

Chapter 6:

Congruence





Introduction

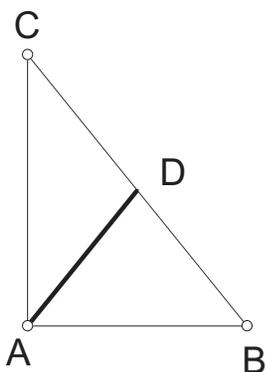
Application problems can provide the context for using geometry and for applying problem-solving techniques. The six problems in this chapter make connections among geometric concepts. The students will use the concept of congruence to solve problems. For example, The Shortest Cable Line involves a reflection of a triangle. Tell Me Everything You Can About... requires constructions. The School Flag assesses area concepts. Median to the Hypotenuse of a Right Triangle requires conjecturing and justification using axiomatic or coordinate methods.





Median to the Hypotenuse of a Right Triangle

1. On a right triangle how does the length of the median drawn to the hypotenuse compare with the length of the hypotenuse? Use at least two approaches—drawings, constructions, or appropriate geometry technology—to investigate and conjecture a relationship between these lengths.



Triangle ABC is a right triangle with $\overline{AC} \perp \overline{AB}$. \overline{AD} is the median to the hypotenuse.

2. Prove your conjecture using one of these methods:
 - a) congruence transformations (isometries)
 - b) a Euclidean argument
 - c) a coordinate proof
3. Is your conjecture for right triangles valid for other triangles? Why or why not?
4. Is the converse of your conjecture true? Why or why not?



Materials:

One compass and ruler per student

Unlined paper, patty paper, and geometry software

Connections to Geometry**TEKS:**

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

Teacher Notes

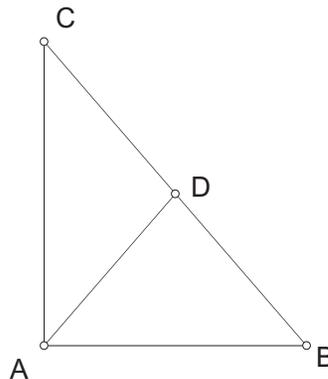
Scaffolding Questions:

- What is meant by a median of a triangle?
- Construct a scalene right triangle, an isosceles right triangle, and a 30-60-90 triangle. Construct the median to the hypotenuse. What appears to be true about these medians? How can you use measurements to investigate your observation?
- What is meant by congruence transformations? What are these transformations?
- What is meant by a Euclidean argument?
- A right triangle can always be viewed as “half” of what polygon? How could you extend (add auxiliary segments) the diagram of the right triangle to show this?

Sample Solutions:

Use geometry technology or paper and pencil constructions and take measurements of the median to the hypotenuse and the hypotenuse. For example,

AD = 3.54 cm
 CD = 3.54 cm
 DB = 3.54 cm
 CB = 7.08 cm



In $\triangle BAC$, $\overline{BA} \perp \overline{AC}$ and \overline{AD} is the median to the hypotenuse.

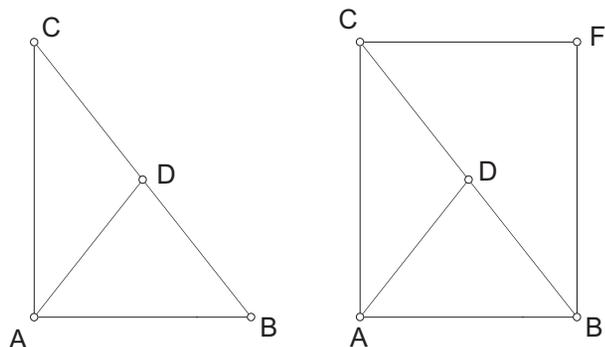
1. Conjecture: The measurements show that the median to the hypotenuse is one-half as long as the hypotenuse, or



the hypotenuse is twice as long as the median drawn to it.

2. Congruence Transformations:

Rotate $\angle ACB$ 180 degrees clockwise about point D and $\angle ABC$ 180 degrees counter-clockwise about point D. Label the point of intersection of the angles point F.



These transformations form $\angle FBC$ and $\angle FCB$ so that $\angle ACB \cong \angle FBC$ and $\angle ABC \cong \angle FCB$

This means that \overline{AC} is parallel to \overline{BF} and \overline{AB} is parallel to \overline{CF} because these angle pairs are congruent alternate interior angles.

This means that quadrilateral ABFC is a parallelogram. Since $\angle CAB$ is a right angle, we know that all four angles are right angles.

Now we know that ABFC is a rectangle. Since the diagonals of a parallelogram bisect each other, and the diagonals of a rectangle are congruent, we now know that

$$AD = \frac{1}{2}AF = \frac{1}{2}BC.$$

This proves our conjecture that on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse.

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;



(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

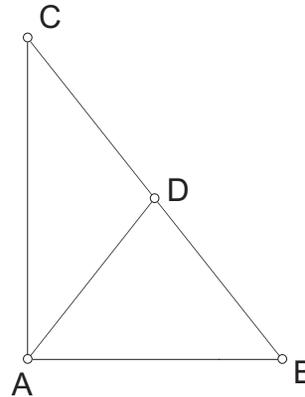
(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

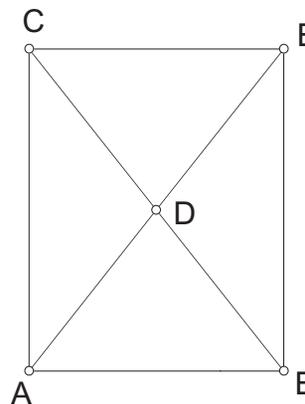
Euclidean Argument:



We are given that $\triangle BAC$ is a right triangle with a right angle at A and that \overline{AD} is the median to the hypotenuse.

By the definition of a median, we know that $CD = DB$.

Extend \overline{AD} through point D to point E so that $AD = DE$, and draw \overline{EC} and \overline{EB} to form quadrilateral ACEB.



By the Vertical Angle Theorem, $\angle ADC \cong \angle EDB$.

Now $\triangle ADC \cong \triangle EDB$ by SAS.

It follows that $\overline{AC} \cong \overline{EB}$ and $\angle CAD \cong \angle BED$ because these are corresponding parts of congruent triangles.

These congruent angles are alternate interior angles formed by transversal \overleftrightarrow{AE} on \overleftrightarrow{AC} and \overleftrightarrow{EB} . Therefore, \overleftrightarrow{AC} and \overleftrightarrow{EB} are parallel.

Now consider quadrilateral ACEB. Since quadrilateral ACEB

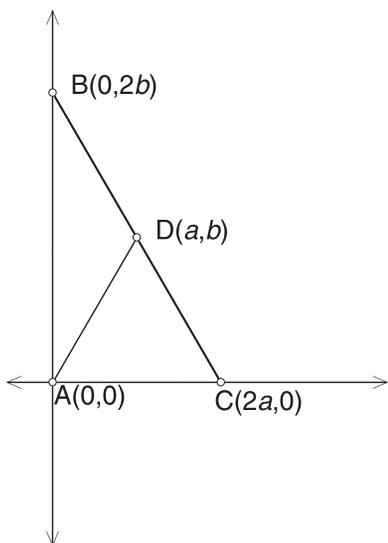


has a pair of opposite sides that are congruent and parallel, the quadrilateral is a parallelogram.

Next, since opposite angles of a parallelogram are congruent, and consecutive angles are supplementary, and since $\angle BAC$ is a right angle, the other angles of the parallelogram are right angles. This makes $ACEB$ a rectangle.

Finally, the diagonals of a rectangle bisect each other and are congruent so that $AD = \frac{1}{2} AE = \frac{1}{2} BC$.

Coordinate Proof:



Draw right triangle BAC with right angle A at the origin and the legs along the axes.

Assign to vertex C the coordinates $(2a, 0)$ and to vertex B the coordinates $(0, 2b)$.

We know that \overline{AD} is the median to the hypotenuse, which means point D is the midpoint of \overline{BC} . We apply the Midpoint Formula to get the coordinates of point D.

$$\left(\frac{2a+0}{2}, \frac{0+2b}{2} \right) = (a, b)$$

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Connections to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Investigating Quadrilateral Properties
- III. Triangles: Pythagorean's Theorem



Now we use the distance formula to find the lengths of the median and the hypotenuse.

$$AD = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

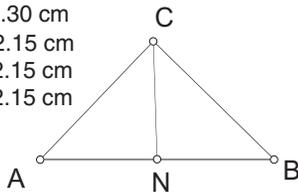
and

$$\begin{aligned} BC &= \sqrt{(2a-0)^2 + (0-2b)^2} \\ &= \sqrt{4a^2 + 4b^2} \\ &= \sqrt{4(a^2 + b^2)} \\ &= 2\sqrt{a^2 + b^2} \end{aligned}$$

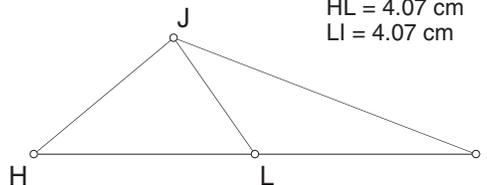
This shows that $AD = \frac{1}{2}BC$; that is, on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse.

3. Other triangles to consider are an acute triangle and an obtuse triangle. Using geometry technology or paper and pencil constructions, we can easily investigate this situation. Possible values are given below.

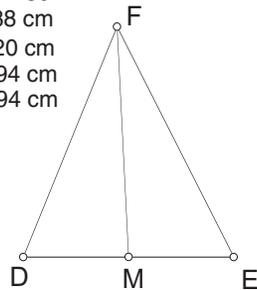
$m\angle ACB = 90^\circ$
 $AB = 4.30$ cm
 $CN = 2.15$ cm
 $AN = 2.15$ cm
 $NB = 2.15$ cm



$m\angle HJI = 120^\circ$
 $HI = 8.15$ cm
 $JL = 2.59$ cm
 $HL = 4.07$ cm
 $LI = 4.07$ cm



$m\angle DFE = 50^\circ$
 $DE = 3.88$ cm
 $FM = 4.20$ cm
 $DM = 1.94$ cm
 $ME = 1.94$ cm



$\triangle ABC$ is a right triangle and demonstrates the conjecture we made and proved.

$\triangle DFE$ shows the median drawn from an acute angle to the opposite side and shows that the median is longer than half of the side to which it is drawn.

$\triangle HJI$ shows the median drawn from an obtuse angle to the opposite side. It shows that the median is shorter than half of the side to which it is drawn, but it is not half the length of that side.



