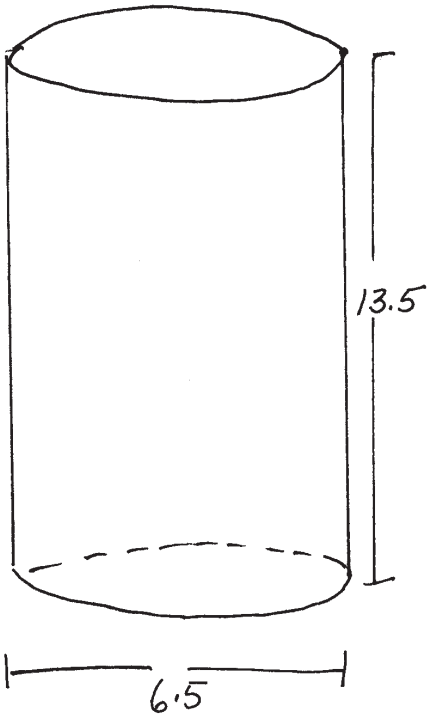
- Communicates clear, detailed, and organized solution strategy.

The students used the words, "first", "to find these measurements we did the following things". They described what they did and why. They listed the formulas used, what the variables in the formula represented and what they were for this particular situation. "Formula of the surface area (SA) of a hexagonal prism is SA = Ph+2B. So we know the perimeter, p, of the prism is ….

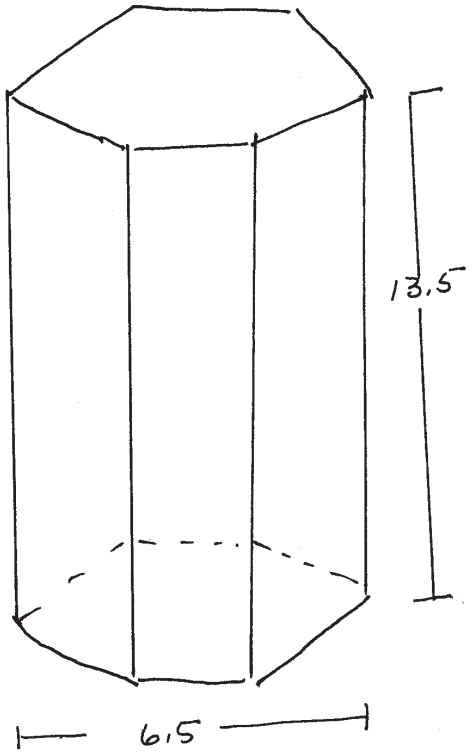


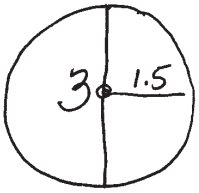
First, we were give the information that this perfume bottle wase 10 cm high, each base edge was 3 cm. wide, and the radius of the sphere was 1.5 cm. So we labeled accordingly.

Option #1



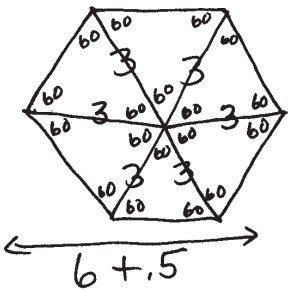
Option #2





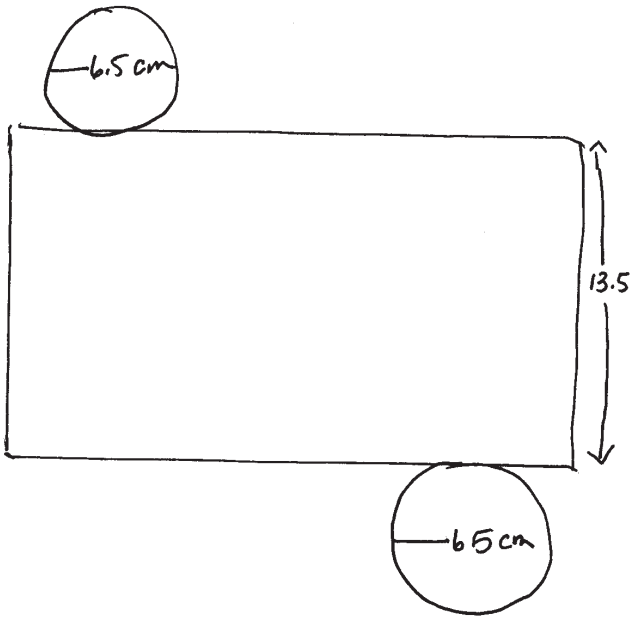
② We were given the information that the box should be .5 cm bigger at the highest and widest parts. To find these measures we did the following things. The radius of the circle at the top of the bottle is 1.5 cm. The diameter would be 3 cm. Therefore the height of the bottle is 13 cm. The height of the box must be 13.5 cm (13 + .5). The width is 6 cm on the bottle.

