

# Geometry Assessments

**The Charles A. Dana Center**  
at the University of Texas at Austin

With funding from  
the Texas Education Agency and  
the National Science Foundation



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The Charles A. Dana Center at The University of Texas at Austin works to support education leaders and policymakers in strengthening Texas education. As a research unit of UT Austin's College of Natural Sciences, the Dana Center maintains a special emphasis on mathematics and science education. We offer professional development institutes and produce research-based mathematics and science resources for educators to use in helping all students achieve academic success. For more information, visit the Dana Center website at [www.utdanacenter.org](http://www.utdanacenter.org).

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## TEKS and TAKS Resources

The mathematics Texas Essential Knowledge and Skills (TEKS) were developed by the state of Texas to clarify what all students should know and be able to do in mathematics in kindergarten through grade 12. Districts are required to provide instruction that is aligned with the mathematics TEKS, which were adopted by the State Board of Education in 1997 and implemented statewide in 1998. The mathematics TEKS also form the objectives and student expectations for the mathematics portion of the Texas Assessment of Knowledge and Skills (TAKS), which will be implemented in spring 2003 for grades 3 through 10 and for the Grade 11 Exit Level assessment.

The mathematics TEKS can be downloaded in printable format, free of charge, from the Texas Education Agency website ([www.tea.state.tx.us/teks](http://www.tea.state.tx.us/teks)). Bound versions of the mathematics and science TEKS are available for a fee from the Charles A. Dana Center at The University of Texas at Austin ([www.utdanacenter.org](http://www.utdanacenter.org)).

Resources for implementing the mathematics TEKS, including professional development opportunities, are available through the Texas Education Agency and the Charles A. Dana Center, the state-designated Mathematics Center for Educator Development. Online resources can be found in the Mathematics TEKS Toolkit at [www.mathtekstoolkit.org](http://www.mathtekstoolkit.org).

Additional products and services that may be of interest are available from the Dana Center at [www.utdanacenter.org](http://www.utdanacenter.org). These include the following:

- TEKS, TAAS, and TAKS: What's Tested at Grades 3–8 charts
- Mathematics Abridged TEKS charts
- Mathematics TEKS “Big Picture” posters
- Mathematics Standards in the Classroom: Resources for Grades 3–5
- Mathematics Standards in the Classroom: Resources for Grades 6–8
- *Algebra I Assessments* and the corresponding professional development
- TEXTEAMS professional development mathematics institutes
- TEKS for Leaders professional development modules for principals and other administrators





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# Introduction

The Dana Center has developed Geometry Assessments as a resource for teachers to use to provide ongoing assessment integrated with high school geometry instruction.

The National Council of Teachers of Mathematics has identified the following six standards to guide classroom assessment<sup>1</sup>:

Standard 1: Assessment should reflect the mathematics that all students need to know and be able to do.

Standard 2: Assessment should enhance mathematics learning.

Standard 3: Assessment should promote equity.

Standard 4: Assessment should be an open process.

Standard 5: Assessment should promote valid inferences about mathematics learning.

Standard 6: Assessment should be a coherent process.

Implementing these assessment standards may require significant changes in how teachers view and use assessment in the mathematics classroom. Teachers should assess frequently to monitor individual performance and guide instruction.

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1. National Council of Teachers of Mathematics. 1995. *Assessment Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.



## What are the Geometry Assessments?

The geometry assessments are problems that reflect what all students need to know and be able to do in a high school geometry course that follows a first-year algebra course. These assessments may be formative, summative, or ongoing. The problems focus on students' understanding as well as their procedural knowledge. The tasks require more than right or wrong answers; they focus on how students are thinking about a situation.

## What is the purpose of the Geometry Assessments?

The purpose of these assessments is to make clear to teachers, students, and parents what is being taught and learned. Teachers should use evidence of student insight, student misconceptions, and student problem-solving strategies to guide their instruction. Teachers may also use the questions included with the assessments to guide learning and to assess student understanding. The use of these assessments should help teachers enhance student learning and provide them with a source of evidence on which they may base their instructional decisions.

## What is the format of the Geometry Assessments?

This book contains 43 problems.

The problems have been divided into seven categories:

- Coordinate Geometry
- Patterns, Conjecture, and Proof
- Properties and Relationships of Geometric Figures
- Area, Perimeter, and Volume
- Solids and Nets
- Congruence
- Similarity

Each problem

- includes a geometry task,
- is aligned with the Geometry Texas Essential Knowledge and Skills (TEKS) student expectations,
- is aligned with the Grade 11 Exit Level Texas Assessment of Knowledge and Skills (TAKS) objectives,
- is aligned with the *TEXTEAMS High School Geometry: Supporting TEKS and TAKS* professional development institute,
- includes “scaffolding” questions that the teacher may use to help the student to analyze the problem,



- provides a sample solution,\* and
- includes extension questions to bring out additional mathematical concepts in a summative discussion of solutions to the problem.

\*The sample solution is only one way that a problem may be approached and is not necessarily the “best” solution. For many of the problems there are other approaches that will also provide a correct analysis of the problem. The authors have attempted to illustrate a variety of methods in the different problem solutions. Several of the problems include samples of anonymous student work.

Following this introduction are alignments of all the problems to the TEKS and to the TAKS Grade 11 Exit Level objectives.

### **What is the solution guide?**

A solution guide is a problem-solving checklist that may be used to understand what is necessary for a complete problem solution. When assigning the problem, the teacher will give the students the solution guide and will indicate which of the criteria should be considered in the problem analysis. In most problems all of the criteria are important, but initially the teacher may want to focus on only two or three criteria. On the page before a student work sample in this book, comments on some of the criteria that are evident from the student’s solution are given. The professional development experience described below will help the teacher use this tool in the classroom and will also help guide the teacher to use other assessment evaluation tools.

### **TEXTEAMS Practice-Based Professional Development: Geometry Assessments**

The Dana Center has developed a three-day TEXTEAMS institute in which participants experience selected assessments, examine the assessments for alignment with the TEKS and TAKS, analyze student work to evaluate student understanding, consider methods for evaluating student work, view a video of students working on the assessments, develop strategies for classroom implementation, and consider how the assessments support the TAKS. Teachers should contact their local school district or regional service center to determine when this institute is offered.



## Geometry Solution Guide

Name of Student \_\_\_\_\_ Name of Problem \_\_\_\_\_

The teacher will mark the criteria to be considered in the solution of this particular problem.