This suggests the following conjecture:

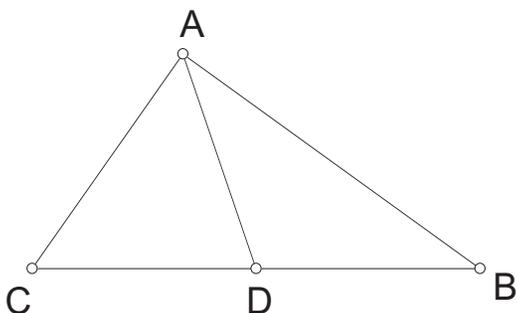
If a median on a triangle is drawn from an acute angle, the median will be longer than half of the side to which it is drawn. If a median on a triangle is drawn from an obtuse angle, the length of the median will be less than half the length of the side to which it is drawn.

4. The converse of our conjecture is the following:

If a median drawn on a triangle is half as long as the side to which it is drawn, then the median is drawn to the hypotenuse of a right triangle.

This is true, as the following argument shows:

Let \overline{AD} be a median on $\triangle CAB$ and $AD = \frac{1}{2}CB$.



Then, by the definition of a median, D is the midpoint of \overline{CB} , and we have $CD = DB = AD$. This makes $\triangle CDA$ and $\triangle BDA$ isosceles triangles with bases \overline{CA} and \overline{BA} . Since base angles of an isosceles triangle are congruent,

$$m\angle DCA = m\angle DAC = x, \text{ and}$$

$$m\angle DBA = m\angle DAB = y.$$

Now the angles of $\triangle CAB$ sum to 180° so that

$$m\angle DCA + m\angle CAB + m\angle DBA = x + (x + y) + y = 180^\circ.$$

This simplifies to

$$2x + 2y = 180$$

$$x + y = 90.$$

This shows that $\angle CAB$ is a right angle, and $\triangle CAB$ is a right triangle with hypotenuse \overline{CB} .



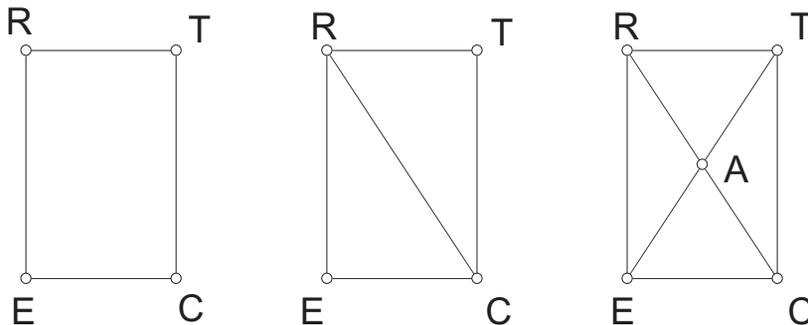
Extension Questions:

- How does this problem relate to your work with parallelograms?

Desired Response: A right triangle is formed when a diagonal is drawn on a rectangle. The diagonals of a rectangle are congruent and bisect each other. Therefore, on a right triangle, the median of a right triangle drawn to the hypotenuse is half the diagonal drawn on a rectangle.

Consider rectangle $RECT$. Draw diagonal \overline{RC} to form a right triangle, $\triangle REC$ with hypotenuse \overline{RC} . Draw diagonal \overline{ET} , and label the point of intersection A . Because of the properties of rectangles, $RA = AC = EA = AT$.

Therefore, \overline{EA} becomes the median drawn to the hypotenuse \overline{RC} of $\triangle REC$ and is half as long as \overline{RC} .



- You have shown that on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse. You were given the choice of three types of proofs. What might be an advantage or disadvantage for each type?

We can write the proof using congruence transformations, a Euclidean argument, or a coordinate proof.

The transformation approach is nice because we can model it with patty paper, and that makes it more visual.

The Euclidean approach is a little harder. We have to start with the right triangle and the median drawn on it and realize that we must draw auxiliary segments on it so that we form a rectangle with its diagonals.

A coordinate proof is nice because we see how we can use algebra to prove geometric concepts.

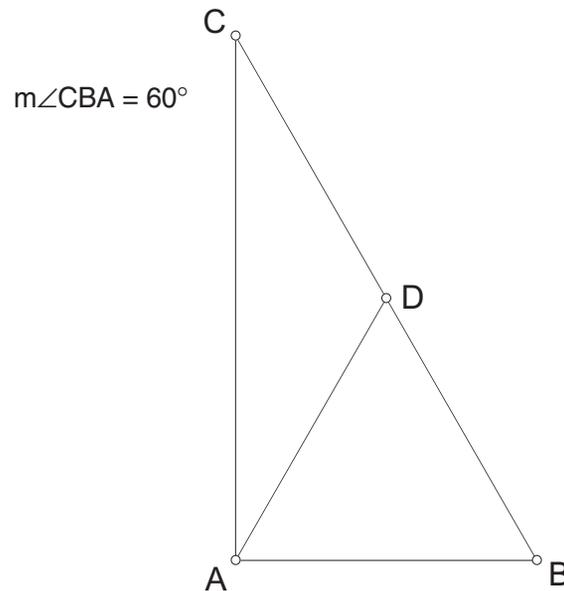
- Suppose that $\triangle BAC$ is a right triangle with $\overline{BA} \perp \overline{AC}$ and \overline{AD} as the median to the hypotenuse. What do we know about BD , AD , and CD ? How does this relate to your knowledge of circles?



Since $BD = AD = CD$ and these 4 points are coplanar, we know that points B , A , and C lie on a circle with center D . Since points B , D , and C are collinear, we know that \overline{BC} is a diameter of the circle.

- What additional patterns emerge if we draw the median to the hypotenuse of the two special right triangles (the triangle with angles that measure 30° , 60° , and 90° or 45° , 45° , and 90°)?

Consider a 30-60-90 triangle:



Triangle BAC is a 30-60-90 triangle with $m\angle BAC = 90^\circ$ and $m\angle B = 60^\circ$. \overline{AD} is the median to the hypotenuse.

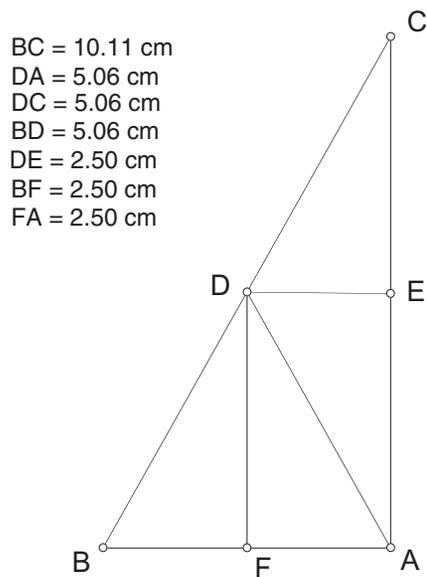
Suppose $BC = 10$. Then $AD = DB = 5$. Also, $AB = 5$. This makes $\triangle ADB$ equilateral.

The other triangle formed, $\triangle ADC$, is an isosceles triangle. The legs are $AD = DC = 5$ and base angles are 30 degrees.

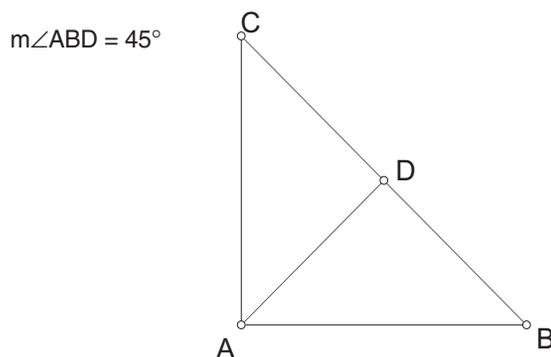
If we draw the median from point D to segment \overline{AB} , this will form two 30-60-90 triangles. Also, if we draw the median from D to \overline{AC} , we will have 30-60-90 triangles.

The second set of medians separates the original triangle into 4 congruent 30-60-90 triangles.





Now consider the isosceles right triangle:



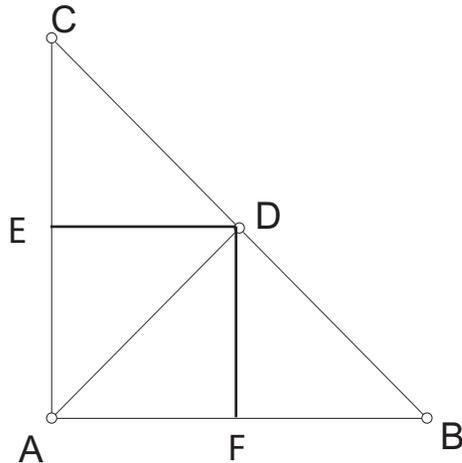
The median to the hypotenuse separates the triangle into two congruent isosceles right triangles:

1. They are isosceles because $AD = DC = DB$.
2. They are right triangles because the median from the vertex angle of an isosceles triangle is also an altitude so that $\overline{AD} \perp \overline{BC}$.
3. $\triangle ADC \cong \triangle ADB$ by SSS because of (1) and because \overline{AC} and \overline{AB} are congruent legs of the original isosceles right triangle.

If we draw the medians from point D to segments \overline{AB} and \overline{AC} , it will separate the original isosceles right triangle into four congruent isosceles right triangles.



$$m\angle ABD = 45^\circ$$

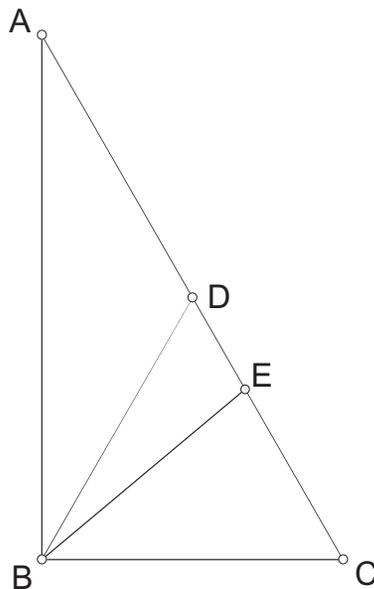


$$\triangle CED \cong \triangle AED \cong \triangle AFD \cong \triangle BFD \text{ by SSS}$$

The isosceles right triangle pattern is different from the 30-60-90 triangle pattern. Drawing the first median to a hypotenuse gives two isosceles right triangles. Drawing the medians on these repeats this pattern.

In the 30-60-90 triangle pattern, the first median drawn to the hypotenuse gives an isosceles triangle (30-30-120) and an equilateral triangle. Drawing medians to the base of isosceles triangle and the equilateral triangle gives four 30-60-90 triangles.

- The median divides the given right triangle into two triangles. Make and prove a conjecture about the areas of the triangles.



The two triangles have equal area. To demonstrate this, drop a perpendicular from B to \overline{AC} to form a segment \overline{BE} . This segment is an altitude to both triangles, with bases \overline{AD} and \overline{DC} . These two segments are equal in length. The triangles have equal bases and the same height, thus they have the same area.



Tell Me Everything You Can About...

1. Given a line segment, \overline{RO} , that is a diagonal of a rhombus RHOM, construct the rhombus. Describe your construction procedure, and explain why your figure is a rhombus.
2. List as many conclusions that you can make about the triangles that result from the construction as possible.

