Chapter 7:
Similarity

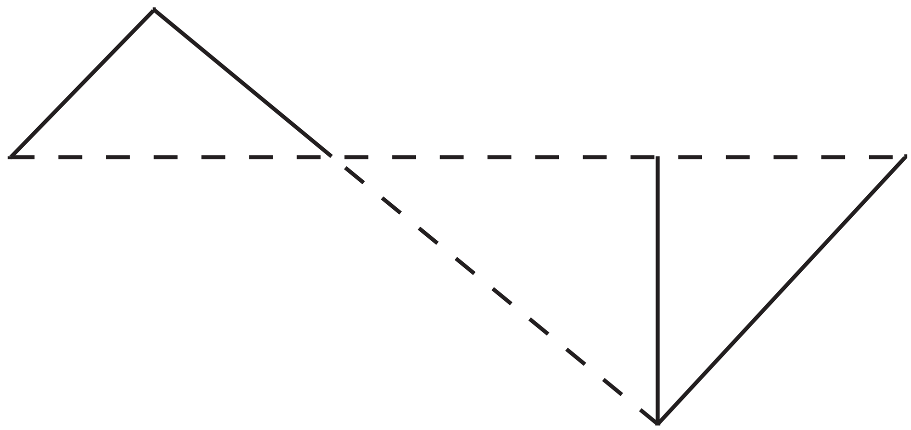
Introduction

In this chapter the problems require the students to apply the properties of similarity to justify properties of figures and to solve problems using these properties.

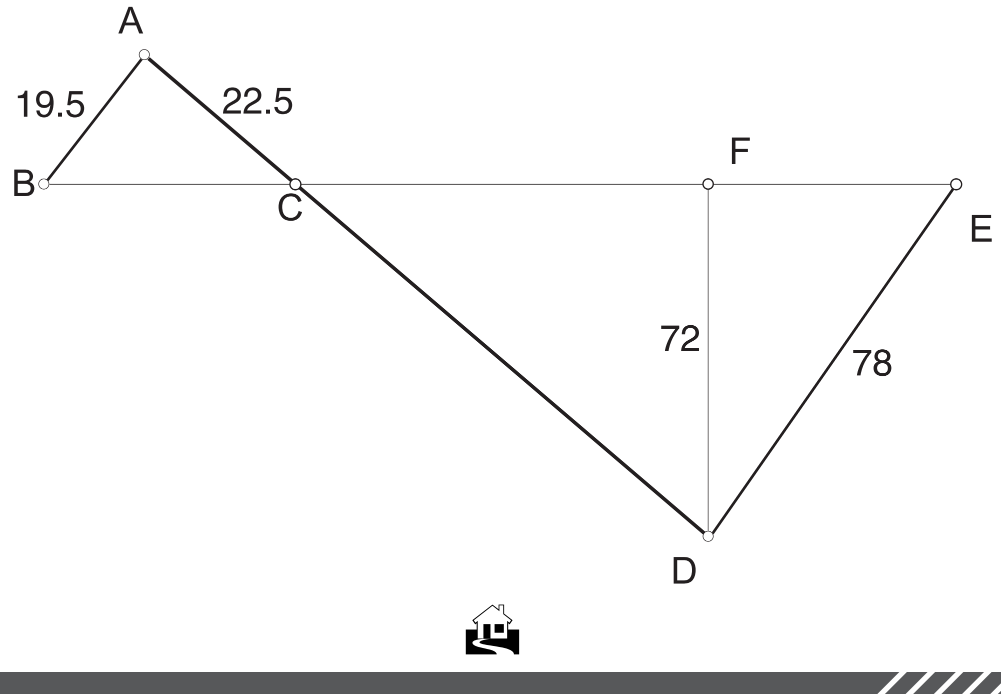
> Geometry consists of the study of geometric figures of zero, one, two, and three dimensions and the relationships among them. Students study properties and relationships having to do with size, shape, location, direction, and orientation of these figures. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999.)

Ancient Ruins

Archaeologists flying over a remote area in the interior of Mexico saw what appeared to be the ruins of an ancient ceremonial temple complex below. This diagram shows what they saw:



Based on their photographs they believed that some of the walls were still intact. These are drawn as solid segments, and the walls that had crumpled or had obviously been there are drawn as dashed segments. Before they can excavate the site they need to construct an accurate scale drawing or model. Based on their altitude flying over the ruins and the measurements made on the photograph, they generated the following drawing of the ruins:



The lengths, in feet, of the walls $\overline{AB}$, $\overline{AC}$, $\overline{DE}$, and $\overline{DF}$ were 19.5, 22.5, 78 and 72, respectively. $\overline{AB} // \overline{DE}$ and $\overline{DF} \perp \overline{CE}$.

To plan for the excavation they need to know a number of things about the site.

1. The archaeologists estimate it will require 45 minutes to an hour to excavate each foot of the exterior walls of the temple site. Approximately how long will it take to complete this task? Explain in detail how you determined this.

2. The entrance into the ceremony preparation room appears to be along wall $\overline{BC}$ and directly opposite vertex A. What would this mean geometrically? What is the distance from point A to the entrance? Were any of the walls of this room perpendicular to each other?