Mark the criteria to be considered in the solution of this particular problem.	<b>Criteria</b>	Check here if the solution satisfies this criteria.
	Identifies the important elements of the problem.	
	Shows an understanding of the relationships among elements.	
	Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.	
	Evaluates reasonableness or significance of the solution in the context of the problem.	
	Demonstrates geometric concepts, processes, and skills.	
	Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.	
	Communicates clear, detailed, and organized solution strategy.	
	States a clear and accurate solution using correct units.	
	Uses appropriate terminology and notation.	
	Uses appropriate tools.	



## Mathematics TEKS Alignment

The charts on the following pages indicate the Geometry Texas Essential Knowledge and Skills (TEKS) student expectations addressed by each problem. The student expectation has been included only if the problem specifically requires mastery of that student expectation.





Problem	TEKS Focus	Congruence and the Geometry of Size			Similarity and the Geometry of Shape
		e(1)ABCD	e(2)ABCD	e(3)AB	
<b>Chapter 1: Coordinate Geometry</b> Cross Country Cable Whitebeard's Treasure Quadrilateral Quandary Quite a Quilt Wearable Art Sea Quest	d(2) d(2) d(2) d(2) d(2) d(2)	A		e(3)AB	f(1), f(2)
<b>Chapter 2: Patterns, Conjecture, and Proof</b> Pizza Delivery Service Regions More Pizza Delivery Conjecture as Discovery and Proof as Explanation Extending the Triangle Midsegment Conjecture Why Doesn't My Conjecture Always Work? Steiner's Point Steiner's Point Revisited Walking the Archimedean Walk Talking the Archimedean Talk Mad as a Hatter or Hat as a Madder Going the Distance in Taxicab Land	b(2) b(2) b(3) b(3) b(3) b(4), c c b(1), c b(1), c b(3) b(1)	A	A	B A, B	f(1), f(2)
<b>Chapter 3: Properties and Relationships of Geometric Figures</b> Angle Bisectors and Parallel Lines Circles and Tangents The Clubhouse Diagonals and Polygons The Most Juice	e(2) e(2) e(2) e(2) e(2)	A, C, D	A A A B B, C, D	A	f(1) f(1)



Problem	TEKS Focus	Geometric Structure				Geometric Patterns	Dimensionality and the Geometry of Location
		b(1)ABC	b(2)AB	b(3)ABCDE	b(4)		
<b>Chapter 4: Area, Perimeter, and Volume</b> Boxing Basketballs Flower Walter and Juanita's Water Troughs Great Pyramids Greenhouse Nesting Hexagons	e(1)				b(4)	c(1), (2), (3)	d(2)ABC
	e(1)					c(3)	A
	e(1)					c(3)	B
	e(1)					c(1), c(2), c(3)	A
	e(1)						
<b>Chapter 5: Solids and Nets</b> Perfume Packaging Playing with Pipes Circular Security Different Views The Slice Is Right!	d(1)				b(4)	c(3)	B, C
	d(1)				b(4)	c(3)	A, C
	d(1)						B, C
	d(1)					c(3)	A, B, C
	d(1)						A
<b>Chapter 6: Congruence</b> Median to the Hypotenuse of Right Triangle Tell Me Everything You Can About... Tiling with Four Congruent Triangles Shadow's Doghouse The School Flag	e(3)	A	A, B	A, B, C, D, E		c(1), c(3)	
	e(3)	A	A	B, C			
	e(3)	A	B	A, B, C, D, E	b(4)	c(2)	
	e(3)	A	A, B	A	b(4)	c(2)	C
	e(3)		B	B	b(4)	c(2)	
<b>Chapter 7: Similarity</b> Ancient Ruins Sightseeing Walk Spotlights Will It Fit?	f					c(1), c(3)	
	f					c(1), c(2), c(3)	A, B
	f						
	f					c(3)	



Problem	TEKS Focus	Congruence and the Geometry of Size			Similarity and the Geometry of Shape
		e(1)ABCD	e(2)ABCD	e(3)AB	
<b>Chapter 4: Area, Perimeter, and Volume</b> Boxing Basketballs Flowers Walter and Juanita's Water Troughs Great Pyramids Greenhouse Nesting Hexagons	e(1) e(1) e(1) e(1) e(1) e(1)	D A, B, C A, C, D C, D B, D A, C	D	e(3)AB	f(1), (2), (3), (4)
<b>Chapter 5: Solids and Nets</b> Perfume Packaging Playing with Pipes Circular Security Different Views The Slice Is Right!	d(1) d(1) d(1) d(1) d(1)	A, C, D A C A, B, C C, D	D		
<b>Chapter 6: Congruence</b> Median to the Hypotenuse of Right Triangle Tell Me Everything You Can About... Tiling with Four Congruent Triangles Shadow's Doghouse The School Flag	e(3) e(3) e(3) e(3) e(3) e(3)	A, B  A, D A C, D	B B B, D B	A, B A, B A A, B A, B A, B	f(4)
<b>Chapter 7: Similarity</b> Ancient Ruins Sightseeing Walk Spotlights Will It Fit?	f f f f	C			f(1), f(2), f(3), f(4) f(1), f(2), f(3), f(4) f(1), f(2), f(3) f(2), f(3)



## Mathematics Grade 11 Exit Level TAKS Alignment

This chart shows the problems that have been aligned to the Grade 11 Exit Level Texas Assessment of Knowledge and Skills (TAKS).

### TAKS Objective 1:

Diagonals and Polygons  
Nesting Hexagons  
The Most Juice

### TAKS Objective 4:

Diagonals and Polygons  
More Pizza Delivery  
Quadrilateral Quandary  
Quite a Quilt  
Sea Quest  
The Most Juice

### TAKS Objective 6:

Angle Bisectors and Parallel Lines  
More Pizza Delivery  
Median to the Hypotenuse of a Right Triangle  
Nesting Hexagons  
Pizza Delivery Service Regions  
Shadow's Doghouse  
Sightseeing Walk  
Talking the Archimedean Talk  
Tell Me Everything You Can About...  
The School Flag  
The Shortest Cable Line  
Tiling with Four Congruent Triangles  
Why Doesn't My Conjecture Always Work?