$$+10$$
$$13$$
$$+.5$$
$$13.5$$



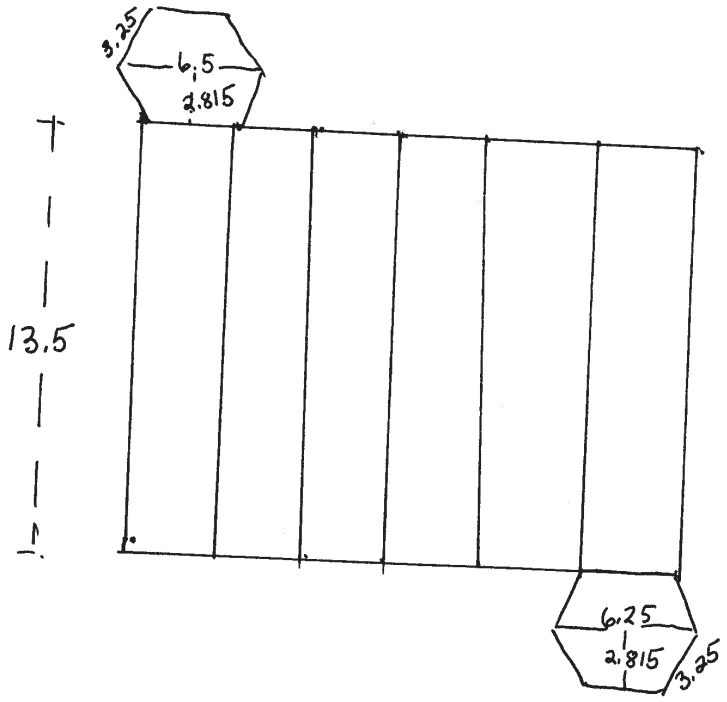
We found that by splitting the hexagon into equilateral triangles the width was 6 cm (3 + 3) to make it as big as needed the width would need to be 6.5 cm on the packing box.

Angles $\frac{180(6-2)}{6} = \frac{720}{6} = 120$ $\frac{120}{2} = 60^\circ$ each angle.

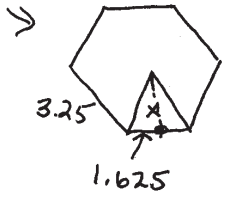


① Formula of Surface Area (SA) of a cylinder is $2\pi rh + \pi r^2$. So we know the height is $h = 13.5$ and the radius is $r = 3.25$ So we fill in the formula accordingly, and solve.

$$SA = 2\pi(3.25)(13.5) + 2\pi(3.25)^2$$
$$SA = 2\pi(43.875) + 2\pi(10.563)$$
$$SA = (137.8) + 33.2$$
$$SA = 341.9 \text{ cm}^2$$



Formula of surface Area (SA) of a hexagonal prism is $SA = Ph + 2B$. So we know the perimeter, P, of the prism is $6(3.25) = 19.5$ and the height $h = 13.5$. In addition, area of a hexagon equals $SA = \frac{1}{2}Pa$ where P stands for perimeter a stands for apothem. Since $P = 19.5$ and $a = 2.815$



then we use Pythagorean Theorem

$$3.25^2 = 1.625^2 + x^2$$
$$x^2 = 7.421875$$
$$x = \sqrt{7.421875}$$
$$x = 2.815$$

So we fill in the formula accordingly...

Area of the hexagon $= \frac{1}{2}Pa$

$$A = \frac{1}{2}(19.5)(2.815)$$
$$A = 27.4.$$

Now we know the area of the hexagon or base we fill in the SA of the prism formula. $SA = Ph + 2B$

$$SA = (19.5)(13.5) + 2(27.4)$$
$$SA = 318.1\ cm^2$$

The hexagon prism would serve as the most cost efficient container because it has a lesser surface area than the cylinder.

Playing with Pipes

R&B Engineers have been hired to design a piping system for an industrial plant. The design calls for 4 different shaped pipes that will carry water to the industrial plant. The pipes will be constructed out of four rectangular pieces of sheet metal, each with a width of 360 cm and a length of 100 cm. The metal will be folded or rolled to make the 4 pipes, each having cross sections with the same perimeter, 360 cm. One of the pipes has a rectangular cross section with dimensions of 60 cm by 120 cm. One pipe cross section is a square; one is a regular hexagon, and one is circular. The length of each pipe will be 100 cm.

1. Draw a sketch of each pipe, and label the dimensions on each cross section of pipe.

2. The pipe with the greatest flow is considered to be the pipe with the greatest cross section area. Determine which cross section of pipe will allow for the greatest flow of water, and list them in order from greatest to least. Justify your answers.

Teacher Notes

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(b.4) Geometric structure. The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) Geometric patterns. The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(d.1) Dimensionality and the geometry of location. The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

Scaffolding Questions:

- How can the perimeter of the square cross section be determined? The regular hexagonal cross section? The circular cross section?
- How can you determine the amount of water flow through the pipes?