Justify your conclusions.

3. Explain why and how the diagonals may be used to determine the area of the rhombus.



Teacher Notes

Materials:

One compass and straightedge per student

Unlined paper, patty paper, and construction tools (including geometry software)

Connections to Geometry

TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties; and

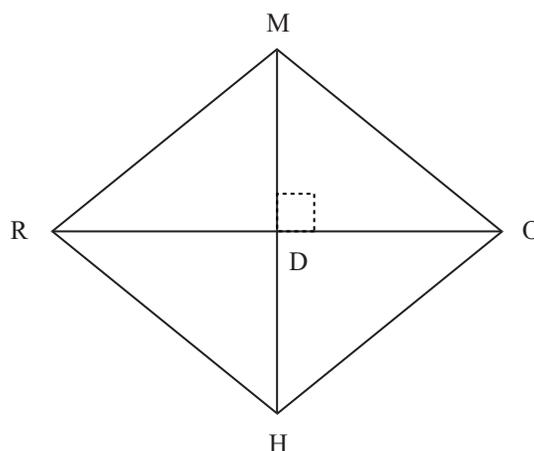
(C) demonstrates what it means to prove mathematically that statements are true;

Scaffolding Questions:

- What is the definition of a rhombus?
- How is a square related to a rhombus?
- How are the diagonals of a square related?
- How are the diagonals of a rhombus related?
- What special triangles do you see on your constructed rhombus?
- What kind of triangle congruencies do you see?

Sample Solutions:

1.



Construction process using patty paper:

Step 1: To construct the perpendicular bisector of \overline{RO} , fold the patty paper so that R reflects onto O. The fold line is the perpendicular bisector.

Step 2: Mark a point, M, on this perpendicular bisector. A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment, so $RM = MO$.

Step 3: Locate point H on the perpendicular bisector of \overline{RO} , on the opposite side from point M, so that $RH = RM$. Then, since H is equidistant from R and O, $HO = RH$.

Because $HD = DM$, and \overline{RO} is perpendicular to \overline{HM} , \overline{RO}



is the perpendicular bisector of \overline{HM} . Any point on \overline{RO} is equidistant from H and M. Thus, $HO = MO$. Hence, $HO = MO = RM = RH$. The figure is a rhombus because it is a quadrilateral with four congruent sides.

2. Possible answers:

Triangles RMO, RHO, HRM, and HOM are isosceles triangles because $HO = MO = RM = RH$.

Triangles RMO and RHO are congruent by SAS.

Triangles HRM and HOM are congruent by SAS because $HO = MO = RM = RH$.

Since a rhombus is a parallelogram, its opposite angles are congruent.

Therefore, $\angle RMO \cong \angle RHO$, and $\angle HRM \cong \angle HOM$.

$\triangle RMD \cong \triangle OMD \cong \triangle HOD \cong \triangle HRD$

Since the diagonals of the rhombus are perpendicular, they divide the rhombus into 4 right triangles with $\angle RMD \cong \angle OMD$ and $\angle HOD \cong \angle HRD$. \overline{HM} and \overline{RO} are also perpendicular bisectors of each other. $HD = MD$ and $RD = DO$. Thus the 4 right triangles are congruent by SAS.

3. The diagonals of the rhombus may be used to find the area of the rhombus. The diagonals of the rhombus divide it into 4 congruent right triangles. Therefore, the area of the rhombus is 4 times the area of any one of the right triangles. The area of a right triangle can be found by taking one-half the product of its legs. One leg is one-half the length of one diagonal of the rhombus (diagonal 1). The other leg is one-half the length of the other diagonal of the rhombus (diagonal 2).

$$\text{Area of Rhombus} = 4\left(\frac{1}{2}\right)\left(\frac{\text{diagonal 1}}{2}\right)\left(\frac{\text{diagonal 2}}{2}\right)$$

so

$$\text{Area of Rhombus} = \frac{1}{2}(\text{diagonal 1})(\text{diagonal 2})$$

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Investigating Quadrilateral Properties



Extension Questions:

- What characteristics of a rhombus helped you decide how to construct RHOM?

A rhombus is a parallelogram with 4 congruent sides.

Responses would vary from this point on. The following response represents the construction approach used in the sample solution.

This leads to the diagonals of the rhombus being perpendicular bisectors of each other. Therefore, start by constructing a segment and its perpendicular bisector. Then decide the length you want the rhombus' sides to be and locate points on the perpendicular bisector of your segment to complete the rhombus.

- How does a square compare with a rhombus?

A square is a rhombus with congruent diagonals. You could also say it is a rhombus with adjacent angles being right angles.

- How does a kite compare with a rhombus?

A kite is a quadrilateral with perpendicular diagonals. It has to have two pairs of adjacent sides that are congruent, but it does not necessarily have congruent opposite sides.

A kite does not have to be a parallelogram.



Tiling with Four Congruent Triangles

Shannon is helping her mom decide on triangular tile patterns for the kitchen they are going to remodel. Using colored construction paper cutouts, they saw they could easily use right triangles (isosceles or scalene), since two congruent right triangles form a rectangle. Shannon began to wonder about using congruent acute or congruent obtuse triangles, which would give a very different look. This led to her question:

1. Can 4 congruent triangles always be arranged to form a triangle?
Describe how you would explore this question.
2. If your response to # 1 is no, give a counter-example.
3. If your response to # 1 is yes, state a conjecture about how the original triangle and its 3 copies are related to the newly formed triangle?
4. If your response to # 1 is yes, write a proof that uses transformations to justify your conjecture.



Teacher Notes

Materials:

Construction paper, patty paper, unlined paper, and construction tools (including geometry software)

Connections to Geometry

TEKS:

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The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

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(A) determines if the converse of a conditional statement is true or false;

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Scaffolding Questions:

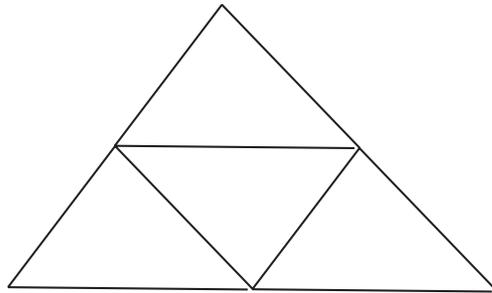
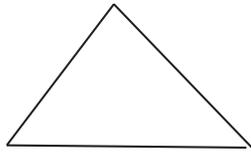
- How can you arrange the triangles so that sides are collinear?
- How do you know that the sides of the smaller triangles are collinear?
- After you have arranged the 4 triangles, where are their vertices located?
- One triangle could be described as an inner triangle. What can you say about its vertices?
- As you arrange the 4 triangles, think about transformations. What transformations are you using?
- How can you label information on the triangles to make it easier for you to describe these transformations?
- Recall the definition of a triangle midsegment. Recall the triangle midsegment properties we investigated in class. How does this activity relate to these investigations?

Sample Solutions:

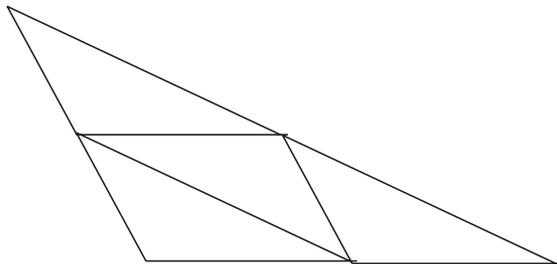
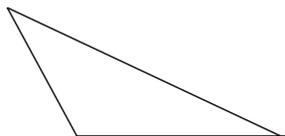
1. To explore this question cut out 4 copies of an acute triangle using different colors for each triangle. Arrange the triangles so that they do not overlap and they form a triangle. Repeat this using 4 copies of an obtuse triangle. You could also draw a triangle on patty paper and make 3 copies. Then arrange the 4 triangles to form the larger triangle.



Acute triangle example:



Obtuse triangle example:



2. Since a triangle may be formed with the four triangles, there is no apparent counterexample.

3. In both cases the 4 congruent triangles form a larger triangle that is similar to the original triangle and its copies.

The sides of the larger triangle are twice as long as the corresponding sides of the original triangle. The area of the large triangle is four times the area of the original triangle because it is made up of the 4 non-overlapping congruent triangles.

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;



(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures.

If the large triangle is similar to the original triangle, the ratio of the area of the large triangle to the area of the original triangle is the square of the ratio of the corresponding sides.

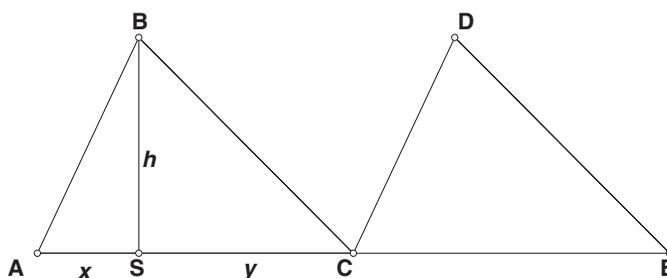
$$\frac{\text{Area of large triangle}}{\text{Area of original triangle}} = \left(\frac{2}{1}\right)^2 = \frac{4}{1}$$

4. Transformational Proof:

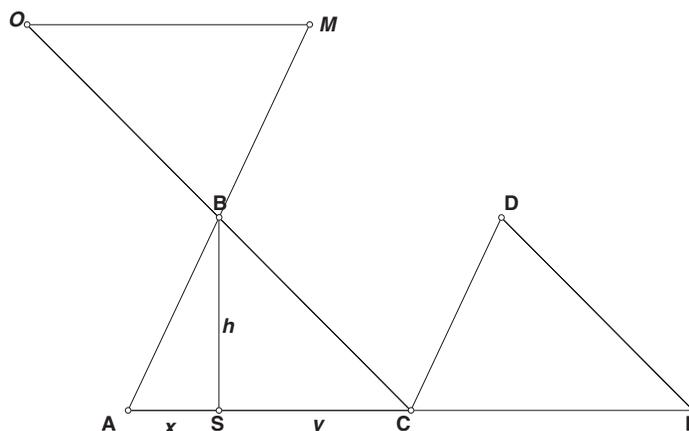
Step 1: Draw $\triangle ABC$ so that segment \overline{AC} is horizontal. Draw altitude \overline{BS} .

Let $BS = h$, $AS = x$, and $SC = y$.

Step 2: Translate $\triangle ABC$ $x + y$ units to the right to form $\triangle CDE \cong \triangle ABC$. A , C , and E are collinear.