### TAKS Objective 2:

Diagonals and Polygons  
Nesting Hexagons  
Quite a Quilt  
The Most Juice

### TAKS Objective 5:

None (Refer to the problems  
in *Algebra Assessments*)

### TAKS Objective 3:

Cross Country Cable  
Diagonals and Polygons  
Quadrilateral Quandary  
Quite a Quilt  
The Most Juice  
Wearable Art  
Whitebeard's Treasure

## Mathematics Grade 11 Exit Level TAKS Alignment

This chart shows the problems that have been aligned to the Grade 11 Exit Level Texas Assessment of Knowledge and Skills (TAKS).

TAKS Objective 7:	TAKS Objective 8:	TAKS Objective 10:
<ul style="list-style-type: none"> <li>Angle Bisectors and Parallel Lines</li> <li>Circles and Tangents</li> <li>Circular Security</li> <li>Cross Country Cable</li> <li>Different Views</li> <li>Extending the Triangle</li> <li>Midsegment Conjecture</li> <li>Going the Distance in Taxicab Land</li> <li>More Pizza Delivery</li> <li>Perfume Packaging</li> <li>Playing with Pipes</li> <li>Quadrilateral Quandary</li> <li>Quite a Quilt</li> <li>Sea Quest</li> <li>Spotlights</li> <li>Steiner's Points</li> <li>The Clubhouse</li> <li>The Most Juice</li> <li>The Slice Is Right!</li> <li>Walking the Archimedean Walk</li> <li>Wearable Art</li> <li>Whitebeard's Treasure</li> </ul>	<ul style="list-style-type: none"> <li>Ancient Ruins</li> <li>Boxing Basketballs</li> <li>Circular Security</li> <li>Different Views</li> <li>Extending the Triangle</li> <li>Midsegment Conjecture</li> <li>Flower</li> <li>Great Pyramid</li> <li>Nesting Hexagons</li> <li>Perfume Packaging</li> <li>Playing with Pipes</li> <li>Sightseeing Walk</li> <li>Spotlights</li> <li>The Most Juice</li> <li>The Shortest Cable Line</li> <li>The Slice Is Right!</li> <li>Tiling with Four Congruent Triangles</li> <li>Walter and Juanita's Water Troughs</li> <li>Will It Fit?</li> </ul>	<ul style="list-style-type: none"> <li>Ancient Ruins</li> <li>Angle Bisectors and Parallel Lines</li> <li>Boxing Basketballs</li> <li>Circles and Tangents</li> <li>Conjecture as Discovery and Proof as Explanation</li> <li>Diagonals and Polygons</li> <li>Extending the Triangle</li> <li>Midsegment Conjecture</li> <li>Greenhouse</li> <li>Mad as Hatter or Hat as a Madder</li> <li>Nesting Hexagons</li> <li>Pizza Delivery Service Regions</li> <li>Quadrilateral Quandary</li> <li>Quite a Quilt</li> <li>Spotlights</li> <li>Steiner's Points</li> <li>Steiner's Points Revisited</li> <li>Talking the Archimedean Talk</li> <li>Walking the Archimedean Walk</li> <li>Walter and Juanita's Water Troughs</li> <li>Whitebeard's Treasure</li> <li>Why Doesn't My Conjecture Always Work?</li> </ul>
	<p><b>TAKS Objective 9:</b></p> <p>None</p>	





**Chapter 1:**  
*Coordinate  
Geometry*





# Introduction

The assessments in this chapter use the coordinate system as a convenient and efficient way of representing geometric figures and investigating geometric relationships. These problems provide the connections between what students learned in Algebra I and the geometric concepts. For example, Whitebeard's Treasure may be used as a problem to assess the student's knowledge of algebraic concepts and concepts learned about geometric figures in earlier mathematics courses.

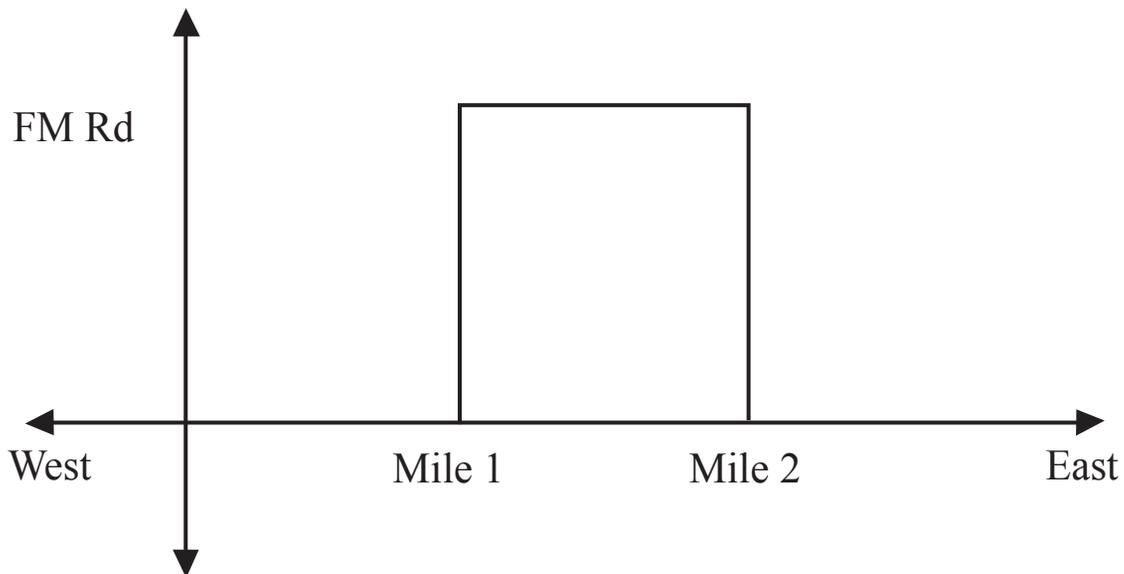
Geometry can be used to model and present many mathematical and real-world situations. Students perceive connections between geometry and the real and mathematical worlds and use geometric ideas, relationships, and properties to solve real problems. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999.)





## Cross Country Cable

Jose owns a square parcel of land on the north side of West–East Road. The property is on the edge of the road between the one- and two-mile markers. The Farm-to-Market (FM) road runs perpendicular to the West–East Road and one mile west of the one-mile marker.



The cable company wants to run a cable through Jose's property from the intersection of the Farm-to-Market (FM) Road and West–East Road. The cable must divide Jose's parcel of land into two parts that are equal in area.