Sample Solutions:

1. All of the pipes have a cross section with the same perimeter. The given perimeter is 360 cm. The dimensions of the rectangular cross section are given in the problem. The dimensions of the other cross sections can be calculated as follows:

Square: $\frac{360 \text{ cm}}{4} = 90$ cm per side

Regular Hexagon: $\frac{360 \text{ cm}}{6} = 60$ cm per side

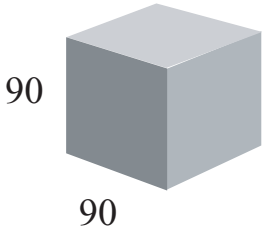
Circle: $2\pi r = 360$

$\pi r = 180$

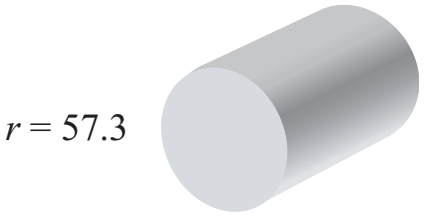
$\pi r \approx 57.3$ cm

The pipes are shown below.

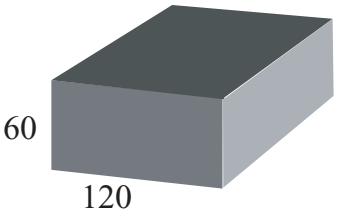
Square



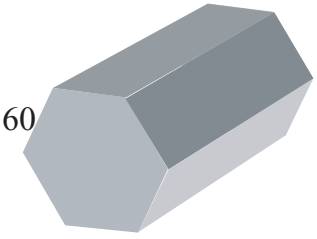
Circular



Rectangle



Regular Hexagon



2. In order to determine the pipe with the greatest water flow, the area of each cross section must be calculated.

Listed in order, from greatest flow to least flow:

The area of the circular region will produce an opening that is: $A = \pi r^2 = \pi(57.3)^2 = 10{,}314 \text{ cm}^2$.

The area of the regular hexagonal region will produce an opening that is: $A = \frac{1}{2}ap = \frac{1}{2} \bullet 30\sqrt{3} \bullet 360 \approx 9353 \text{ cm}^2$.

The area of the square region will produce an opening that is: $A = s^2 = 90 \bullet 90 = 8100$ square cm.

The area of the rectangular region will produce an opening that is: $A = lw = 60 \bullet 120 = 7200$ square cm.

Extension Questions:

- A pipe having a circular cross section with an inside diameter of 10 inches is to carry water from a reservoir to a small city in the desert. Neglecting the friction and turbulence of the water against the inside of the pipes, which is the minimum number of 2" inside-diameter pipes of the same length that is needed to carry the same volume of water to the small city in the desert? Explain your answer.

In order to solve this problem, it will be necessary to find the number of smaller pipes that it will take to provide the same cross sectional area as that of the larger pipe.

The cross sectional areas of the pipes are circles having the area $A = \pi r^2$.

The radius of the 10-inch pipe is 5 inches, and the radius of the 2-inch pipe is 1 inch.

The area of the 10-inch pipe is $A_1 = \pi 5^2 = 25\pi \text{ in}^2$

The area of the 2-inch pipe is $A_2 = \pi 1^2 = 1\pi \text{ in}^2$

The ratio of the areas is $25\pi : 1\pi$ *or 25 : 1. This implies that the cross sectional area of the 10-inch pipe is 25 times larger than the cross sectional area of the smaller pipe. It will require 25 of the 2-inch pipes to provide the same cross sectional area as the 10-inch pipe.*

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

(e.1) Congruence and the geometry of size. The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Bayou City Dome

An alternate approach is to realize that the ratio of the diameters is 10 to 2 or 5 to 1. *Thus, the ratio of the areas is 5^2 to 1^2 or 25 to 1. Therefore, the requirement is 25 of the 2-inch pipes.*

- Suppose the cross sections of the pipes are squares rather than circles. The diagonal of one square is 10 inches, and the diagonal of the second square is 2 inches. Will the shape of the pipes affect the number of pipes required to produce equal areas? Justify your answer.

It makes no difference what shapes the pipes are, provided that the cross sectional areas are the same. If the pipes had square cross sections, and the diagonals were 10 inches and 2 inches, it will still require 25 smaller pipes to provide the same cross sectional area as the larger pipe. The pipe with the 10-inch diagonal forms two 45-45-90 triangles with leg lengths of $5\sqrt{2}$ inches. Therefore, the square cross section will have side lengths of $5\sqrt{2}$ inches. The area of the pipe with the 10-inch diagonal will be $A = \left(5\sqrt{2}\right)^2 = 50 \ in^2$.

The pipe with the 2-inch diagonal forms two 45-45-90 triangles with leg lengths of $1 \bullet \sqrt{2}$ inches. Therefore the square cross section will have side lengths of $1 \bullet \sqrt{2}$ inches. The area of the pipe with the 2-inch diagonal will be $A = \left(1 \bullet \sqrt{2}\right)^2 = 50 \ in^2$.

The ratio of the areas of the square cross section is 50 : 2 or 25: 1. Therefore, it will require 25 of the smaller square pipes to provide the same cross sectional area as the larger pipe.