3. The archaeologists believe that about 9 square feet of space was needed in the main temple for each person. How many people could occupy the temple (triangle CDE)? How many priests and assistants could occupy what appears to be the ceremony preparation room (triangle ABC)?

Materials:
One calculator per student

Unlined paper and construction tools

Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

Teacher Notes

Scaffolding Questions:

- What information do you need to determine in order to compute the perimeters?
- What appears to be true about triangles ABC and CDE?
- What special segment on a triangle is question 3 referring to?
- Can the Pythagorean Theorem or Pythagorean Triples help you find answers to the questions?
- How are the ratios of corresponding linear dimensions on similar triangles related to the ratios of their areas? The ratio of their volumes?

Sample Solution:

1. To find the perimeters the missing sides, segments $\overline{BC}, \overline{DC},$ and $\overline{EC},$ need to be found. Triangles ABC and DEC appear to be similar triangles.

   Segments AB and DE are parallel, so $\angle ABC \cong \angle DEC$ because they are alternate interior angles on parallel lines. Because they are vertical angles, $\angle ACB \cong \angle DCE$.

   Thus, $\Delta ABC \approx \Delta DEC$ because two angles of one triangle are congruent to two angles of the other triangle. Similar triangles have proportional corresponding sides:
   $\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC} = \frac{1}{4}$ since AB = 19.5 and DE = 78.
   Use the proportion to get DC: $\frac{AC}{DC} = \frac{22.5}{DC} = \frac{1}{4}$ so DC = 90.

   To get EC, find FE and FC. Consider the right triangles DFE and DFC. Pythagorean Triples may be used.

   Triangle DFE has DE = 78 = 6 times 13, and DF = 72 = 6 times 12, so FE = 6 times 5 = 30.

   Triangle DFC has DF = 72 = 18 times 4, and DC = 90 = 18 times 5, so FC = 18 times 3 = 54.

Thus, EC = 30 + 54 = 84.

To get BC, use the proportion: $\frac{BC}{EC} = \frac{BC}{84} = \frac{1}{4}$, so BC = 21 .

Perimeter of Triangle ABC = 19.5 + 22.5 + 21 = 63 feet.

Perimeter of Triangle DEC = 78 + 84 + 90 = 252 feet.

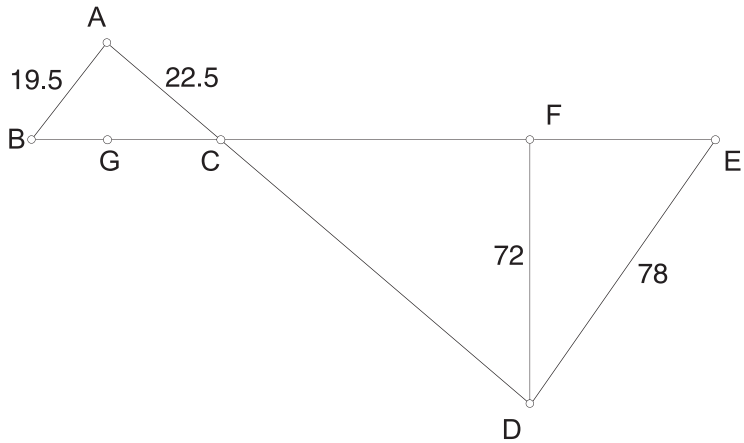
The time it will take to excavate the perimeters is

(63 ft. +252 ft.)•(45 min. per foot) = 14175 min. = 236.25 hours

(63 ft. +252 ft.)•(1 hour per foot) = 315 hours

The excavation time will be between 236.25 hours and 315 hours.

2. Since the entrance along wall $\overline{BC}$ is directly opposite vertex A, the distance from point A to the entrance should be along the perpendicular from A to the line containing $\overline{BC}$. Let point G be the entrance. Then segment $\overline{AG}$ is the altitude to segment $\overline{BC}$ (since triangle ABC is acute).



To find the length of this altitude, use the fact that the ratio of corresponding altitudes on similar triangles is equal to the ratio of the lengths of the corresponding sides, so

$\frac{AG}{DF} = \frac{AG}{72} = \frac{1}{4}$, and AG = 18 feet.

No pair of walls in this room is perpendicular to each other since $(19.5)^2 + (21)^2 \neq (22.5)^2$

3. Since $\overline{DF} \perp \overline{EC}$, $\overline{DF}$ is the altitude to $\overline{EC}$.

Thus, the area of $\Delta DEC = \frac{1}{2}(72)(84) = 3024$ square feet

(2) uses ratios to solve problems involving similar figures;

(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples; and

(4) describes and applies the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

II. Transformations: Dilations

Dividing the area by 9 square feet of space per person shows that the temple could have housed 336 people.

To get the area of $\triangle ABC$, either use the triangle area formula or the property that the ratio of the areas of similar triangles is the square of the ratio of the corresponding sides:

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle DEC} = \frac{A}{3024} = \left(\frac{1}{4}\right)^2.$$

Solving this for A shows the area to be 189 square feet. Dividing this by 9 square feet reveals that the ceremony preparation room could have housed 21 priests and assistants.

Extension Questions:

- In the proposal that the archaeologists must write to excavate the site, they must compare the perimeters and the areas of the ceremony preparation room to those of the main temple. How could this be done?