1. Determine the equation of the line that will represent the path of the buried cable.
2. Determine the length of the portion of the cable that runs across Jose's property.



## Teacher Notes

### Materials:

One graphing calculator per student

### Connections to Geometry

#### TEKS:

#### (d.2) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

### Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

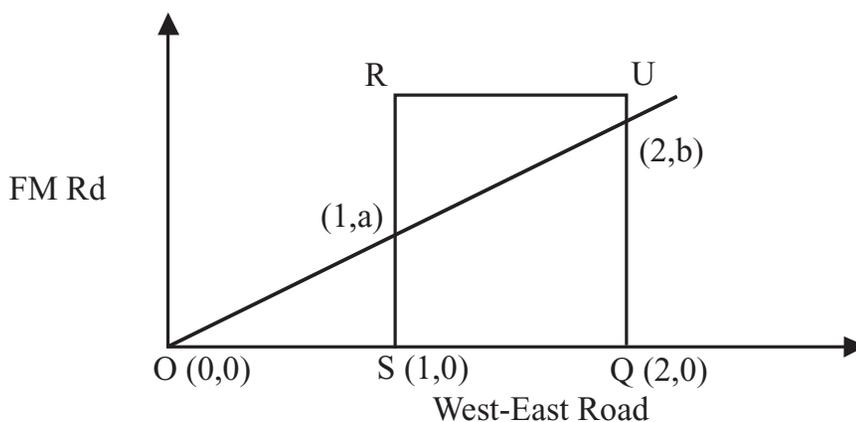
Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

### Scaffolding Questions:

- What information do you know about the roads near Jose's property and the location of his parcel of land?
- What needs to be added to the given figure before the equation of the line can be found?
- Where will the cable line originate? How can you label this point?
- In how many different ways might the cable intersect his property?
- Which way must the cable be drawn to divide it into two portions that are equal in area?
- How can you label the points where the cable will cross the boundaries of Jose's property?
- How can you determine the slope of the cable line?

### Sample Solution:

1. Draw a figure in the coordinate plane to represent Jose's land.



Since  $SQRU$  is a square, the coordinates at  $R$  are  $(1,1)$ , and the coordinates at  $U$  are  $(2,1)$ . The coordinates of the



intersection of  $\overline{SR}$  and the cable line are  $(1,a)$ , and the point of intersection of  $\overline{UQ}$  and the cable line are  $(2,b)$ .

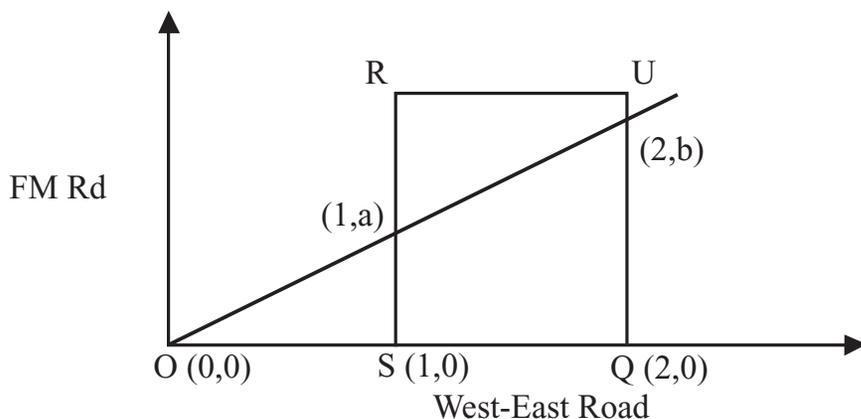
The cable originated at the intersection of the FM Road and the West-East Road. This point is the origin. The cable is going to begin at the origin, therefore one point on the line is  $(0,0)$ . The  $y$ -intercept of the line will also be 0. The line will also contain point  $(1,a)$ .

The slope of the line is  $\frac{a-0}{1-0} = a$ . The equation of this line is  $y = ax$ .

By using points  $(2,b)$  and  $(1,a)$ , the slope of the line can also be expressed as  $\frac{b-a}{2-1} = b-a$ .

When the two slopes are set equal to each other,  $a = b - a$ , the equation can be simplified into  $2a = b$  or  $b = 2a$ .

The cable divides the area of the square into two equal parts. Each upper and lower area is in the shape of a trapezoid.



We know that the area of the land is 1 square mile, and the area of one trapezoid is  $\frac{1}{2}$  square mile. Using this information the area of one trapezoid can be calculated, and the numerical value of the slope can be found.

The lower trapezoid (below the cable line) has vertical bases with lengths  $a$  and  $b$ . The height of the trapezoid is the length of the side running along the West-East Road. The length is 1 mile. The area of the trapezoid is found by

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

- III. Triangles: Pythagorean Theorem
- III. Triangles: Right triangle Ratios

**Teacher's Comment:**

"To introduce the problem to the students, go over the criteria on the solution guide and allow students to read and decide how to solve the problem. Tell the students to read the problem silently before working as a group--it made it easier to work."



multiplying one-half by the product of the sum of the bases and the height.

$$A = \frac{1}{2}(a+b) \cdot 1 = \frac{a+b}{2}$$

We know this area is equal to one-half of the area of the mile square, therefore

$$\begin{aligned}\frac{1}{2} &= \frac{a+b}{2} \\ a+b &= 1\end{aligned}$$

By substituting the slope  $b = 2a$  into the area  $1 = a + b$ , we find that

$$\begin{aligned}1 &= a + 2a \\ 1 &= 3a \\ \frac{1}{3} &= a\end{aligned}$$

The value of the slope for the equation  $y = ax$  is  $\frac{1}{3}$ , so the equation of the cable line that bisects Jose's property is  $y = \frac{1}{3}x$ .

2. The length of the cable is represented by the distance from  $(1,a)$  to  $(2,b)$ . The value of  $a = \frac{1}{3}$ , and the value of  $b = 2 \cdot \frac{1}{3} = \frac{2}{3}$

The coordinates of the vertices representing the opposite corners of Jose's land are  $(1, \frac{1}{3})$  and  $(2, \frac{2}{3})$ .