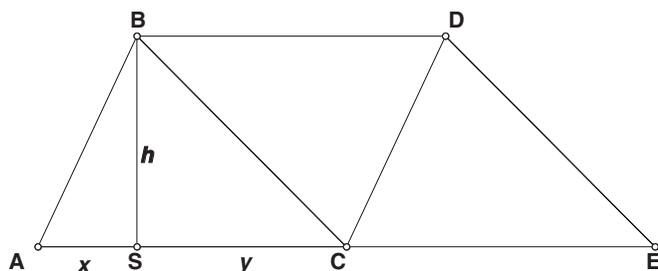


Step 3: Rotate $\triangle ABC$ 180° counterclockwise about point B to form $\triangle BMO \cong \triangle ABC$, $\overline{BO} \cong \overline{CB}$, and $\overline{BM} \cong \overline{CD}$.

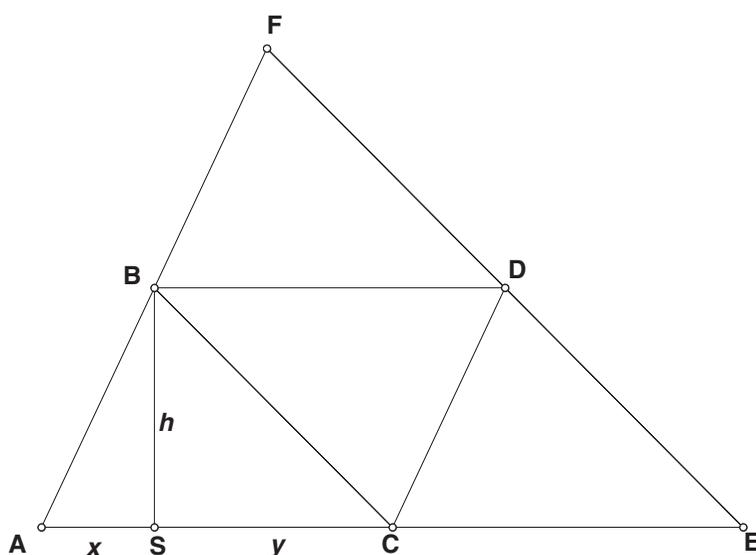


Step 4: Translate $\triangle BMO$ y units to the right and h units down to form $\triangle DCB \cong \triangle ABC$.





Step 5: Translate $\triangle ABC$ x units to the right and h units up to form $\triangle BFD \cong \triangle ABC$. A , B , and F are collinear. F , D , and E are collinear.



The four triangles are congruent. The three angles of the smaller triangle are congruent to the three corresponding angles of the larger triangle.

$$\angle A \cong \angle A, \angle ABC \cong \angle F, \angle ACB \cong \angle E$$

The large triangle is similar to the original triangle by AAA similarity.

$$\triangle AFE \approx \triangle ABC$$

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Triangles Tell It All



Extension Questions:

- In this activity what have you shown to be true?

Triangles of any shape can be used to tile a plane because the four congruent triangles can be arranged in a non-overlapping way to form a larger triangle.

- How are the vertices of the 4 original triangles related to the vertices of the larger triangle that is formed?

Three of the triangles may be considered to be outer triangles, and the fourth triangle is inside the larger triangle. One vertex from each of the outer triangles is a vertex of the larger triangle.

The vertices of the inner triangle are on the sides of the larger triangle and will become the midpoints of the sides of the larger triangle. Each side of the large triangle is formed by two corresponding sides of the congruent triangles. Therefore, each side of the large triangle is twice as long as a side of the original triangle.

- How does this result relate to the Triangle Midsegment Theorems?

It is the converse of the following theorem:

If the midsegments of a triangle are drawn, then four triangles are formed, and the four triangles are congruent.

- How can you prove the result of this activity using algebraic thinking?

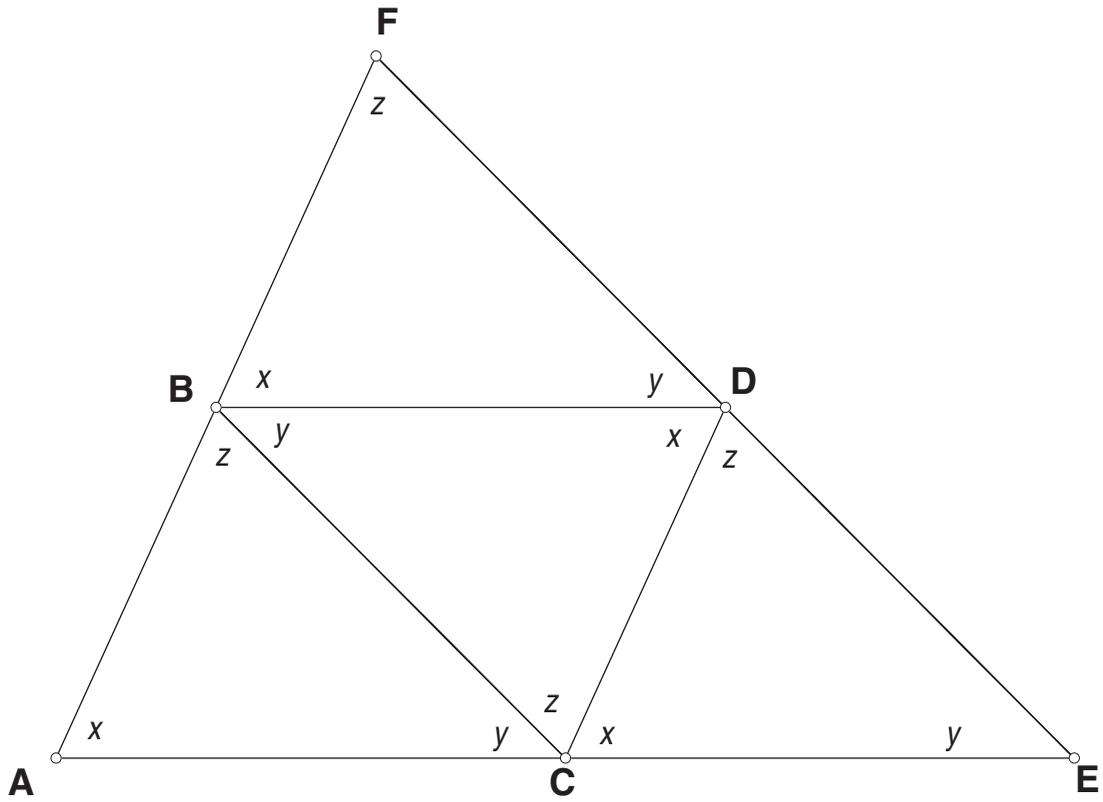
To be sure the 4 congruent triangles are forming a triangle, we need to show that points A, B, and F are collinear; points F, D, and E are collinear; and points C, E, and A are collinear.

Since $\triangle ABC \cong \triangle DCB \cong \triangle CDE \cong \triangle BFD$, let the measures of the congruent corresponding angles be x , y , and z , as shown on the diagram.

We know that the sum of the angles of a triangle equal 180° . Therefore, we know that $x + y + z = 180$.

This shows that B is collinear with A and F, D is collinear with F and E, and C is collinear with E and A. Therefore, the 4 congruent triangles form the large triangle with no overlap.





Student Work Sample

As an extension to the problem, a student was asked to justify his conjecture using a coordinate proof. The student started with and labeled the coordinates. A more complete proof would have included an explanation of why he chose the coordinates of the other points on his diagram. The student used E and G interchangeably from time to time.

Some of the solution guide criteria exemplified by this work are the following:

- Demonstrates geometric concepts, processes, and skills.

The student shows the correct use of the distance formula. He correctly simplifies the radicals and determines the ratio of corresponding sides in the smaller and larger triangles.

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

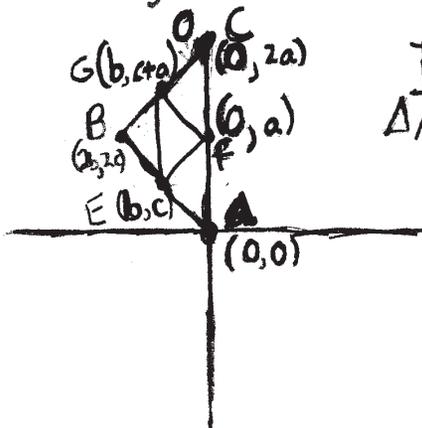
The student correctly used the distance formula to get the length of each side of the triangles. Also, he compared the length of the sides to get the ratio to prove similarity of the triangles.

- Communicates a clear, detailed, and organized solution strategy.

The student states his conjecture that the smaller triangles are similar to the larger triangle. He indicates that he will use the distance formula. He shows all the necessary steps in using the distance formula and in simplifying the ratios of the lengths of the corresponding sides. He states his conclusion that the triangles are similar and gives a reason for this conclusion.



Tiling with 4 Congruent Triangles



I must prove that $\triangle CEF$, $\triangle AEF$, $\triangle BEG$, and $\triangle EFG$ are similar to $\triangle ABC$. Using the distance formula to do this,

$$\overline{AC} = \sqrt{0^2 + (2a)^2} \text{ or } \sqrt{4a^2} \text{ or } 2a$$

$$\overline{CE} = \sqrt{b^2 + a^2} \text{ or } \sqrt{a^2} \text{ or } a$$

$$\overline{AF} = \sqrt{0^2 + a^2} \text{ or } a$$

$$\overline{EG} = \sqrt{(b-b)^2 + (c-a)^2} \text{ or } \sqrt{0^2 + a^2} \text{ or } a$$

$$\frac{\overline{CE}}{\overline{AC}} = \frac{\overline{AF}}{\overline{AC}} = \frac{\overline{EG}}{\overline{AC}} = \frac{a}{2a} = \frac{1}{2}$$

$$\overline{AB} = \sqrt{(2b)^2 + c^2} \text{ or } \sqrt{4b^2 + 4c^2} \text{ or } 2\sqrt{b^2 + c^2}$$

$$\overline{AG} = \sqrt{b^2 + c^2}$$

$$\overline{BG} = \sqrt{(b-b)^2 + (c-c)^2} \text{ or } \sqrt{b^2 + c^2}$$

$$\overline{EF} = \sqrt{b^2 + (c-a-a)^2} \text{ or } \sqrt{b^2 + c^2}$$

$$\frac{\overline{AG}}{\overline{AB}} = \frac{\overline{BG}}{\overline{AB}} = \frac{\overline{EF}}{\overline{AB}} = \frac{\sqrt{b^2 + c^2}}{2\sqrt{b^2 + c^2}} = \frac{1}{2}$$

$$\overline{BC} = \sqrt{(2b-0)^2 + (2c-2a)^2} \text{ or } \sqrt{4b^2 + 4c^2 - 8ac + 4a^2} \text{ or } 2\sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{BE} = \sqrt{(2b-b)^2 + (2c-c+a)^2} \text{ or } \sqrt{b^2 + (c-a)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{CE} = \sqrt{b^2 + (2a-a+c)^2} \text{ or } \sqrt{b^2 + (a-c)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{FG} = \sqrt{b^2 + (c-a)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\frac{\overline{BE}}{\overline{BC}} = \frac{\overline{CE}}{\overline{BC}} = \frac{\overline{FG}}{\overline{BC}} = \frac{\sqrt{b^2 + c^2 - 2ac + a^2}}{2\sqrt{b^2 + c^2 - 2ac + a^2}} = \frac{1}{2}$$

This means $\triangle CEF$, $\triangle AEF$, $\triangle BEG$, $\triangle EFG$, are congruent and similar to $\triangle ABC$ because the corresponding sides are 1:2.