Circular Security

The Circular Security Company manufactures metal cans for small amounts of medical hazardous waste. They have contracted your marketing firm to create a label for the waste can. The can is a right circular cylinder with a base radius of 9 inches and a height of 15 inches.

The slogan for Circular Security is "*Circular* — For that all-around sense of security!" As an employee of the marketing company, your task is to design a label for the can that has a relationship to the slogan. You have decided that a spiral stripe is to be painted on the label of the can, winding around it exactly once as it reaches from bottom to top. It will reach the top exactly above the spot where it left the bottom.

1. Create a net showing the dimensions of the can and the placement of the stripe. Round dimensions to the nearest tenth of an inch if needed.
2. Determine the length (in inches) of the stripe. Round your answer to the nearest tenth of an inch if needed.
3. Justify your solution.

Teacher Notes

Materials:
One graphing calculator per student

Connections to Geometry TEKS:

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(B) uses nets to represent and construct three-dimensional objects; and

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem.

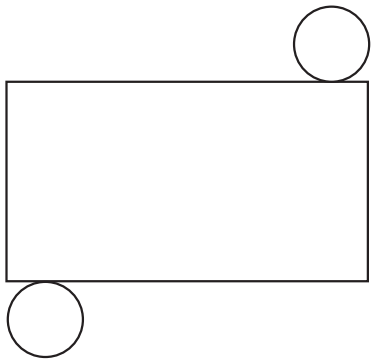
Scaffolding Questions:

- What 2-dimensional figures comprise a net of a cylinder?
- If the radius of the cylinder is 9 inches, what is the circumference of the cylinder?
- How will knowing the circumference of the cylinder help you determine the dimensions of the net?
- Where would the stripe be on the net of the cylinder?
- How can you calculate the length of the stripe?

Sample Solution:

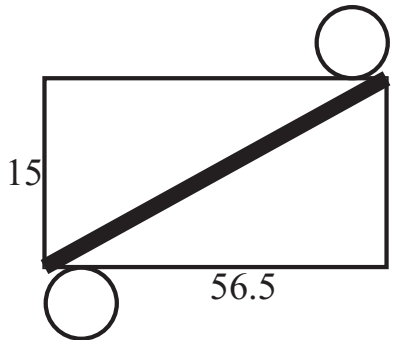
This problem begins as a 3-dimensional problem but can be reduced to a 2-dimensional problem to make the solution easier. Some cans, like ones that might be found in a grocery store, have a label all the way around them. If the label is removed and flattened out, the shape of the label is a rectangle. The top and the bottom of the can are circles.

The net for the can is:



The height of the can is 15 inches; this will also be the height of the rectangle representing the label. The radius of the circular base of the can is used to find the circumference of the circle, which will turn out to be the width of the rectangular label.

If $r = 9$ inches, then the circumference of the circle is $2\pi(9) \approx 56.5$ inches. Therefore the label's width is also 56.5 inches.



Let the length of the diagonal be represented by x. The diagonal black line in the net above represents the stripe. The problem states that the stripe goes from the top to the bottom of the can in exactly one circumference; on the rectangle this is the length of the diagonal.

The Pythagorean theorem can be used to find the length of the diagonal as follows:

$$15^2 + 56.5^2 = x^2$$
$$225 + 3192.25 = x^2$$
$$58.5 \approx x$$

The length of the diagonal stripe, from corner to corner, is approximately 58.5 inches.

Extension Questions:

- Suppose a Circular Security customer wants to place a special order for a can that is similar to the original can but will have a height of 10 inches. Determine the dimensions of the new can and the length of the stripe that will encircle the can.

If a customer wants to order a similar can with a height of 10 inches, the ratio of the corresponding sides is 15 to 10, or 3 to 2. To find the radius of the base, solve the proportion.

$$\frac{3}{2} = \frac{9}{x}$$
$$x = 6 \text{ inches.}$$

The radius of the special-order cylinder is 6 inches.

The circumference of the base is the same as the width of the rectangle.

Texas Assessment of Knowledge and Skills:

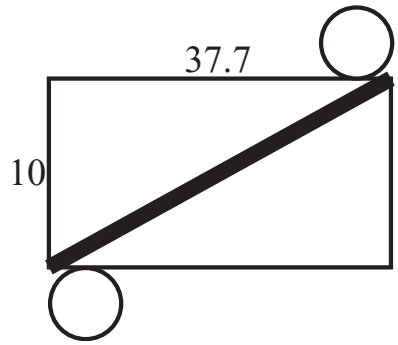
Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Dwellings

$C = 2\pi\ (6)$
$C \approx 37.7$ inches



The length of the stripe encircling the special order can is represented by x:

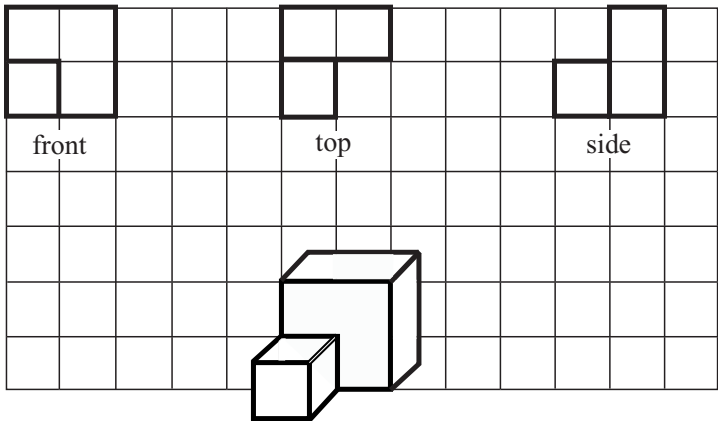
$$10^2 + 37.7^2 = x^2$$
$$100 + 1421.29 = x^2$$
$$1521.29 = x^2$$
$$x \approx 39.0 \text{ inches}$$

- Compare the stripe of the original can to this new can's stripe-length.

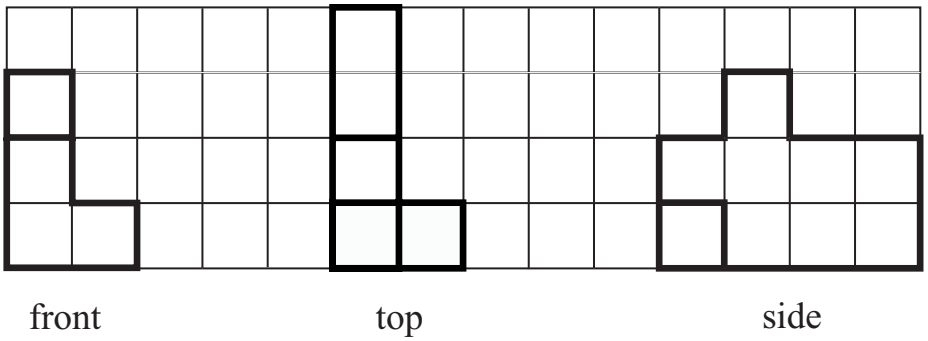
The ratio of the original stripe-length to the new stripe-length is 58.5 in. to 39.0 in. $\frac{58.5}{39.0} = \frac{1.5}{1} = \frac{3}{2}$ *The ratio of the stripes is the same as the ratio that compares the dimensions of the two cans or 3 to 2.*

Different Views

The picture below shows the front, top, and side views of an object as well as the object itself. This object is made of five unit cubes (four in the back row and one in the front), so the volume is 5 units3. The surface area is 20 units^2.



Below are the front, top, and side views of another object. Draw this object, and explain how the given views helped you construct your final object. Calculate the volume of this object, and carefully explain how you got your answer.



Materials:
Cubes or blocks for each student (optional)

Connections to Geometry TEKS:

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems**.**

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

(B) uses nets to represent and construct three-dimensional objects; and

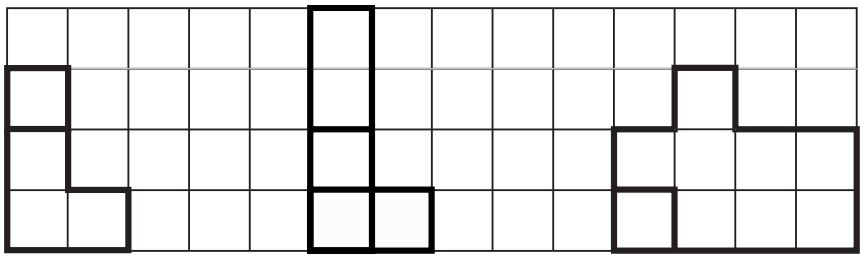
(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

Teacher Notes

Scaffolding Questions:

- What does the square on the front view of the very first figure tell you?
- Demonstrate your answer above by using cubes.
- Build the final figure in the first set of diagrams.
- How many cubes make up the final figure?
- What is the volume of one of the cubes?
- How do you know that the volume of the first final figure really is 5 cubic units?
- Determine the surface area of one cube.
- Explain how you know the surface area of the first final figure really is 20 square units.

Sample Solution:

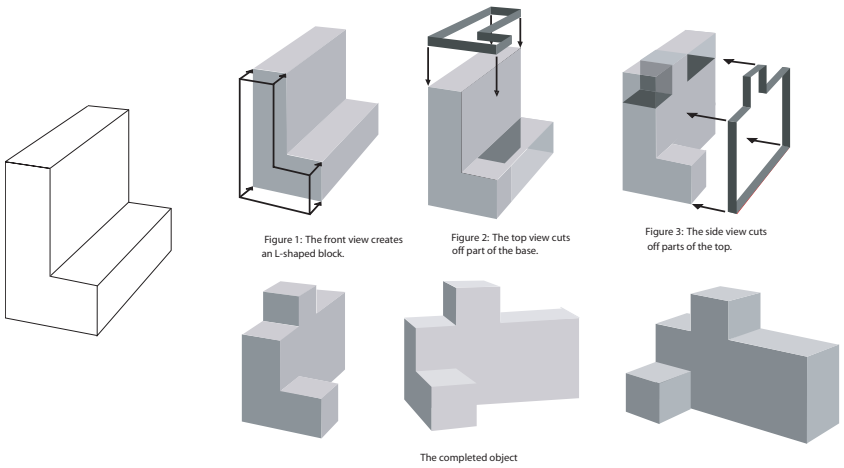


front top side

The front view of the object has 3 blocks in an L shape and another block on top that is either farther in or farther out.