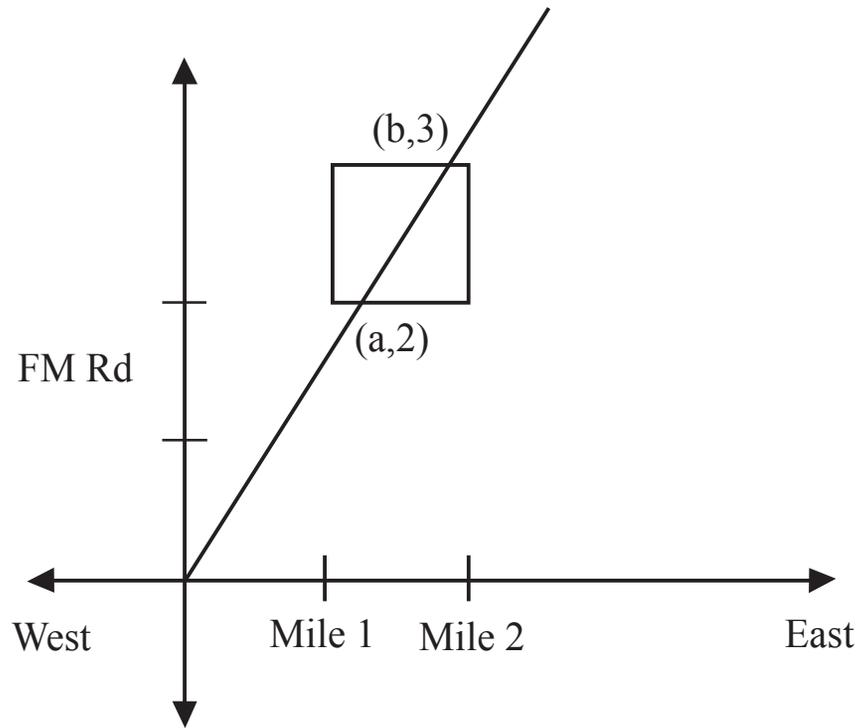
The distance can be found using the distance formula:

$$\begin{aligned}\sqrt{(2-1)^2 + \left(\frac{2}{3} - \frac{1}{3}\right)^2} &= \sqrt{(1)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{10}{9}} = \frac{\sqrt{10}}{3} \text{ miles} \\ &\approx 1.05 \text{ miles}\end{aligned}$$



### Extension Questions:

- Describe how this situation would change if the one square mile parcel of land were two miles north of West–East Road and one mile east of the Farm-to-Market Road. Justify your description with a diagram, and show your work.



The slope of the line can be represented as

$$\frac{3 - 0}{b - 0} = \frac{3}{b} \text{ or}$$

$$\frac{2 - 0}{a - 0} = \frac{2}{a}$$

$$\frac{3}{b} = \frac{2}{a}$$

$$b = \frac{3}{2}a$$



Consider the portion to the left of the line. The height of the square is 1 unit. The two bases of the trapezoid may be expressed as  $a - 1$  and  $b - 1$ . The area of the left half of the square parcel of land can be represented in this way:

$$\frac{1}{2} = \frac{1}{2}[(a-1) + (b-1)] \cdot 1$$

$$1 = a + b - 2$$

$$3 = a + b$$

$$b = \frac{3}{2}a$$

$$3 = a + \frac{3}{2}a$$

$$3 = \frac{5}{2}a$$

$$a = \frac{6}{5}$$

$$b = \frac{3}{2}a = \frac{3}{2}\left(\frac{6}{5}\right) = \frac{9}{5}$$

The points are  $\left(\frac{6}{5}, 2\right)$  and  $\left(\frac{9}{5}, 3\right)$ .

The slope of the line is  $\frac{3 - 2}{\left(\frac{9}{5} - \frac{6}{5}\right)} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$ .

The equation of the line is  $y = \frac{5}{3}x$ .

To verify that the line divides the rectangle into two polygons of equal area, determine the area of the two trapezoids.

Left trapezoid:

$$\frac{1}{2}\left(\frac{6}{5} - 1 + \frac{9}{5} - 1\right) \cdot 1 = \frac{1}{2}\left(\frac{15}{5} - 2\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$



Right trapezoid:

$$\frac{1}{2}\left(2 - \frac{6}{5} + 2 - \frac{9}{5}\right) \cdot 1 = \frac{1}{2}\left(4 - \frac{15}{5}\right) = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

The land has been divided into two equal portions by the line  $y = \frac{5}{3}x$ .





## Whitebeard's Treasure

Whitebeard, the notorious pirate of the West Bay, buried treasure on Tiki Island over 200 years ago. Archeologists recently discovered a map showing the location of the treasure. The location has generated quite a bit of media attention, much to the dismay of the archeologists. In order to allow both the media and archeologists to work together, officials have decided to erect two fences around the location, allowing the media access to the site, yet allowing the archeologists room to work. One fence encloses the actual area where the archeologists will work. Another fence surrounds the enclosed dig area.

Descriptions of the fencing locations have been provided to the media so they may indicate accessible areas for their employees. Use the given information to draw and label a quadrilateral on graph paper indicating the location of the two fences.

1. Corners of the first fence are located at points  $A(11,3)$ ,  $B(3,-11)$ ,  $C(-13,-9)$  and  $D(-5,9)$ . The media must stay within this fenced area. Connect the points in alphabetical order, and then join point D to Point A.
2. Find and label the midpoints of each segment of quadrilateral ABCD, showing all work. Label the midpoints of the segments as follows:

$\overline{AB}$  has midpoint Q,

$\overline{BC}$  has midpoint R,

$\overline{CD}$  has midpoint S,

$\overline{DA}$  has midpoint T.

3. Connect the four midpoints in alphabetical order to create a new quadrilateral QRST. This quadrilateral represents the fence surrounding the archeological dig site.
4. Quadrilateral ABCD was an ordinary quadrilateral, but QRST is a special one. Determine the special name for quadrilateral QRST, and justify your answer using coordinate geometry in two different ways.