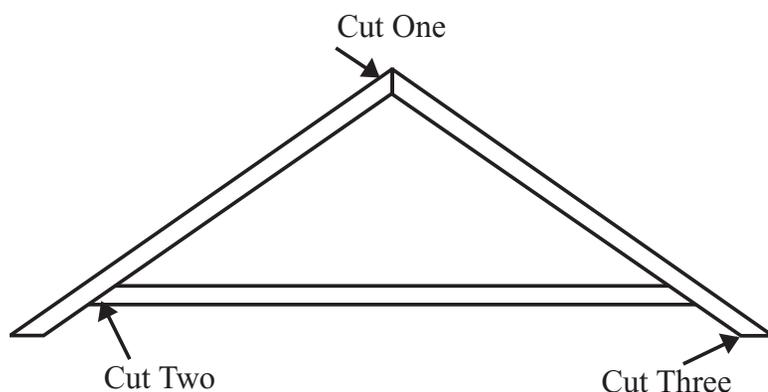
Shadow's Doghouse

You are planning to build a new doghouse for your German Shepherd, Shadow. The doghouse will be a rectangular solid: 24 inches across the front, 36 inches deep, and 30 inches high. The cross section of the pitched roof is an isosceles triangle. The distance from the floor of the doghouse to the peak of the roof is 39 inches.

There will be 4 roof trusses (shown below), which are isosceles triangles, to support the pitched roof.

To determine the materials you need to purchase and how you will construct the frame, you make careful plans on paper before you begin construction.

1. Make a sketch of the doghouse showing the front, side, and roof.
2. How could you ensure the 4 roof trusses are precisely the same shape and size using the least number of measurements? Describe at least two ways. What geometry concepts are being used?
3. The roof trusses are to be cut from 1-inch by 2-inch boards, as diagrammed below:



At what angles should cuts one, two, and three be made? What geometry concepts are being used?

4. The roof is to have a 3-inch overhang. That means that there will be a 3-inch extension of the roof at the front, back, and sides. The roof is to be made from two pieces of plywood and covered with shingles. The shingles



are laid so that they overlap, with the exposed (visible) area of a shingle measuring 6 inches by 6 inches. How many shingles are needed? Explain how you determine this, citing the geometry concepts you use.

5. The roof, sides, front, back, and floor of the doghouse will be made of plywood, which is available in 4-foot by 8-foot sheets and half-sheets (2 feet by 8 feet or 4 feet by 4 feet). The opening on the front of the doghouse will be 12 inches wide by 18 inches high. The front and back pieces are to be cut as pentagons, not as rectangles with an added triangle. How many sheets and/or half-sheets of plywood are needed for the entire doghouse, and how would you lay out the pieces to be cut? What geometry concepts are being used to ensure opposite walls are precisely the same size?
6. After estimating your total costs, you decide to consider making Shadow's doghouse smaller, with a floor 20 inches wide and 30 inches deep. The shape of the new doghouse would be similar to the original doghouse. A dog that is the size of Shadow requires a doghouse with a floor area of at least 4 square feet and a capacity of at least 9 cubic feet. Would the smaller doghouse be large enough for Shadow? Explain.





Teacher Notes

Materials:

One ruler, calculator, and protractor per student

Unlined paper for drawing

Connections to Geometry

TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

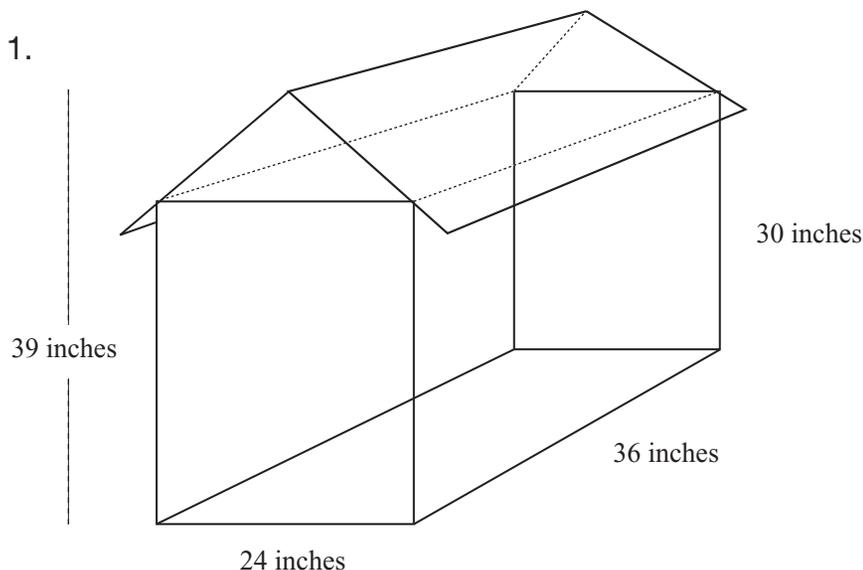
(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

Scaffolding Questions:

- What does it mean for geometric figures to be congruent?
- What are the possible ways to prove triangles are congruent?
- What would be reasonable ways to build the roof trusses so that they are congruent triangles?
- How can Pythagorean Triples help you determine measurements in this problem?
- What are the angle characteristics for various triangles?
- What special triangles are formed by the roof trusses?
- What do you know about the altitude to the base of an isosceles triangle?
- What are the shape and dimensions of the plywood pieces that will form the roof?
- How can you plan the plywood pieces you will need to cut for the doghouse?
- If figures are similar, what do you know about the ratio of the corresponding linear measurements, the ratio of areas, and the ratio of volumes?

Sample Solutions:

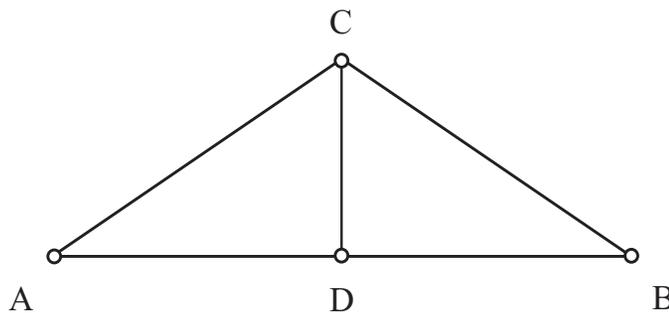
1.



2. For the triangular roof trusses to be the same shape and size, we must design and build them so that we have congruent triangles. Triangles are congruent by *SSS*, *SAS*, *ASA*, or *SAA*. The most reasonable congruence relationship to use in building the roof trusses is probably *SSS*. We would determine how long the boards need to be and where they will be joined. Then we would cut the boards.

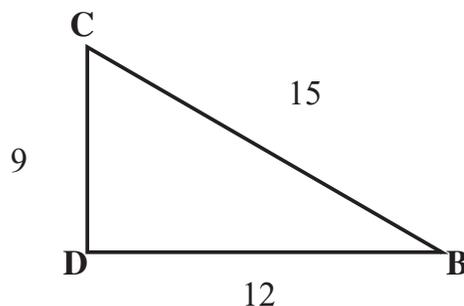
We know the beams will be 24 inches across and 9 inches high.

$AB = 24$, $CD = 9$, \overline{CD} is perpendicular to \overline{AB} , $AC = CB$



Since the beams form an isosceles triangle, we know that the sides that will support the roof are congruent, and that the altitude segment \overline{CD} to the base \overline{AB} bisects segment \overline{AB} .

Using right triangles and Pythagorean Triples, we find the lengths of the congruent sides. Segments AC and BC are 15 inches.



Measuring the angles with a protractor, we find that $m\angle A = m\angle B \approx 37^\circ$ and $m\angle ACB \approx 106^\circ$.

If we build the triangular trusses to be congruent by *SSS*, we would cut four 24-inch boards and eight 18-inch boards. Three inches of each board is for the roof overhang.



The student:

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(D) analyzes the characteristics of three-dimensional figures and their component parts.

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

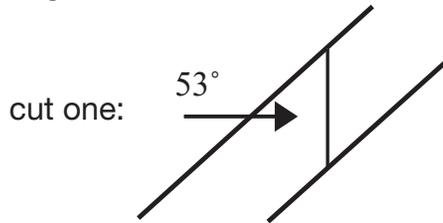
(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(4) describes and applies the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

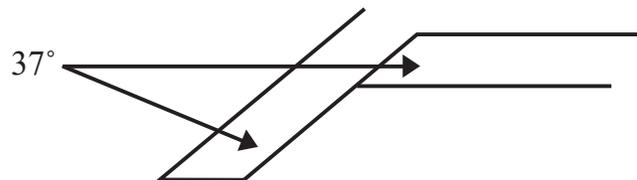
If we build the triangular trusses to be congruent by ASA, we would cut the four 24-inch boards and then measure the base angles on each end to be 37 degrees.

3. Since the vertex angle of the isosceles triangle formed by the outer edges of the boards measures 106 degrees, and the angled edge lies along the altitude to the base of the isosceles triangle, cut one needs to be made at an angle of 53 degrees. This is because the altitude from the vertex angle of an isosceles triangle is also the bisector of that angle.



The top and bottom edges of the baseboard of the truss (the base of the isosceles triangle) and the angled edge of the bottom of the boards that are the sides of the beam are parallel. Since corresponding angles formed by transversals on parallel lines are congruent, cuts two and three need to be made at angles of 37 degrees.

cuts two and three:



4. To determine the number of shingles needed, we compute the surface area to be covered. The surface area is $42(18) = 756$ square inches. The number of shingles needed for one side is

$$\frac{756 \text{ square inches}}{36 \text{ square inches for one shingle}} = 21 \text{ shingles.}$$

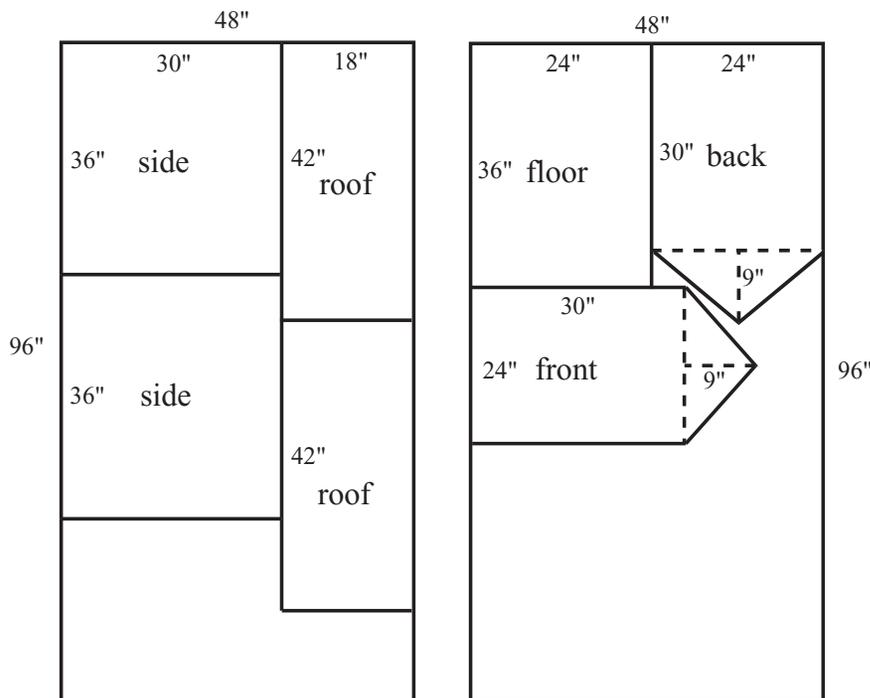
5. The total square area of plywood needed for the doghouse is



$$2[30(36) + 18(42) + 24(30) + 0.5(9)(24)] + 24(36) = 6192 \text{ square inches.}$$

One full sheet of plywood has an area of $48(96) = 4608$ square inches. One and one-half sheets would provide an area of 6912 square inches. However, the layout of the pieces (side, front, back, roof, and floor) must be considered.