The ratio of the areas is the square of the ratio of the corresponding sides of the similar figures. Here is why:

On the first triangle let S_1 and h_1 be a side and the altitude to that side, respectively.

On the second triangle let S_2 and h_2 be the corresponding side and the altitude to those in the first triangle, respectively. Then $\frac{S_2}{S_1} = \frac{h_2}{h_1} = r$ *so that* $S_2 = rS_1$ *and* $h_2 = rh_1$.

Now $\frac{\text{area of triangle 2}}{\text{area of triangle 1}} = \frac{\frac{1}{2}h_2S_2}{\frac{1}{2}h_1S_1} = \frac{(rh_1)(rS_1)}{h_1S_1} = r^2,$

which is the square of the ratio of the sides.

- It is believed that each of the two rooms were built in the shape of a triangular pyramid. What can be said about the capacity of the rooms? Is it possible to determine their capacity?

If it is assumed that the triangular pyramids were similar figures, then the ratio of their volumes is the cube of the ratio of corresponding sides.

In this case, the ratio of the volumes would be $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$.

To compute the volumes, compute the following:

$\frac{1}{3}$*(altitude to pyramid base)(base area).*

Even though the ratio of the volumes is known, there is not enough information to find the volumes. Both base areas can be computed. The altitude of one of the triangular pyramids must be known to find the altitude of the other. If one of the altitudes were known, the volumes could be computed.

Student Work Sample

A student working individually did the work sample on the next page.

- Demonstrates geometric concepts, process and skills

The student applied a ratio for similarity and the Pythagorean theorem to find the length of the missing sides in the problem. He also found correctly corresponding sides to set up equations for the ratio.

Although the way the solution is organized is clear in both symbolic and verbal forms, it is still missing an explanation of why the student could apply certain mathematical principles in the situation. For example, the student used a ratio to find the length of CD without explaining why he could use a ratio. He should have explained that the triangles ABC and CDE are similar so that he can use a ratio to find the missing length.

Ancient Ruins

1) $\frac{19.5}{22.5} = \frac{78}{x}$

$19.5x = 1755$

$x = 90 = CD$

$90^2 - 72^2 = \overline{CF}^2$

$8100 - 5184 = \overline{CF}^2$

$\sqrt{2916} = 54 = \overline{CF}$

$78^2 - 72^2 = \overline{FE}^2$

$6084 - 5184 = \overline{FE}^2$

$\sqrt{900} = 30 = \overline{FE}$

$\frac{19.5}{x} = \frac{78}{84}$

$\frac{78x}{x} = \frac{1638}{x}$ $x = 21 = BC$

time = 236 hrs 15 mins

first we setup a ratio of $\frac{AB}{AC} = \frac{ED}{CD}$ to find CD. Then we used pythagorean theorem to find CF + FE to get CE. Next we set up the proportion $\frac{AB}{BC}/\frac{ED}{CE}$ to get 21. Finally we found the sum of all sides that multiplied it by 215 min then divided it by 60, and converted it into time.

2) That AX ⊥ BC

$\frac{19.5}{x} = \frac{78}{72}$

$78x = 1404$

$x = 18 = AX \perp BC$

3) $A = \frac{B \times H}{2} = \Delta CDE$

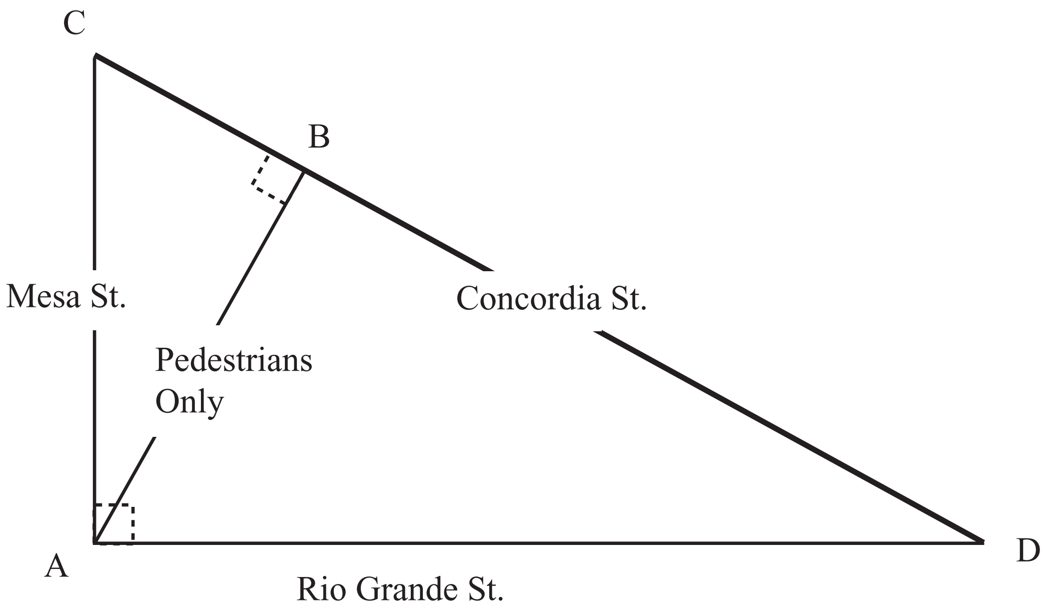
$A = 3024 = \Delta CDE$

$336\, ft^2$ per person in ΔCDE

$A = \frac{bh}{2} = \Delta ABC$

$A = \frac{18 \times 21}{2} = \frac{189\, ft^2}{9}$ $21\, ft^2$ per person in ΔABC