## Teacher Notes

### Materials:

One graphing calculator per student

### Connections to Geometry

#### TEKS:

#### (b.3) Geometric structure.

The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

#### (d.2) Dimensionality and the geometry of location.

The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

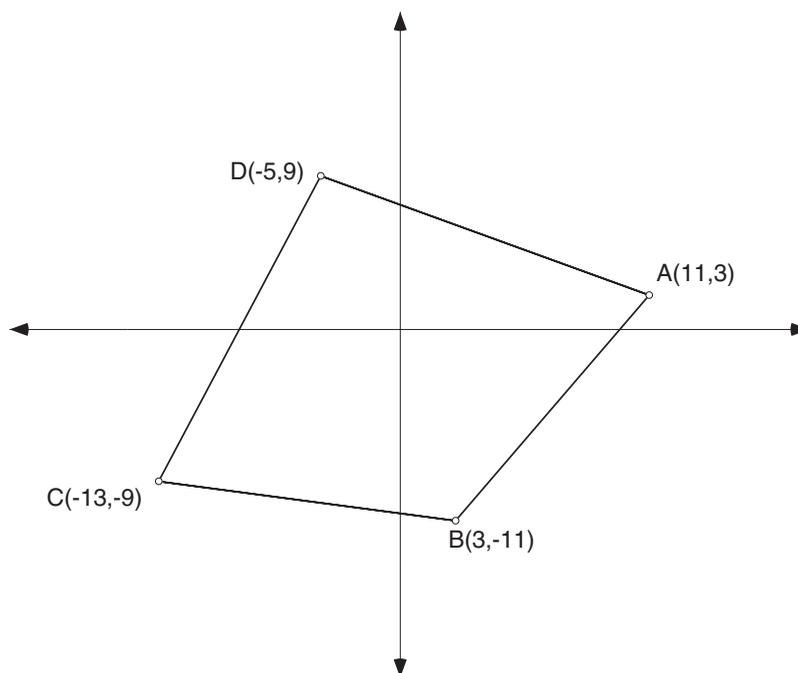
This problem addresses the same mathematical concepts as the problem *Wearable Art*. *Whitebeard's Treasure* gives the numerical coordinates. In *Wearable Art* the coordinates are given and the student must represent the situation using variable coordinates for the points. The teacher may choose to use one or both of these problems.

### Scaffolding Questions:

- What is the formula for finding the midpoint of a line segment?
- Which of the quadrilaterals are special quadrilaterals?
- What are the characteristics of each special quadrilateral?
- What characteristics does quadrilateral QRST appear to possess that matches one of the special quadrilaterals?
- How can you prove these special characteristics?

### Sample Solution:

Quadrilateral ABCD is graphed as shown. This is the outer fence.



To find the midpoint of each segment of quadrilateral ABCD, use the midpoint formula.

The midpoint of the segment with endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$ .

To find the midpoint of each segment, substitute the  $x$  and  $y$  values from the endpoints of the segment into the formula as follows:

Midpoint of  $\overline{AB}$  (Point Q)

$$\left(\frac{11 + 3}{2}, \frac{3 + (-11)}{2}\right) = (7, -4)$$

Midpoint of  $\overline{BC}$  (Point R)

$$\left(\frac{3 + (-13)}{2}, \frac{-11 + (-9)}{2}\right) = (-5, -10)$$

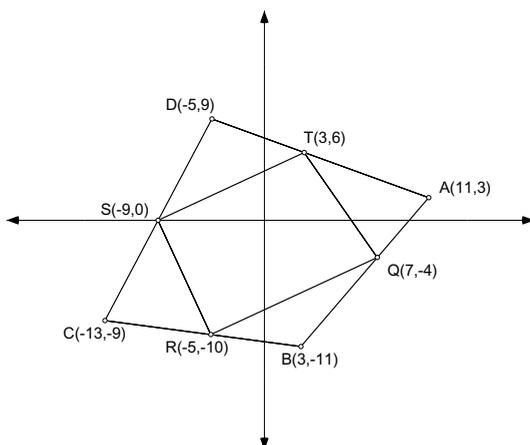
Midpoint of  $\overline{CD}$  (Point S)

$$\left(\frac{-13 + (-5)}{2}, \frac{-9 + 9}{2}\right) = (-9, 0)$$

Midpoint of  $\overline{DA}$  (Point T)

$$\left(\frac{-5 + 11}{2}, \frac{9 + 3}{2}\right) = (3, 6)$$

Graph the midpoints and connect them in alphabetical order to form a new quadrilateral QRST.



### Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

### Connections to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Midpoint Quadrilaterals

IV. Planar Figures: Investigating Quadrilateral Properties

### Teacher's Comment:

"I had the students get out their notes before they started. I gave them two minutes to read the problem and think of questions they wanted to ask the class or me. Some of the scaffolding questions I asked were:

What types of quadrilaterals are there?

How do you know what type of quadrilateral it is?

How do you know when two lines are parallel?

How do you know when two lines are perpendicular?"

Quadrilateral QRST would be the fence that encloses the archeologists' dig site.

Quadrilateral QRST appears to be a parallelogram because the opposite sides of the newly formed quadrilateral appear to be parallel. One way to prove that a quadrilateral is a parallelogram is to prove that both pairs of opposite sides are parallel. Lines that have the same slope are parallel lines.

Use the slope formula:

The slope of the line through points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .

The slope of  $\overleftrightarrow{RS}$  is  $\frac{0 - (-10)}{-9 - (-5)} = \frac{10}{-4} = -\frac{5}{2}$ .

The slope of  $\overleftrightarrow{QT}$  is  $\frac{6 - (-4)}{3 - 7} = \frac{10}{-4} = -\frac{5}{2}$ .

$\overleftrightarrow{RS} // \overleftrightarrow{QT}$  because both lines have the same slope.

The slope of  $\overleftrightarrow{ST}$  is  $\frac{6 - 0}{3 - (-9)} = \frac{6}{12} = \frac{1}{2}$ .

The slope of  $\overleftrightarrow{RQ}$  is  $\frac{-4 - (-10)}{7 - (-5)} = \frac{6}{12} = \frac{1}{2}$ .

$\overleftrightarrow{ST} // \overleftrightarrow{RQ}$  because both lines have the same slope.

Quadrilateral QRST is a parallelogram by definition because both pairs of opposite sides are parallel.

Another way to show that QRST is a parallelogram is to prove that both sides of opposite sides are congruent (using the distance formula to find the lengths of each side).



$$QR = \sqrt{(7 - (-5))^2 + (-4 - (-10))^2} = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}$$

$$ST = \sqrt{(-9 - 3)^2 + (0 - 6)^2} = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}$$

$$QR = ST = 6\sqrt{5}$$

$$RS = \sqrt{(-5 - (-9))^2 + (-10 - 0)^2} = \sqrt{16 + 100} = \sqrt{116} = 2\sqrt{29}$$

$$QT = \sqrt{(7 - 3)^2 + (-4 - 6)^2} = \sqrt{16 + 100} = \sqrt{116} = 2\sqrt{29}$$

$$RS = QT = 2\sqrt{29}$$

Both pairs of opposite sides of quadrilateral QRST are congruent, therefore it is a parallelogram.