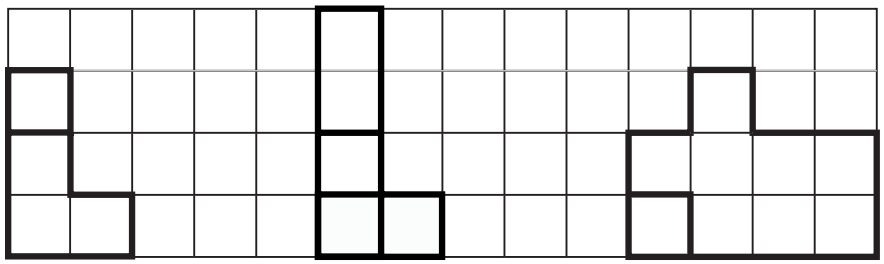
The top view indicates that the front 3 squares have different heights and that there are blocks at the end that are of the same height. The top view also produces an overall L shape.

The side view indicates that there is 1 block that sticks out. This is also visible from the front and top views. In the side view there is also a wall 2 blocks high and 4 blocks long. There is a tower on top of the wall consisting of 1 block. From this information, a picture of the figure is drawn.



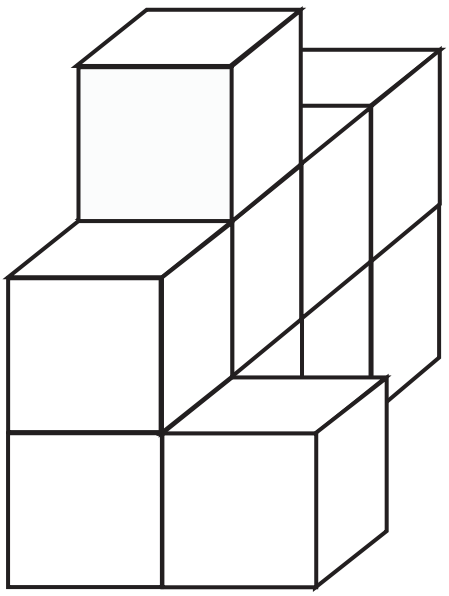
Using the given information in the original diagrams, a sketch is drawn showing the individual blocks.

Given Diagram:



front top side

New figure with individual blocks drawn:



Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

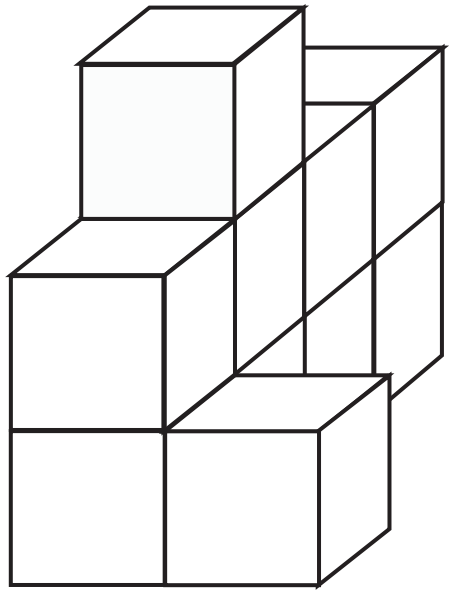
V. Solid Figures: What's Your Perspective

Using the new figure, the total number of blocks is determined as follows. Consider each row to be a tower if the blocks are separated. There are 3 blocks in the front row (the L-shaped front view), 3 blocks are in the second row (a 3-block tower), 2 bocks in the third row, and 2 blocks in the fourth row. The total number of blocks is 3 + 3 + 2 + 2 = 10 blocks. Each block is 1 cubic unit; therefore the figure has a volume of 10 cubic units.

Extension Questions:

- Find the surface area of the second object. Justify your answer.

To find the surface area, the figure is divided into rows, front to back.



Each face on a cube that is showing adds a square unit to the surface area of the figure. Faces that touch each other are not considered part of the surface area. The bottom, back, and left side of the figure is part of the surface area, even though they are not showing.

There are 3 blocks in the front row (the L-shaped front view.) This tower has 3 faces showing on the front, 1 face showing on the back, 2 on the left side, 2 on the right side, and 2 on top. There are 2 faces on the bottom of the tower. There are a total of 12 faces on the front tower.

The second row is a 3-block tower. This tower has 1 face showing from the front, 1 face on the back, 1 face on the bottom, 1 face on the top, 3 faces on the left side, and 3 faces on the right side. There are a total of 10 faces on the second tower.