Sightseeing Walk



Yuma City has an historic region downtown between Mesa St., Rio Grande St., and Concordia St. Mesa and Rio Grande Streets intersect to form a right angle. There is a Pedestrians Only path from the intersection of Mesa and Rio Grande to Concordia that intersects Concordia at a right angle. A sightseer started at the intersection of Mesa and Rio Grande and walked the 6 blocks long path to Concordia. She then walked 4 blocks along Concordia to Mesa and back to the intersection of Mesa and Rio Grande. After that she walked to the intersection of Rio Grande and Concordia.

Answer the following questions, completely justifying your answers with geometric explanations.

1. How far did the sightseer walk?
2. If another sightseer had started at the intersection of Mesa and Concordia and walked along Concordia Street to the intersection of Concordia and Rio Grande, how far would he have walked?

Materials:
One ruler and compass per student

Unlined paper, patty paper, and geometry software

Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

(2) uses ratios to solve problems involving similar figures;

Teacher Notes

Scaffolding Questions:

- What segments represent the paths taken by the sightseer?
- What are the known distances?
- What are the unknown distances?
- What types of triangles do you see in this problem?
- What relationships about this/these triangle types can help you find unknown quantities?

Sample Solutions:

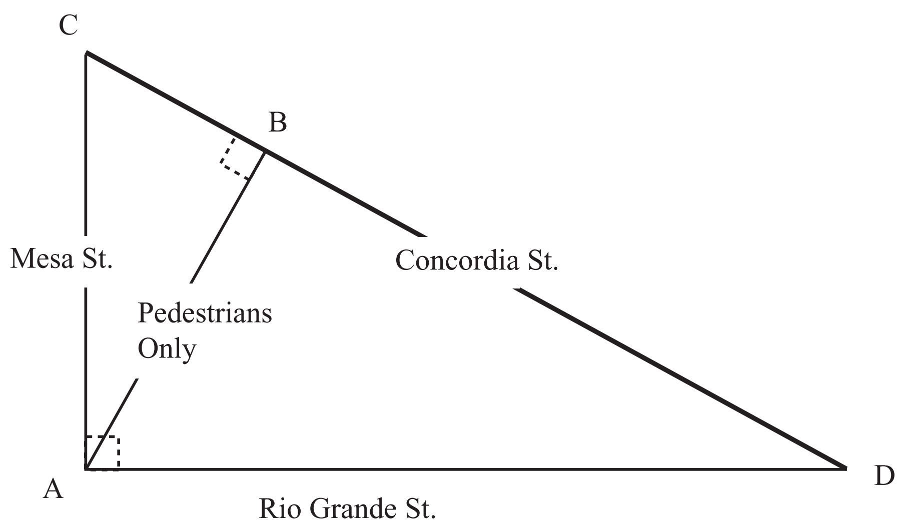
1. The distance the sightseer walks is given by AB + BC + CA + AD.

   We know that AB = 6 and BC = 4. To find CA we apply the Pythagorean Formula to right triangle ABC.

   $CA = \sqrt{AB^2 + BC^2} = \sqrt{36 + 16} = \sqrt{52} \approx 7.21$ blocks

   There are three right triangles: ΔCAD, ΔCBA, and ΔABD, with right angles $\angle CAD$, $\angle CBA$, and $\angle ABD$.

   In the large right triangle CAD, let the measure of acute angle C be x degrees. Then, since the acute angles of a right triangle are complementary, the measure of angle D is $90 - x$ degrees.



In triangle CBA, the measure of angle C is x degrees, so the measure of angle CAB is $90 - x$ degrees.

In triangle ABD, the measure of angle D is $90 - x$ degrees, so the measure of angle BAD is x degrees Now we have $\Delta CAD \sim \Delta CBA \sim \Delta ABD$, because if three angles of one triangle are congruent to the corresponding three angles of another triangle, the triangles are similar. (Angle-Angle-Angle similarity)

Using triangles CBA and ABD, we have

$$\frac{CB}{AB} = \frac{CA}{AD}$$

$$\frac{4}{6} = \frac{\sqrt{52}}{AD}$$

$$AD = \frac{6\sqrt{52}}{4}$$

$$AD \approx 10.82 \text{ blocks}$$

The sightseer walked AB + BC + CA + AD = 6 + 4 + 7.21 + 10.82 = 28.03 ª 28 blocks.

2. The distance from the intersection of Mesa and Concordia to the intersection of Concordia and Rio Grande is given by

CD = CB + BD = 4 + BD.

To find BD we can use the same similar triangles as in Problem 1:

$$\frac{CB}{AB} = \frac{AB}{BD}$$

$$\frac{4}{6} = \frac{6}{BD}$$

$$BD = 9 \text{ blocks}$$

The distance from Mesa and Concordia to Concordia and Rio Grande is 4 plus 9 or 13 blocks.