The following possible layout shows that two full pieces of plywood are needed to efficiently cut the pieces as planned.



The sides are congruent rectangles, and the front and back are congruent pentagons since corresponding sides and corresponding angles are congruent.

6. The floor space of the new doghouse would be 20 inches by 30 inches, or 600 square inches. Convert to square feet.

$$600 \text{ in}^2 \cdot \frac{1 \text{ ft}^2}{144 \text{ in}^2} = 4.167 \text{ ft}^2$$

The floor space is more than 4 ft².

Since the two doghouses are similar in shape, the other dimension of the doghouse may be determined by using ratios.

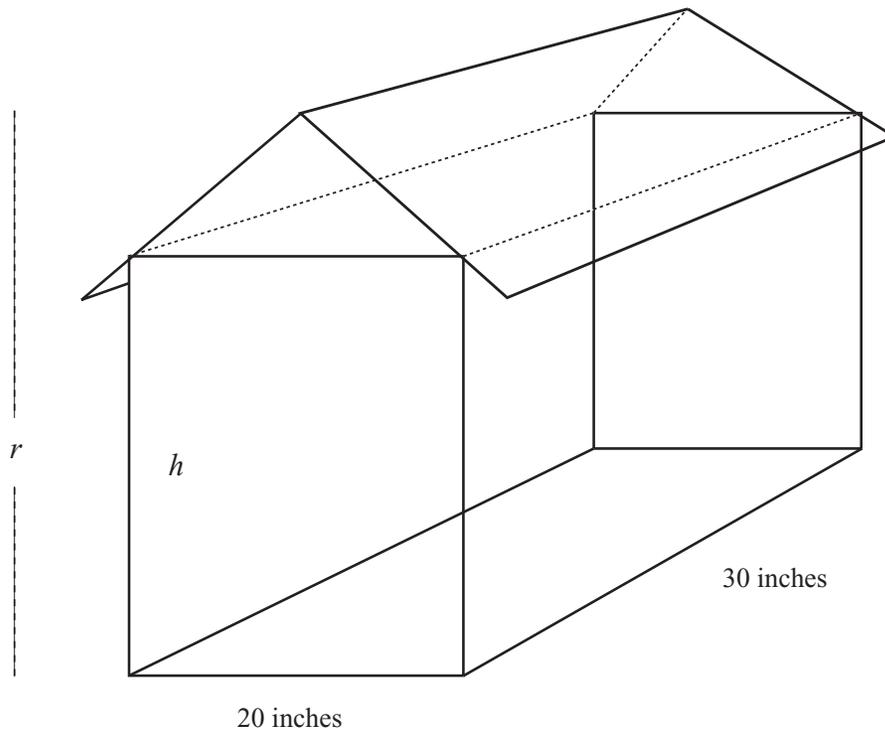
Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Connections to High School Geometry: Supporting TEKS and TAKS Institute:

- III. Triangles: Foundations and Trussess
- V. Solid Figures: Bayou City Dome





If the shapes are similar, then

$$\frac{h}{20} = \frac{30}{24} \quad \text{and} \quad \frac{r}{20} = \frac{39}{24}$$

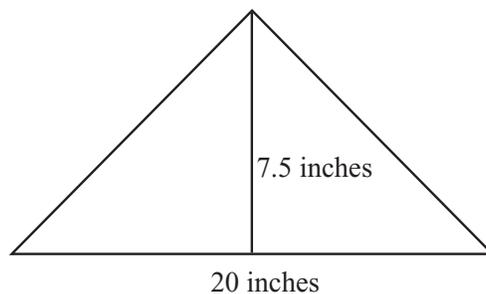
$$h = 20 \left(\frac{30}{24} \right) = 25 \text{ in} \quad r = 20 \left(\frac{39}{24} \right) = 32.5 \text{ in}$$

The capacity of the original doghouse is the volume of the rectangular prism portion plus the volume of the triangular prism of the roof.

The volume of the rectangular prism is 20 inches by 30 inches by 25 inches, or 15000 in³.

$$15000 \text{ in}^3 \cdot \frac{1 \text{ ft}^3}{12 \text{ in} \cdot 12 \text{ in} \cdot 12 \text{ in}} = 8.68 \text{ ft}^3$$

The volume of the triangular prism is the area of the triangle times the length. The height of the triangle is 32.5 – 25 = 7.5 inches.



$$\text{Area of the triangle} = \frac{1}{2} 7.5(20) = 75 \text{ in}^2$$

The volume of the prism is $75(30) = 2250 \text{ in}^3$.

$$2250 \text{ in}^3 \cdot \frac{1 \text{ ft}^3}{12 \text{ in} \cdot 12 \text{ in} \cdot 12 \text{ in}} = 1.30 \text{ ft}^3$$

The volume of the new doghouse would be $8.68 + 1.30 \text{ ft}^3$, or 9.98 ft^3 .

This doghouse would be large enough for Shadow.

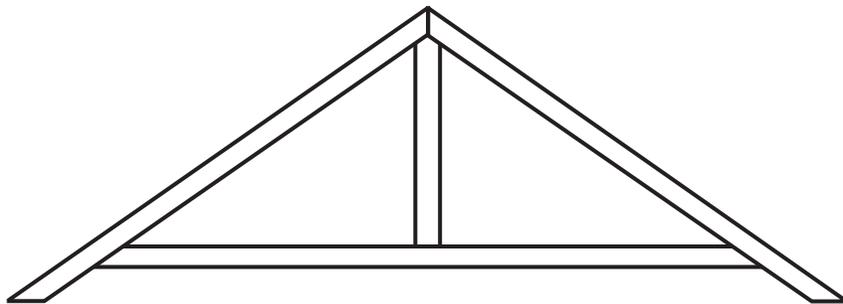
Extension Questions:

- Build a scale model of Shadow's doghouse.

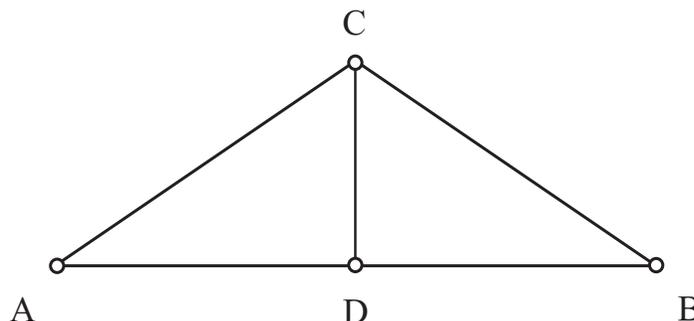
Answers will vary.

- Describe how you could build the roof trusses using congruent scalene triangles.

Assuming the peak of the roof will be centered on the doghouse, each truss would consist of two congruent scalene triangles as shown below:



The truss would consist of the 24-inch base, a 9-inch board that is perpendicular to the base, and the 15-inch roof supports. This is using two scalene triangles that are congruent by SSS:



$$AD = BD = 12, CD = 9 \text{ and } AC = BC = 15.$$

For the remaining questions, suppose that you decide to build the doghouse so that the floor is a regular hexagon.

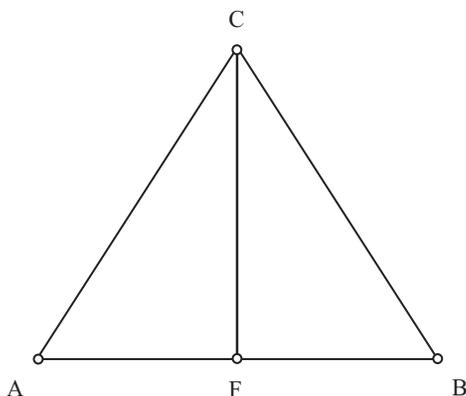
- Determine the dimensions, to the nearest inch of the hexagon, and describe how you will draw the hexagon on the plywood in order to cut it. Remember that you need at least 4 square feet of floor space.

The hexagon will consist of 6 congruent equilateral triangles. To determine the side of the equilateral triangle, solve the following area problem:

Area of polygon:

One-half the apothem of the polygon times its perimeter = one-half times the altitude of the triangle space times six times the base of the triangle.

Since the apothem is the altitude of the equilateral triangle, and the altitude divides the equilateral triangle into two 30-60-90 triangles, use the Pythagorean Formula to express the altitude, a , in terms of a side, s , of the equilateral triangle.



$\triangle ABC$ is an equilateral triangle, and \overline{CF} is the altitude to \overline{AB} .

Let $CF = a$, and $BC = s$. Then $FB = \frac{s}{2}$, and

$$a^2 + \left(\frac{s}{2}\right)^2 = s^2$$

$$a^2 = s^2 - \frac{s^2}{4}$$

$$= \frac{3s^2}{4}$$

$$\text{so that } a = \frac{\sqrt{3}}{2}s.$$



Next, get an expression for the area of the hexagon:

$$\begin{aligned}A &= \frac{1}{2}ap \\ &= \frac{1}{2}a(6s) \\ &= 3\left(\frac{\sqrt{3}}{2}s\right)s \\ &= \frac{3\sqrt{3}}{2}s^2.\end{aligned}$$

Now

$$\begin{aligned}\text{let } \frac{3\sqrt{3}}{2}s^2 &= 4 \\ 3\sqrt{3}s^2 &= 8 \\ s^2 &= \frac{8}{3\sqrt{3}} \\ &\approx 1.54 \text{ sq. ft.} \\ \text{so that } s &\approx 1.24 \text{ ft} \\ &= 14.88 \text{ inches.}\end{aligned}$$

Rounding to the nearest inch, we need to cut a regular hexagon with a side that is 15 inches.