### Extension Questions:

- Use algebra to find the point of intersection of the diagonals of quadrilateral QRST .

To find the point where the diagonals intersect, the equations of lines  $\overleftrightarrow{RT}$  and  $\overleftrightarrow{SQ}$  must be identified and then used to find the point of intersection.

$$\text{The slope of } \overleftrightarrow{RT} \text{ is } \frac{6 - (-10)}{3 - (-5)} = \frac{16}{8} = 2$$

$$\text{The slope of } \overleftrightarrow{SQ} \text{ is } \frac{0 - (-4)}{-9 - 7} = \frac{4}{-16} = -\frac{1}{4}$$

$$\text{The equation of } \overleftrightarrow{RT} \text{ is } (y - 6) = 2(x - 3) \text{ or } y = 2x$$

The equation of  $\overleftrightarrow{SQ}$  is

$$y = -\frac{1}{4}(x - (-9))$$

$$y = -\frac{1}{4}x - \frac{9}{4}$$



The point where the diagonals intersect can be found by using linear combination.

$$y = 2x$$

$$y = -\frac{1}{4}x - \frac{9}{4}$$

$$2x = -\frac{1}{4}x - \frac{9}{4}$$

$$8x = -1x - 9$$

$$9x = -9$$

$$x = -1$$

$$y = 2x = 2(-1) = -2$$

The point of intersection is  $(-1, -2)$ .

- Use coordinate geometry to prove the diagonals of quadrilateral QRST bisect each other.

The midpoint of QS is  $\overline{QS}$  is  $\left(\frac{7+(-9)}{2}, \frac{-4+0}{2}\right) = (-1, -2)$ .

The midpoint of RT is  $\overline{RT}$  is  $\left(\frac{-5+3}{2}, \frac{-10+6}{2}\right) = (-1, -2)$ .

The midpoints of the segment are the same point as the intersection point. The diagonals bisect each other.





## Student Work Sample

Field Test Teacher's Comment:

This was the first problem I had the students do using a poster. I enjoyed the poster and I feel most of the students did, too. I would like to have done this problem during the quadrilaterals section and will do so next year. One thing I did different this time was to have the students write on the back of their solution guide exactly what to put on their posters for the 3 criteria we emphasized.

Written on the back of one of her student's solution guide:

Shows an understanding of the relationships among elements.

- Statement showing how the elements are related.
- Can the history teacher understand your steps?

Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

- Drawing the pictures
- Make appropriate markings on the picture

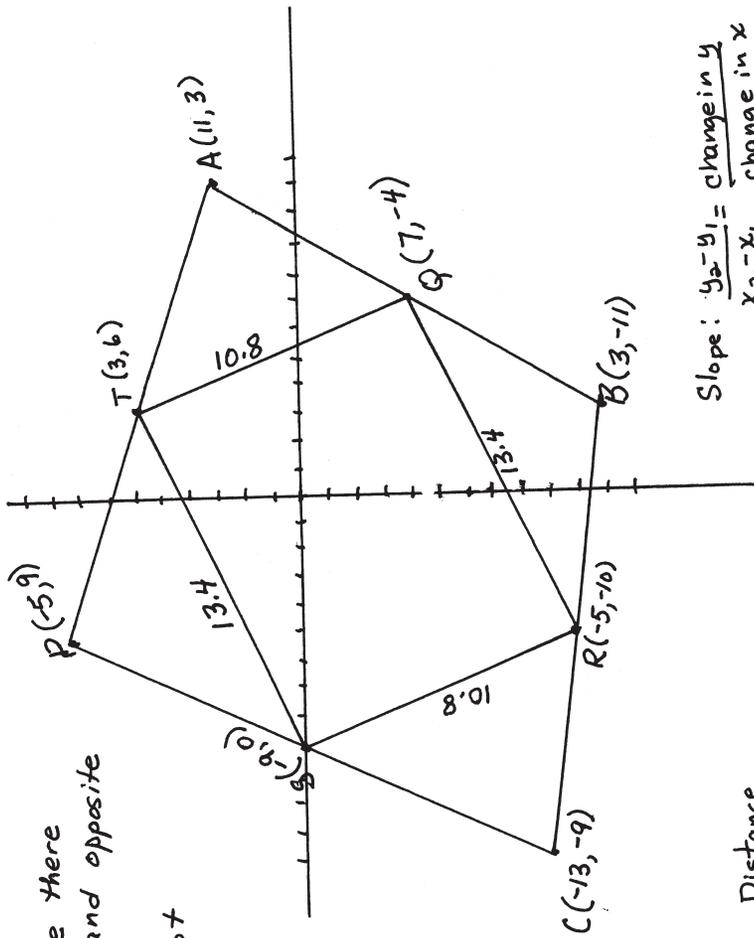
Communicates clear, detailed, and organized solution strategy.

- Step by step details that can be followed
- Don't plug in a number without showing why/how
- Must have justification
- Explaining your thinking!!!

A copy of the poster from this student's group appears on the next page.



A parallelogram - because there are two parallel sides and opposite sides are equal.  
 The diagonals are also not equal so it is not a rectangle or a square.



Midpoints on graph.

$$RS = \sqrt{(-9-5)^2 + (0-70)^2} = \sqrt{16+100} = \sqrt{116} \approx 10.8$$

$$TQ = \sqrt{(7-3)^2 + (-4-6)^2} = \sqrt{16+100} = \sqrt{116} \approx 10.8$$

Opposite sides are equal.  
 $RS = TQ$

$$QR = \sqrt{(-5-7)^2 + (-10-4)^2} = \sqrt{144+36} = \sqrt{180} \approx 13.4$$

$$ST = \sqrt{(-9-3)^2 + (0-6)^2} = \sqrt{144+36} = \sqrt{180} \approx 13.4$$

Opposite sides are equal.  
 $QR = ST$

Distance

$$RT = \sqrt{(-5-3)^2 + (-10-6)^2} = \sqrt{64+256} = \sqrt{320} \approx 17.9$$

$$SQ = \sqrt{(-9-7)^2 + (0-4)^2} = \sqrt{256+16} = \sqrt{272} \approx 16.5$$

The diagonals are not equal.