(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples; and

(4) describes and applies the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

III. Triangles: Pythagorean Theorem

Teacher's Comment:
"It surprised me how hard they worked on this problem, because they are usually difficult to keep on task. I was pleased with how quickly they recognized similar triangles and how well they set up the proportions because the concept was still new to them. I was disappointed in their written explanations because they were usually incomplete although they understood the problem. The instructional strategies I will modify to improve student success is that I would insist on better written explanations for all problems."

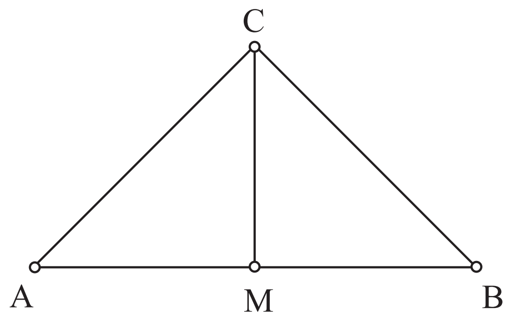
Extension Questions:

- What important geometric concepts did you consider in this problem?

  To find the missing distances along the sightseer's route, we showed we had 3 similar right triangles. Using the resulting proportions we were able to find all missing dimensions along the streets.

- On the other side of Concordia St. is an old cemetery in which a famous sheriff, Mo Robbins, is believed to be buried. The cemetery forms an isosceles right triangle with legs 10 blocks long. The hypotenuse of the triangular cemetery runs along Concordia St. Sheriff Robbins' gravesite is believed be located at the centroid of the triangular cemetery, but there is no tombstone to mark his grave. How would you locate the gravesite of this famous person?

Cemetery:



The centroid is the intersection of the medians. Its distance from each vertex is two-thirds the length of that median. Since triangle ACB is an isosceles right triangle with right angle C, the median from vertex C is the median to the hypotenuse. Triangle CMB is also a right isosceles triangle. CM = MB. Therefore, the median CM is half the length of the hypotenuse, AB.

Because the legs of isosceles right triangle ACB are 10 blocks long,

$$AB = 10\sqrt{2} \text{ and } CM = \frac{1}{2}AB = 5\sqrt{2}.$$

The centroid is located $\frac{2}{3}(5\sqrt{2}) = 4.71$ *blocks along segment CM from point C.*

- Suppose the triangular region formed by the streets were an equilateral triangle and the distance between any two intersections were 10 blocks. How will your answers to Problem 1 change?

Since the triangular region formed by the streets is equilateral, the pathway, segment $\overline{AB}$, *is an altitude and divides the region into two 30-60-90 triangles. This is because an altitude on an equilateral triangle is also an angle bisector and a median.*

In 30-60-90 triangle ACB, the hypotenuse AC = 10, so the shorter leg CB = 5 and the longer leg $AB = 5\sqrt{3} \approx 8.66$ *blocks. Also AC = DA.*

The sightseer walked AB + BC + AC + AD = 8.66 + 5 + 10 + 10 = 33.66 blocks.

Student Work Sample

An individual student produced the work on the next page. The work exemplifies the following criteria:

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem

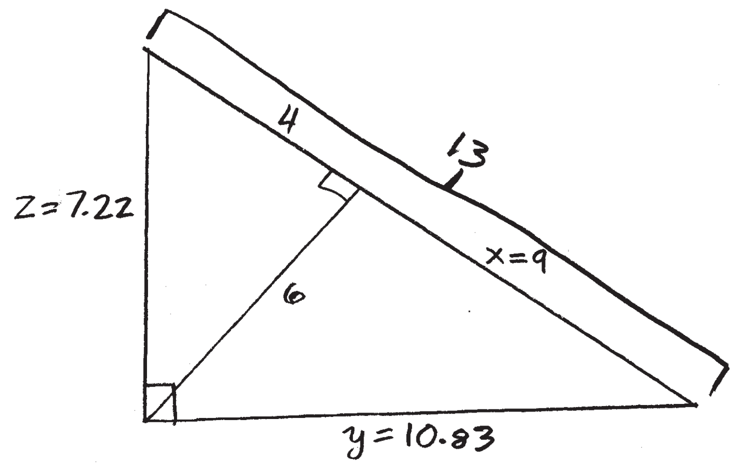
The student applied theorems and corollaries correctly to get answers in that she substituted the correct numbers in the equations provided by corollaries. The student referenced specific corollaries from the book, but she did write what they meant. The student also provided good combination of verbal explanation and symbolic process.

The following criteria was not met:

- Uses appropriate terminology and notation

The student inappropriately used equality signs and used x to represent two different things (one for the altitude and the other for the missing part of hypotenuse).