The third row is a tower, 2 blocks high. There are no faces showing from the front or the back. There is 1 face on the top, 1 face on the bottom, 2 faces on the left side, and 2 faces on the right side. There are a total of 6 faces on the third tower.

The fourth row is a tower that is also 2 blocks high. There are no faces showing from the front. There are 2 faces on the back, 1 face on the bottom, 1 face on the top, 2 faces on the left, and 2 faces on the right. There are a total of 8 faces on the fourth row.

The total number of faces that comprises the surface area is found by adding the total from each row: 12 + 10 + 6 + 8 = 36 faces. The surface area of this figure is 36 square units.

In an alternate approach, the surface area can be found by subtracting the number of faces that are facing each other from the total number of faces of the 10 cubes.

The total number of faces of 10 cubes is 10•6 because each cube has 6 faces.

There are 12 cube faces that are touching another face. The cube at the front right faces another cube. When row two is stacked on row one there are 4 horizontal faces touching each other and there are 6 vertical faces touching each other. There is one more face touching another face when row three is stacked. Thus, the total number is 1 + 4 + 6 + 1 or 12 faces.

The total number of facing cubes is 12•2 or 24 faces.

The number of faces in the surface area is 60 – 24 or 36 faces. Therefore, the surface area of the figure is 36 square units.

The Slice Is Right!

A cube has edges that are 4 units in length. At each vertex of the cube, a plane slices off the corner of the cube and hits each of the three edges at a point 1 unit from the vertex of the cube.

1. Draw the cube showing how the corners will be sliced.
2. Draw and label a figure that represents the solid that is cut from each corner of the cube.
3. How many cubic units are in the volume of the solid that remains? Justify your answer.

Materials:
One graphing calculator per student

Styrofoam or play dough cubes, fishing line to cut off corners

Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem; and

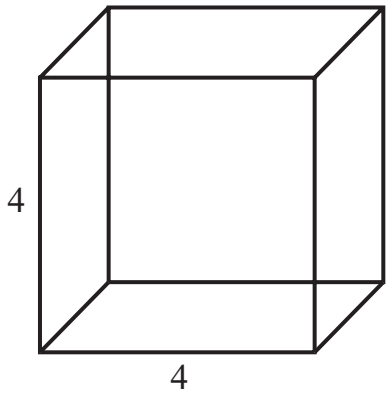
Teacher Notes:

Scaffolding Questions:

- If the plane cuts off the corner one unit from the edge of the cube, what is the shape of the piece that is cut off?
- How many of the corners are cut off?
- What are the dimensions of each piece that is cut off?
- Determine the volume of the original cube.
- Determine the volume of each cut off piece.

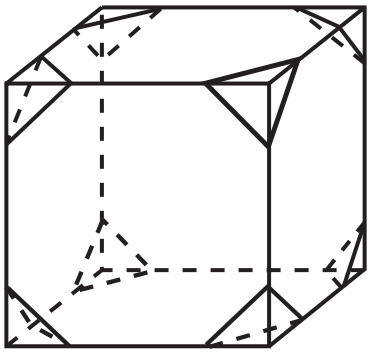
Sample Solutions:

1. Original Cube:

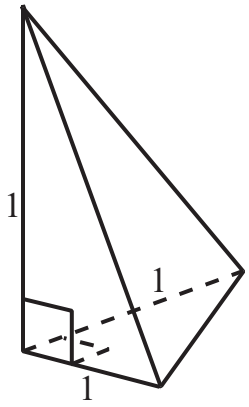


When the corners of the cube are cut off, eight congruent pyramids with four triangular faces are formed. Three of the faces are isosceles right triangles with legs that are 1 unit in length. The fourth face is an equilateral triangle. The side of this triangle is also the hypotenuse of the isosceles right triangle.

A cube with all corners sliced by a plane at a distance of 1 unit from each corner:



2. Each triangle pyramid may be positioned so that its base is one of the isosceles right triangles. The height of the pyramid is also 1 unit. The corner of the cube forms a right angle on each face of the pyramid.



3. Each triangular pyramid that is sliced off has volume:

$$V = \frac{1}{3}Bh = \frac{1}{3}\left(\frac{1}{2} \bullet 1 \bullet 1\right) \bullet 1 = \frac{1}{6}.$$

The volume of the 8 pyramids that are sliced off the cube is:

$$8 \bullet \frac{1}{6} = \frac{8}{6} = \frac{4}{3}.$$

The volume of the original cube is: $V = 4^3 = 64$ cubic units.

The volume of the solid after the 8 corners are removed can be found by subtracting the volume of the 8

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Bayou City Dome

Teacher's Comment:

"Some students who don't do as well on traditional assignments were in their element with this problem. It pleased me that several students saw how you would turn the corner pieces around in order to find the volume before I asked any scaffolding questions. It worked well to have them put their solutions on chart paper. They got a chance to be thorough with their thinking as well as their creativity."