One way to draw the hexagonal floor is to draw a circle of radius 15 inches. Then mark off and draw on the circle chords that are 15 inches long.

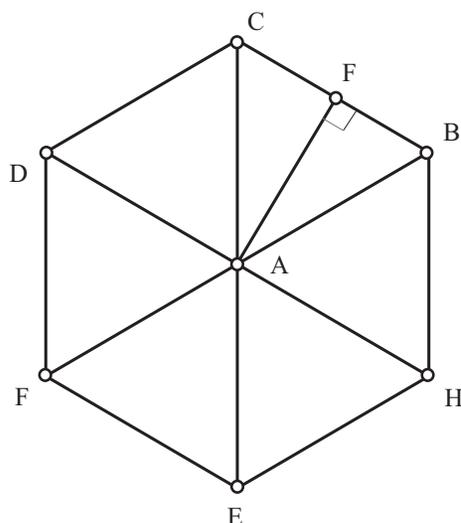
- Assume the floor-to-roof edge must still be 30 inches, and the door opening must be at least 12 inches wide by 18 inches high. How much plywood will be needed for the sides?

You would need to cut six congruent rectangles that are 15 inches wide by 30 inches high and then cut the opening in one of them. One 4-foot by 8-foot piece of plywood would be enough, measuring 15 inches across the 4 feet width twice and 30 inches along the 8 feet length three times.

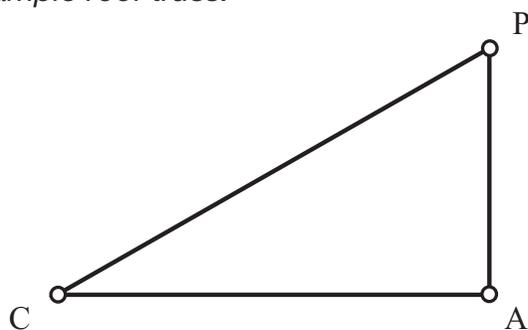
- The roof for the redesigned doghouse will be a hexagonal pyramid with a height (altitude) of 9 inches. Describe how you would design the beams to support the roof and the plywood sections that will form the roof.



The base of the hexagonal pyramid that will be the roof will be congruent to the floor of the doghouse. The base is shown below.

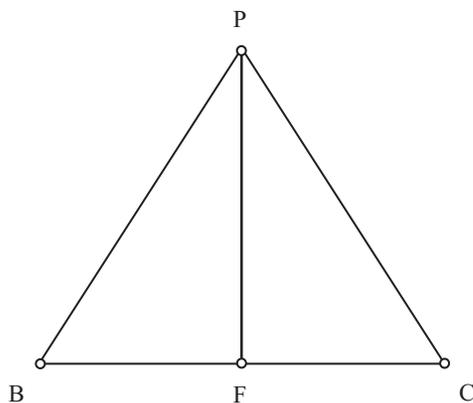


Six congruent triangular roof trusses can be made. Let P be the point at the top of the roof. The triangle CAP is a sample roof truss.



$CA = 15$ inches, $PA = 9$ inches, and $CP = 17.49$ inches (approximately) by the Pythagorean Formula. (The board corresponding to segment PA needs to be a hexagonal post, and it is a common leg of all six right triangles forming the roof trusses.)

The 6 congruent, plywood triangles that would form the roof would look like the following triangle:



$BC = 15$ inches. $BP = PC = 17.49$ inches, and the segments are the lateral edges of one of the six congruent triangle faces of the pyramid that forms the roof.

- How will the capacity (volume) of the new doghouse compare with that of the original doghouse?

The capacity of the new doghouse will be the volume of the regular hexagonal prism that is Shadow's room plus the volume of the hexagonal pyramid that is the roof:

Prism Volume = height times base area

Pyramid Volume = one-third height times base area.

Base Area:

$$\text{Find the apothem of the hexagon: } a = \frac{\sqrt{3}}{2} s = \frac{\sqrt{3}}{2} \cdot 15 \approx 12.99$$

$$A = \frac{1}{2} ap = 0.5(12.99)(6 \cdot 15) = 584.55 \text{ square inches}$$

Prism Volume:

$$V = 30 \cdot 584.55 = 17536.5 \text{ cubic inches}$$

Pyramid Volume:

$$V = \frac{1}{3} \cdot 9 \cdot 584.55 = 1753.65 \text{ cubic inches}$$

Total Volume:

$$V = 17536.5 + 1753.65 = 19290.15 \text{ cubic inches}$$

Convert to cubic feet.

$$19290.15 \text{ cubic inches} \cdot \frac{1 \text{ cubic foot}}{12 \text{ cubic inches}} = 11.16 \text{ cubic feet}$$

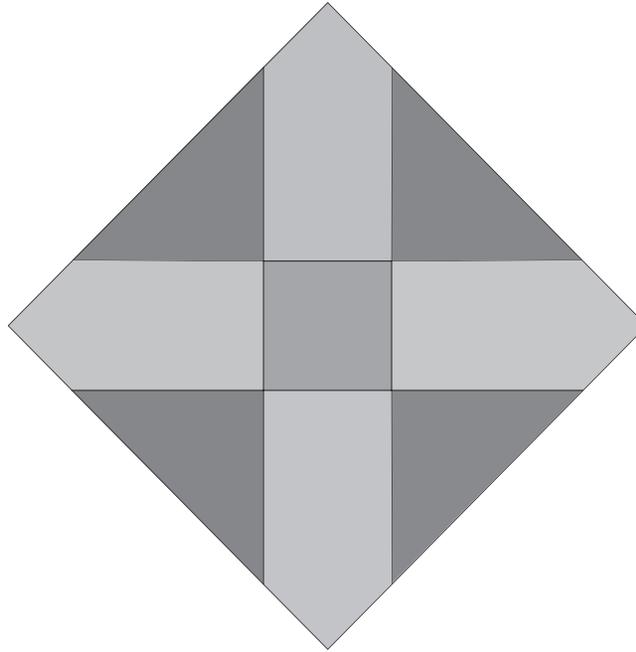
The volume is 11.16 cubic feet.





The School Flag

A flag for a high school displays a cross on a square as shown below:



The square in the center of the cross will be silver, the rest of the cross will be red, and the remainder of the flag is to be black.

The total area of the flag will be 400 square inches, and the cross should fill 36% of the square flag.

The total cost of a flag is based on the parts: silver (most expensive), red (next in cost), and black (least expensive). These cost issues prompt the question:

What percent of the area of the flag is each colored section?



Materials:

One calculator per student

Construction paper in different colors, unlined paper, and construction tools (including geometry software)

Connections to Geometry**TEKS:**

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

Teacher Notes

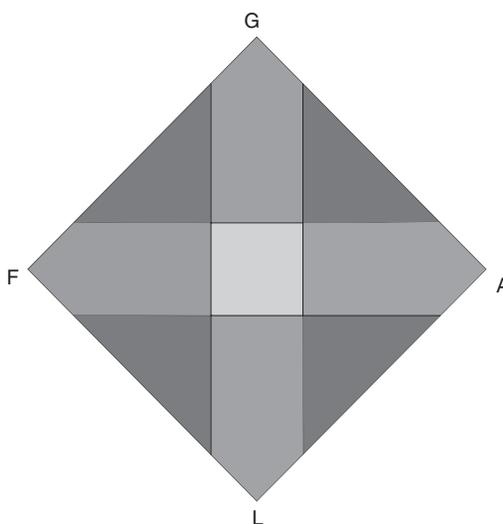
Scaffolding Questions:

- Make a model using different colors of construction paper.
- If you break the flag apart into pieces, what kinds of polygons do you see?
- What are the dimensions of the shapes that could vary?
- What information does the problem give you that restricts the dimensions?
- How could you dissect your model and rearrange the pieces so that you have the same colored pieces grouped together to form rectangles or squares?
- What properties of a square help you find congruent triangles?
- As you rearrange pieces think about transformations. What transformations are you using?

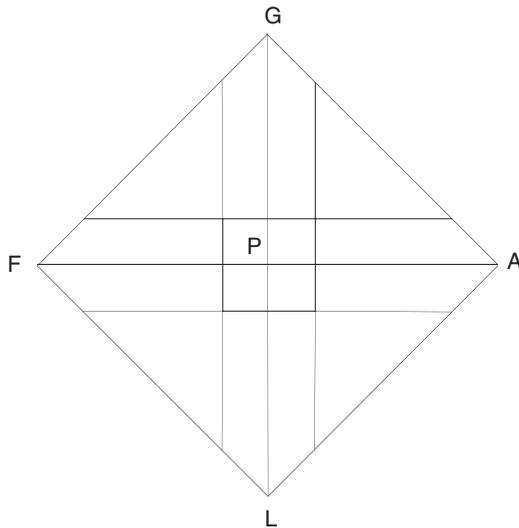
Sample Solution:

The flag can be cut into pieces, and the pieces can be rearranged so that each color forms a rectangular region.

Label the original square FLAG.

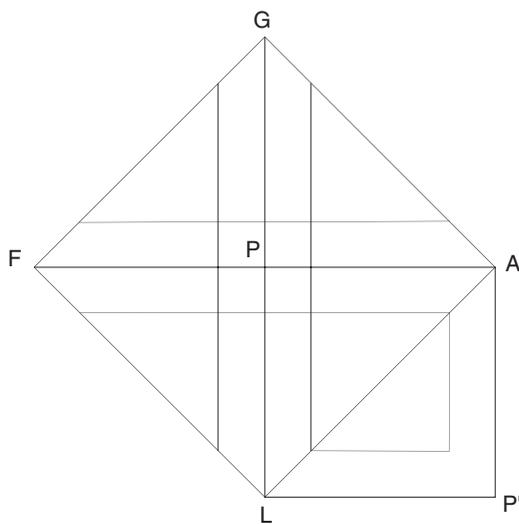


Draw diagonals \overline{FA} and \overline{LG} . Label their point of intersection P.



Since the diagonals of the square are congruent, bisect each other, and are perpendicular, triangles FPL, LPA, APG, and GPF are congruent by SAS.

Translate $\triangle FPG$ so that segment \overline{FG} coincides with segment \overline{LA} to form the following:



Triangles LPA and LP'A form a square that is one-half of the flag. Therefore, the area of square PLP'A is 200 square inches.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.



(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures.