Sight Seeing Walk



①

- I used corollary theorem 10.2, which states, if the altitude is drawn to the hypotenuse of the right triangle, then its (altitude) length is the geometric mean of the length of two segments of the hypotenuse. $\frac{\text{one side}}{x} = \frac{x}{\text{2nd side}}$, this allowed me to swap it and find x (one of the segments of the hypotenuse). $\frac{4}{6} = \frac{6}{x} = 36 = 4x =$ **9=x**
- I also used the pythagorem theorem, which states if a triangle is a right triangle with legs of length A+B and hypotenuse of length c, then $a^2+b^2=c^2$

$6^2+4^2=c^2$
$36+16=c^2$ **z=7.22**
$\sqrt{52}=\sqrt{c^2}$
$7.22=c$

- I used corollary 10.3, which states, the leg of a right triangle is the geometric mean of the total and adjacent part of the hypotenuse

$\frac{\text{tot hyp}}{\text{leg}} = \frac{\text{leg}}{\text{Adjacent segment of hypotenuse}}$ $\frac{13}{y} = \frac{y}{9} =$ $y^2 = 117$
$\sqrt{y^2} = \sqrt{117}$
$y = 10.83$

- I added the length of Mesa St. + Concord Street + Pedestrian + Rio Grande to get the distance she walked = 28.05 blocks

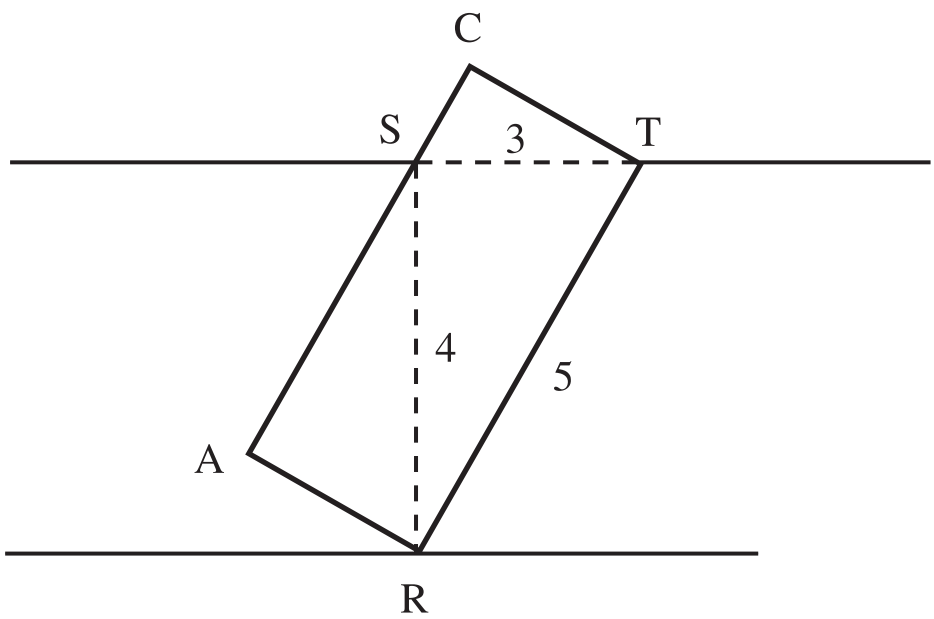
②

- I added the length of the two segments of Concordia St. that are divided by Pedestrian to get the total length of Concordia St.

$9 + 4 = 13$ (13 blocks)

Will It Fit?

Rectangular cartons that are 5 feet long need to be placed in a storeroom that is located at the end of a hallway. The walls of the hallway are parallel. The door into the hallway is 3 feet wide and the width of the hallway is 4 feet. The cartons must be carried face up. They may not be tilted. Investigate the width and carton top area that will fit through the doorway.



Teacher Notes

Materials:
One calculator per student

Connections to Geometry TEKS :

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem; and

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

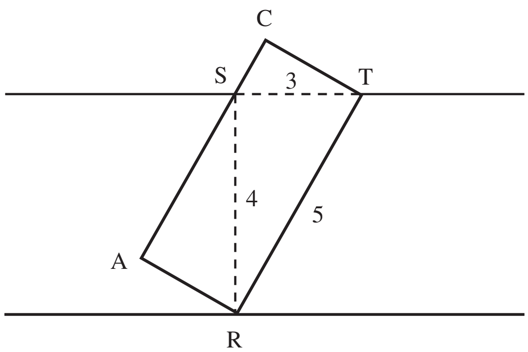
(2) uses ratios to solve problems involving similar figures;

(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

Scaffolding Questions:

- What segments represent the width of the trunk?
- How does knowing the dimensions of triangle TSR help you?
- What triangles involve the width of the trunk?
- What triangles involve the width of the hallway?
- Explain any angle relationships in the triangles.
- How are the triangles you see related? How does this help you?
- How can you use your solution to Problem 1 to help you with Problem 2?

Sample Solution:



Triangle SRT is a right triangle because it has three sides that are Pythagorean Triples. That is, the sum of the squares of two sides is the square of the third side. $3^2 + 4^2 = 5^2$

Triangle SCT is a right triangle because $\overline{AC} \perp \overline{CT}$.

$\overline{AC} // \overline{RT}$ so the alternate interior angles are congruent. $\angle RSA \cong \angle TRS$.

$\angle TRS$ and $\angle STR$ are complementary angles because they are the two acute angles of a right triangle SRT.

$\angle CTS$ and $\angle STR$ are complementary angles because the figure CTRA is a rectangle with right angle CTR.

$\angle CTS \cong \angle TRS$ because they are two angles that are complementary to the same angle.

Thus, the three right triangles, ΔCTS, ΔSRT, and ΔASR, have congruent acute angles: $\angle CTS \cong \angle TRS \cong \angle RSA$.

Therefore, $\Delta CTS \approx \Delta SRT \approx \Delta ASR$ because they are right triangles with acute angles that are congruent.