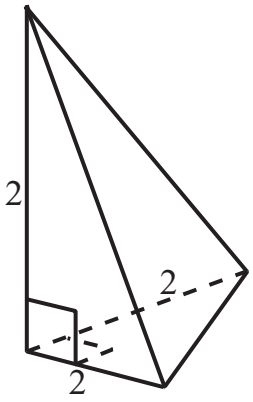
pyramids from the volume of the original cube.

V = original cube – 8 pyramid volumes

$V = 64 - \frac{4}{3} = 62\frac{2}{3}$ cubic units.

Extension Questions:

- Suppose the plane slices off sections that are 2 units from each vertex of the cube. How does the volume of the remaining figure compare to the original slice (1 unit from each vertex)? Calculate the percent of decrease of the volume. Round your answer to the nearest percent. Justify your answer.



There are 8 corners on the original cube; therefore 8 triangular pyramids are formed. Each triangle pyramid has an isosceles right triangle for the base and 2 faces. The area of the base triangle is $\frac{1}{2} \bullet 2 \bullet 2$ *or* 2 *square units.*

The volume of one pyramid is $\frac{1}{3} \bullet 2 \bullet 2$ *or* $\frac{4}{3}$ *cubic units.*

The volume of the 8 pyramids is $8\left(\frac{4}{3}\right)$ *or* $\frac{32}{3}$ *or* $10\frac{2}{3}$ *cubic units.*

The volume of the solid after the eight unit corners have been removed is $64 - 10\frac{2}{3}$ *or* $53\frac{1}{3}$.

The difference of the volume of the solid after the eight 2-unit corners are removed and the cube with the eight 1-unit pyramids removed can be found by subtracting the volumes.

V = volume of cube with 8 1-unit pyramids – volume of cube with 8 2-unit pyramids

$V = 62\frac{2}{3} - 53\frac{1}{3} = 9\frac{1}{3}$ *cubic units*

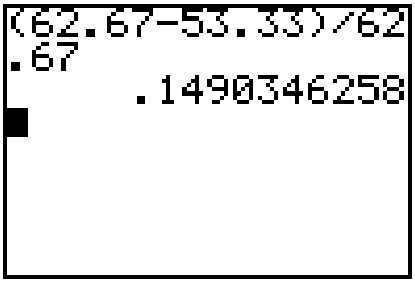
The percent of decrease in the volume of the cubes is found by calculating the following:

$$\frac{\text{(original volume)} - \text{(new volume)} \bullet 100}{\text{(original volume)}}$$

Fractions were converted to decimals for ease of use in calculation:

Original volume with 1-inch slices: $62\frac{2}{3} = 62.67$ *cubic units*

New volume with 2-inch slices: $53\frac{1}{3} = 53.33$ *cubic units*



The percent of volume decrease is rounded to 15%.

Student Work Sample

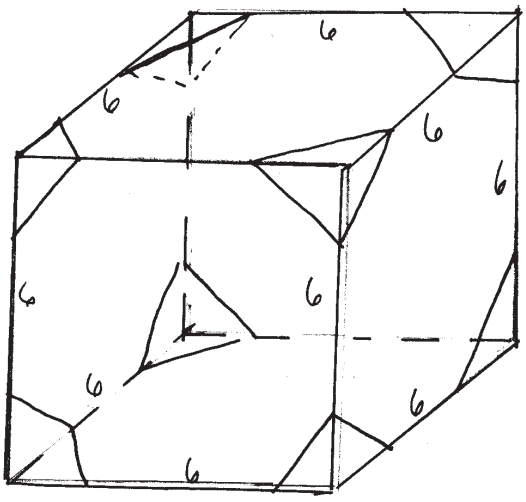
The student who created the work on the next page

- Shows an understanding of the relationships among elements.

The student understands what the given information means and how to interpret the given to create the solid that is cut off from the corners of the cube. He correctly draws the resulting triangular pyramid.

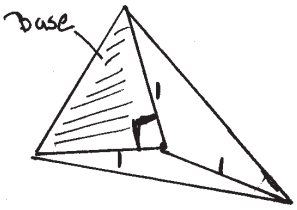
- Demonstrates geometric concepts, processes, and skills.

The student shows the formulas used to determine area and volume. He shows the triangular pyramid, shows the base of the pyramid, and how to determine the volume using the right triangle base.



A cube has edges of length 6 units at each vertex of the cube a plane slices off the corner of the cube and hits each of three edges at a point 1 unit from the vertex of the cube.

The volume of the original cube is: $6^3 = 216$ units^3



Area of one of the cut off corners.
+
Volume

Area of Base: $\frac{\text{Base} \cdot \text{High}}{2}$: $\frac{1 \cdot 1}{2} = 0.5$ units^2

Volume: $0.5 \cdot 1 = \frac{0.5}{3} = 1/6$ units^3

The Volume of one of these pyramids is $1/6$ units^3

So the added volume of all these pyramids combined is $\boxed{1 \cdot 1/3}$ units^3

- Subtract the volume of the corners from the volume of the cube

$$\begin{array}{r} 216 \\ -\ 1\,1/3 \\ \hline 214\,2/3 \end{array}$$ cubic units