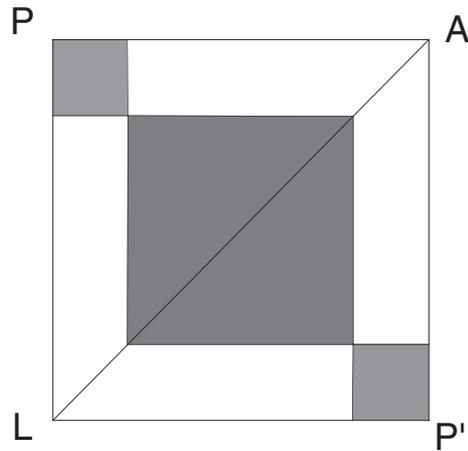
(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

Now focus on the colors of the regions of square $PLP'A$:



The original square flag has an area of 400 square inches. Square $PLP'A$ has an area of 200 square inches. If we let s be the side of square $PLP'A$, then

$$\begin{aligned} s^2 &= 200 \\ s &= 10\sqrt{2} \\ &\approx 14.14 \text{ inches} \end{aligned}$$

Since the cross fills 36% of the flag, the black portion is 64% of the flag. Therefore, in the half-flag the black portion is 64% of 200 square inches, or 128 square inches.

In the figure above, the black portion is square. If we let b be the side of the black square, then

$$\begin{aligned} b^2 &= 128 \\ b &= 8\sqrt{2} \\ &\approx 11.31 \text{ inches} \end{aligned}$$

Now y , the side of the two small silver squares, must be

$$y = \frac{14.14 - 11.31}{2} \approx 1.42 \text{ inches.}$$

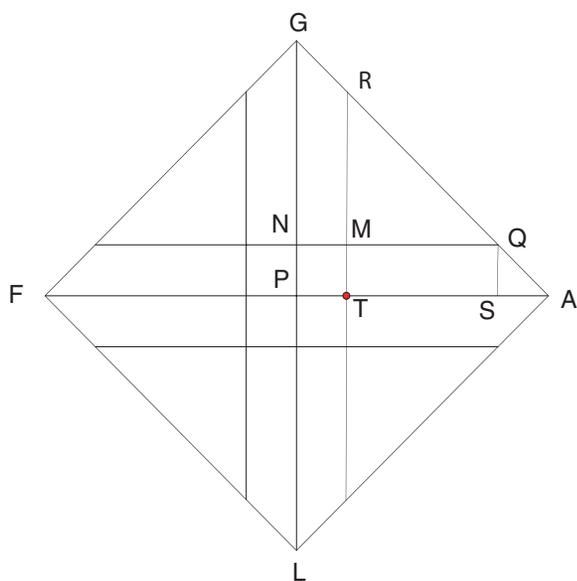
The area of the two small silver squares in the half-flag is $A = 2(1.42)^2 \approx 4$ square inches.



The area of the red portion of the cross in the half-flag must be $200 - 128 - 4 = 68$ square inches.

The color percents of the half-flag will be the same for the original flag. Therefore, 64% of the flag is to be black, $\frac{68}{200} = 34\%$ is to be red, and $\frac{4}{200} = 2\%$ is to be silver.

Another approach to solving this problem would be to recognize that the variable in this situation is the side of the silver square. The lengths may be expressed in terms of this variable.



$NMTQ$ is a square. Let the sides of this square be x units. Let \overline{QS} be perpendicular to \overline{AP} . Because parallel lines are equidistant, QS is also x units. SA would also be x units because QAS is an isosceles right triangle. $MTSQ$ is a rectangle, so $MQ = TS$. As shown in the previous solution, AP is $10\sqrt{2}$ units. MQ is $10\sqrt{2} - 2x$.

RMQ is a right isosceles triangle with area

$$\frac{1}{2}(10\sqrt{2} - 2x)(10\sqrt{2} - 2x).$$

The area of the four black triangles is

$$A = 4 \cdot \frac{1}{2}(10\sqrt{2} - 2x)(10\sqrt{2} - 2x)$$

$$A = 2(10\sqrt{2} - 2x)^2.$$

As indicated earlier, since the cross fills 36% of the flag, the black portion is 64% of the flag. Therefore, the black

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Bayou City Lake

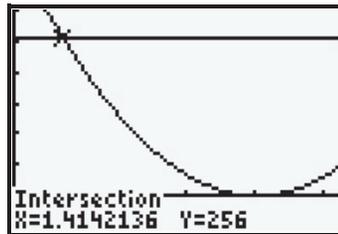
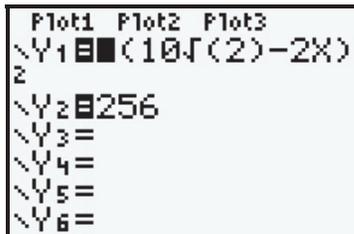


portion is 64% of 400 square inches, or 256 square inches.

$$A = 2(10\sqrt{2} - 2x)^2$$

$$256 = 2(10\sqrt{2} - 2x)^2$$

This equation may be solved using a graphing calculator. The area function may be graphed or a table of values may be found.



X	Y ₁	Y ₂
1.4	257.29	256
1.41	256.38	256
1.42	255.48	256
1.43	254.57	256
1.44	253.67	256
1.45	252.77	256
1.46	251.87	256

$X = 1.41$

x is 1.41.

The area of the silver square is $(2x)^2$, or 7.9524 square inches.

The percent of the total area of 400 is $\frac{7.9524}{400} \approx 0.02$ or 2% of the total area. The red area must be 36% minus 2%, or 34%.



Extension Questions:

- What transformations could you use to rearrange the pieces of the flag to place same colored pieces together?

Only horizontal and vertical translations of the triangular piece FPL were needed.

We translated a distance equal to half the diagonal of the square flag to the right and down.

- What shapes were formed by the colored areas?

The black region was a square formed by the 4 congruent isosceles right triangles. The silver pieces were two congruent squares. The red pieces were 4 congruent trapezoids with one leg being an altitude of the trapezoid. The red pieces could have been cut and rearranged into a rectangle, but that did not make the problem easier to solve.

- What key ideas help you solve the problem?

We worked with half of the flag. We were able to arrange it into a square, so all we needed was the formula for the area of a square to help us get the side length of that square and the black square. The rest was basically computation.

- How could you extend this problem?

Change the shape of the flag to be a non-square rectangle. Change the design on the flag. Look for patterns in the different flags.



Student Work

A copy of a student's work on this problem appears on the next page.

This work exemplifies many of the solution guide criteria. For example:

- Shows an understanding of the relationships among elements.

The student recognized the right triangles and used the Pythagorean Theorem. He recognized that one of the black right triangles is one-fourth of the black area of the square. He recognized that the segment labeled $x + y + z$ was the same as the side of the square—20 cm in length .

- Demonstrates geometric concepts, process, and skills.

The student used the Pythagorean Theorem correctly. He showed the algebraic processes necessary to solve for the lengths of the unknown segments.

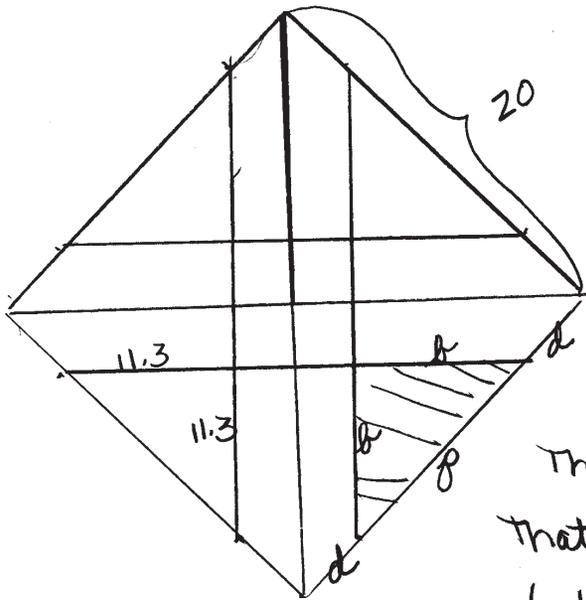
- Communicates clear, detailed, and organized solution strategy

Overall, the solution is organized clearly. However, when setting up equations, there should have been more detailed explanation of how and why the equations were set up the way they were. For example, the student set up $2x^2 = b^2$ but there is no explanation of why.

This student was advised that his solution would have been more complete if he had justified his steps and indicated the reasons for the steps in his solution. A teacher might ask the student some of the following questions to clarify his thinking:

- How did you know that the triangle with legs labeled b had two congruent legs?
- Why does $2b^2 = p^2$?
- How do you know that $20 - p = 2d$?
- What justifies your statement that $2x + y = 20$?





red 34%
 silver 2%
 black 64%

$$\frac{1}{4} \text{ of the square} = \frac{1}{4}(20)^2 = 100 \text{ m}^2$$

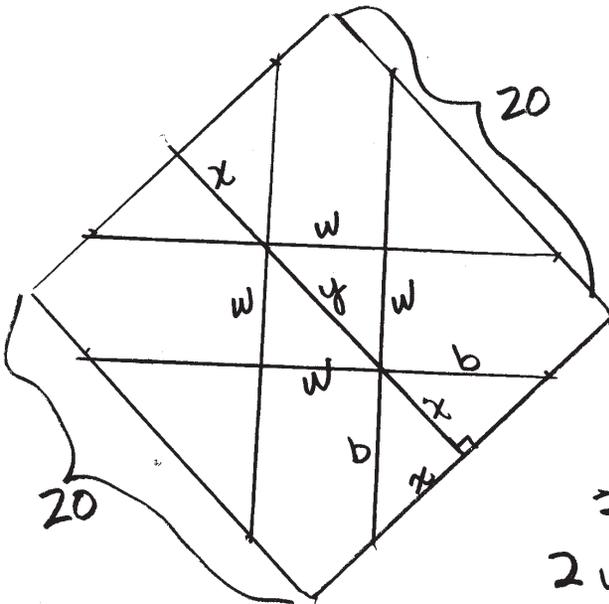
The black part is 64% of it or 64 m^2 .
 That is one of the black triangles

$$\frac{1}{2}bh = 64 \quad b = h$$

$$\frac{1}{2}b^2 = 64 \quad b^2 = 128 \quad b \approx 11.3$$

$$2b^2 = p^2 \quad 2(128) = p^2 \quad p^2 = 256 \quad p = 16$$

$$20 - p = 2d \quad 20 - 16 = 2d \quad 4 = 2d \quad 2 = d$$



$$2x^2 = b^2 \quad 2x^2 = 128$$

$$x^2 = 64 \quad x = 8$$

$$2x + y = 20$$

$$2(8) + y = 20$$

$$16 + y = 20$$

$$y = 4$$

$$2w^2 = 4^2$$

$$w^2 = 8$$

$$2w^2 = y^2$$

$$2w^2 = 16$$

$$w = \sqrt{8}$$

$$w^2 = \text{area of silver} = 8 \quad \frac{8}{400} = 2\%$$

$$100\% - (2\% + 64\%) = \text{area of red} = 100\% - 66\% = 34\%$$





The Shortest Cable Line

Two houses are to be connected to cable TV by running cable lines from the houses to a common point of connection with the main cable. The main cable runs underground along the edge of the street the houses face. Without measuring, determine where that connection point should be placed to minimize the amount of cable run from the houses to the street. Explain your reasoning.