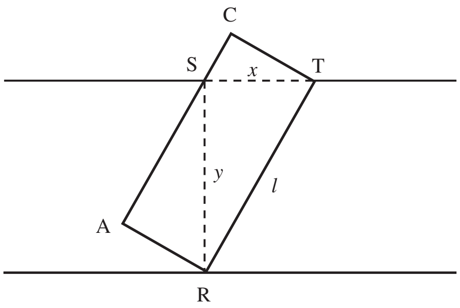
Now write the proportion between the corresponding sides:

$$\frac{CT}{SR} = \frac{ST}{TR}$$
$$\frac{CT}{4} = \frac{3}{5}$$
$$CT = 2.4$$

The widest carton that will fit through the opening has a width of 2.4 feet, and the top surface area is 2.4 feet times 5 feet = 12 square feet.

Extension Questions:

- Generalize your results for Problem 1 for a hallway opening of *x* feet and a hallway width of *y* feet if the maximum carton dimensions are *l* feet length and $x^2 + y^2 = l^2$.



Because $x^2 + y^2 = l^2$, the triangle STR is a right triangle.

$$\frac{CT}{SR} = \frac{ST}{TR}$$
Let $CT = w$.
$$\frac{w}{y} = \frac{x}{l}$$
$$w = \frac{xy}{l}$$

Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

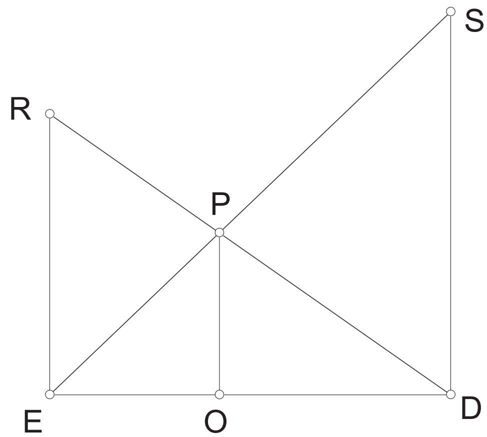
III. Triangles: Pythagorean Theorem

This shows that the maximum carton width is the product of the door width and the hallway width divided by the carton length. The maximum carton surface is the width, w, times the length, l, or $w \bullet l = \frac{xy}{l} \bullet l = xy$. *The maximum carton area is the product of the door width and the hallway width.*

Spotlights

1. The spotlight display for an outdoor rock concert is being planned. At the sides of the stage a red spotlight is mounted on a pole 18 feet high, and a green spotlight is mounted on a pole 27 feet high. The light from each spotlight must hit the base of the other pole as shown in the diagram. How high above the ground should the stage be so that the spotlights meet and highlight the upper body of a performer who is about 6 feet tall?

   The red spotlight is at point R. The green spotlight is at point S. Ground level is segment $\overline{ED}$.



2. If the poles are 30 feet apart and the stage is 20 feet long, how should the stage be positioned so that the spotlights meet on the upper body of the performer when he is center stage?

3. Generalize your results in Problem 1 if the poles for the spotlights are *a* feet and *b* feet long.

Teacher Notes

Materials:
One ruler per student

Unlined paper and geometry software

Connections to Geometry TEKS:

(d.2) **Dimensionality and the geometry of location**. The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons;

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

(2) uses ratios to solve problems involving similar figures;

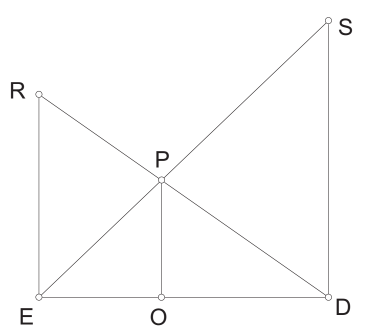
(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples; and

Scaffolding Questions:

- In what figures do you see segment $\overline{PO}$ as a side or special segment?
- Since $\overline{PO}$ is perpendicular to segment $\overline{ED}$, what does this tell you about $\overline{PO}$ and triangle EPD? Does this help? Why or why not?
- Can you find similar triangles involving segment $\overline{PO}$?
- What proportions can you write involving $\overline{PO}$?
- How can you use algebraic notation to make your explanation easier to follow?
- Experiment with equivalent ways of writing a proportion to discover a way to solve for the length of $\overline{PO}$.
- Does the distance between the light poles seem to matter?
- How can you generalize from 18 feet and 27 feet long poles to poles of arbitrary lengths, *a* feet and *b* feet?

Sample Solutions:

1.



Let *h* = PO, which is the height at which the spotlight beams meet.

We know that RE = 18 and SD = 27. We can use two pairs of similar triangles to compare *h* to the lengths of the poles.

$\Delta EOP \sim \Delta EDS$ by Angle-Angle-Angle similarity. The triangles have right angles EOP and EDS, and they share the common angle, $\angle PEO$.

Similarly, $\Delta DOP \sim \Delta DER$.

Let EO = x and OD = y. Consider the following proportions that we get from the similar triangles:

$$1.\quad \frac{PO}{SD} = \frac{EO}{ED} \Rightarrow \frac{h}{27} = \frac{x}{x+y}$$

$$2.\quad \frac{PO}{RE} = \frac{OD}{ED} \Rightarrow \frac{h}{18} = \frac{y}{x+y}$$

Experimenting with properties of proportions gives an equation with h and the pole heights:

$$1.\quad \frac{x}{h} = \frac{x+y}{27}$$

$$2.\quad \frac{y}{h} = \frac{x+y}{18}$$

Adding these equations results in the equation

$$\frac{x}{h} + \frac{y}{h} = \frac{x+y}{27} + \frac{x+y}{18}$$

$$\frac{x+y}{h} = \frac{x+y}{27} + \frac{x+y}{18}$$

$$(x+y)\frac{1}{h} = (x+y)\left(\frac{1}{27} + \frac{1}{18}\right)$$

$$\frac{1}{h} = \frac{1}{27} + \frac{1}{18} = \frac{18+27}{(27)(18)} = \frac{45}{(27)(18)}$$

Take the reciprocal of both sides.

$$h = \frac{(27)(18)}{45} = \frac{54}{5} = 10.8 \text{ feet}$$

This shows that the spotlight beams will meet at a point 10.8 feet above the ground.

Therefore, if the performer is about 6 feet tall, the stage should be 4.8 feet off the ground.

Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Connection to High School Geometry: Supporting TEKS and TAKS Institute:

II. Transformations: Dilations

It also shows that the distance between the poles, $x + y$, does not matter. This distance was not needed to determine $h = 10.8$ feet.

2. The spotlight poles are to be 30 feet apart, so $x + y = 30$. The point at which the spotlights meet must be determined. In other words, find the value of *x or y*.

$$\frac{x}{x+y} = \frac{h}{27}$$

$$\frac{x}{30} = \frac{10.8}{27}$$

$$x = 10.8 \bullet \frac{30}{27}$$

$$x = 12 \text{ feet}$$

The spotlights meet 12 feet in from the red spotlight and, therefore, 18 feet in from the green spotlight. We want this point to correspond to the center of the 20-foot long stage.

We need 10 feet of the stage to the left of this point and 10 feet of this stage to the right of this point. This means the pole for the red spotlight should be 2 feet from the left edge of the stage, and the pole for the green spotlight should be 8 feet from the right edge of the stage.

3. To generalize our results for spotlight poles of lengths *a* feet and *b* feet, we would have the same pairs of similar triangles and the same proportions. We simply need to replace 18 feet with *a* feet and 27 feet with *b* feet.

$$\frac{1}{h} = \frac{1}{b} + \frac{1}{a}$$

$$\frac{1}{h} = \frac{a+b}{ab}$$

$$h = \frac{ab}{a+b}$$

Extension Questions:

- What are the key concepts needed to solve Problem 1?

We needed to relate the height at which the spotlight beams meet to the lengths of the poles. Segment $\overline{PO}$ is the altitude from point P to $\overline{ED}$ in triangle EPD. This does not help since we know nothing about this triangle. By the same reasoning, it does not help to consider $\overline{PO}$ as a leg of right triangles EOP or POD.

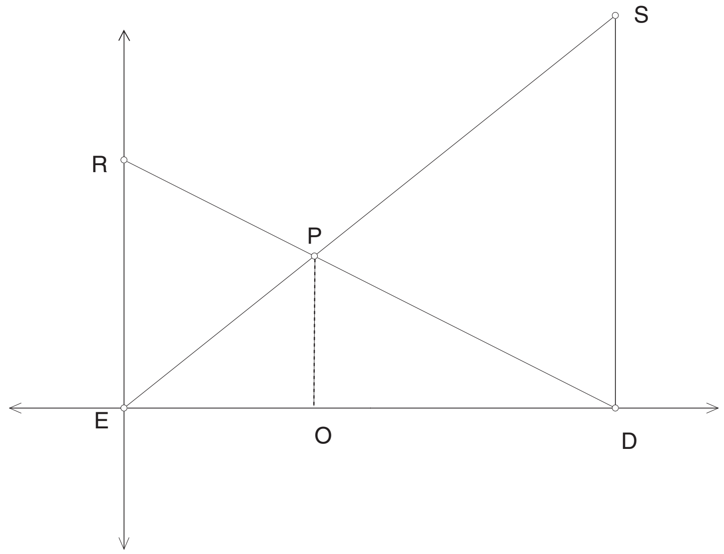
The big idea to use is similar triangles so we can write proportions involving $\overline{PO}$ and the lengths of the light poles.

- Why doesn't the distance between the poles matter?

Segment $\overline{PO}$ divides segment $\overline{ED}$ into two pieces: segments $\overline{EO}$ and $\overline{OD}$. When we solve the proportions for PO, the sum of these unknown lengths is a factor on both sides of the equation.

- How could you use algebra and a coordinate representation to solve this problem?

The coordinates of the base of the stage can be (0,0) and (30,0). The red spotlight will be located at (0,18), and the green spotlight will be located at (30,27). The diagram below shows this.



The slope of segment $\overline{RD}$ is $\frac{-18}{30} = \frac{-3}{5}$, and its y-intercept is 18.

The slope of segment $\overline{ES}$ is $\frac{27}{30} = \frac{9}{10}$, and its y-intercept is 0.

The equations of these two segments are $y = \frac{-3}{5}x + 18$ and $y = \frac{9}{10}x$.

Solve this system to get x = 12. Substitute for x in either equation and solve to get y = 10.8.