Slope:  $\frac{y_2-y_1}{x_2-x_1} = \frac{\text{change in } y}{\text{change in } x}$

QR:  $\frac{10-4}{-5-7} = \frac{-6}{-12} = \frac{1}{2}$  because the slopes are the same.  
 $ST: \frac{6-0}{3-9} = \frac{6}{-12} = \frac{1}{2}$  because the slopes are the same.  
 $RS: \frac{0-10}{-9-5} = \frac{-10}{-14} = \frac{5}{7}$  because the slopes are the same.  
 $QT: \frac{6-4}{3-7} = \frac{2}{-4} = \frac{1}{2}$  because the slopes are the same.

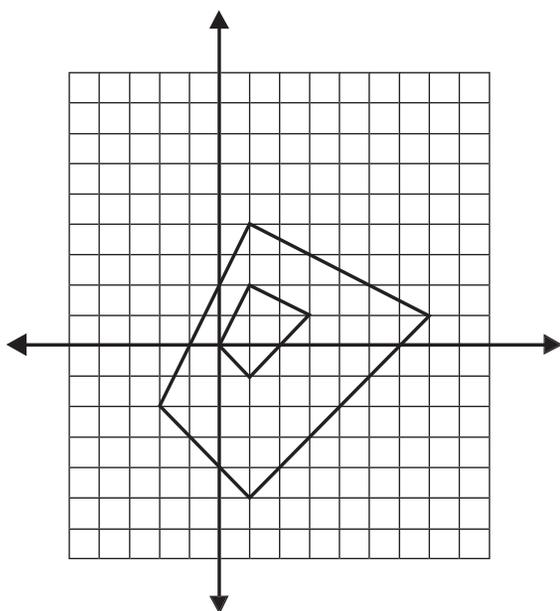




## Quadrilateral Quandary

The Seaside Hotel is going to install new landscaping on the hotel grounds. Plans that included the design and amounts of exact materials needed were purchased immediately after the landscaping project was approved. Due to recent water restrictions, however, the board of directors has decided to modify the project plans so that a smaller landscape bed will be constructed.

The hotel's landscape engineer has modified the size of the bed as shown. She must calculate the amount of the size reduction so that a detailed mathematical explanation can be presented to the board of directors, and the new costs can be calculated.



1. You have been hired to help the landscape engineer prepare her report and have been instructed to justify all of your calculations. You remember that the center of dilation is the intersection of 2 or more lines, each containing a point from the original figure and the corresponding point from the dilated figure. Use this definition and the figures above to calculate the center of dilation.
2. Find the scale factor for this dilation using the center of dilation and corresponding points of the quadrilaterals. Explain your process.
3. Verify that your scale factor is correct using the distance formula and corresponding parts of the figure.



## Teacher Notes

### Materials:

One graphing calculator per student

### Connections to Geometry

#### TEKS:

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

(f) **Similarity and the geometry of shape.** The student applies the concepts of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures; and

(2) uses ratios to solve problems involving similar figures.

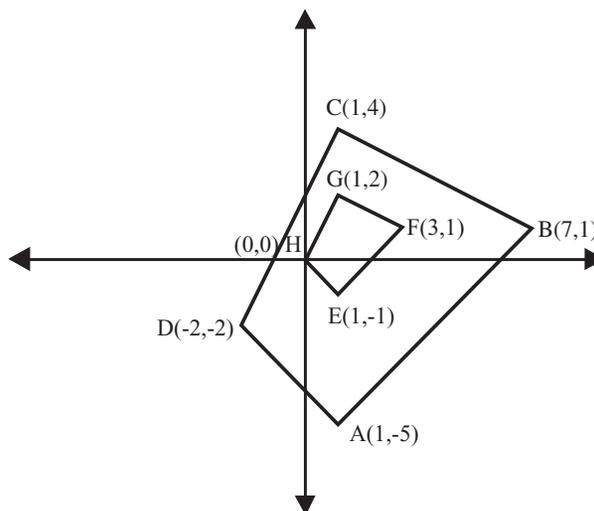
### Scaffolding Questions:

- What kind of transformation is illustrated?
- Why is it necessary to calculate the center of dilation?
- Is the transformation rigid? Why or why not?
- What does the scale factor tell you about the size of the new figure?
- How does the value of the scale factor relate to the center of dilation?
- How do the dimensions of the new figure relate to the scale factor?

### Sample Solutions:

1. The center of dilation is the intersection of 2 or more lines, each containing a point from the original figure and a corresponding point from the dilated figure. The strategy used will be to find the equations of the intersecting lines, enter them into the graphing calculator, and find the point of intersection. That point of intersection will be the center of dilation.

First, label the coordinates of the vertices of the quadrilaterals as follows:  $A(1,-5)$ ;  $B(7,1)$ ;  $C(1,4)$ ;  $D(-2,2)$  and  $E(1,-1)$ ;  $F(3,1)$ ;  $G(1,2)$ ;  $H(0,0)$ .



Using the point-slope form, find the equation of the line that contains points A(1,-5) and E(1,-1) and an equation of the line containing D(-2,-2) and H(0,0).

The calculation for the slope of line  $\overleftrightarrow{DH}$  is  $1: \frac{-2 - 0}{-2 - 0} = 1$ .

The equation of line  $\overleftrightarrow{DH}$  is found using point-slope form with H:

$$y - 0 = 1(x - 0)$$

$$y = x$$

The line  $\overleftrightarrow{AE}$  is the vertical line  $x = 1$ . This line also passes through points C and G.

To find the intersection point of the line  $y = x$  and  $x = 1$  substitute 1 for  $x$ .

$$y = 1.$$

The intersection point of the two lines is (1,1). The vertical line through points B and F is  $y = 1$ . This line also passes through the point (1,1).

The center of dilation is (1,1). Label that point X on the diagram.

2. The center of dilation and the points A(1,-5) and E(1,-1) can be used in determining the scale factor. The ratio of the distances between the center of dilation and the corresponding points on the quadrilateral determines the scale factor. To calculate the scale factor, solve the equation:

$$\text{scale factor} = \frac{XE}{XA}$$

The distance formula is used to find the length of the segments.

$$XE = \sqrt{(1-1)^2 + (1-(-1))^2} = \sqrt{0+2^2} = 2$$

### Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts of measurement and similarity.

### Connections to High School Geometry: Supporting the TEKS and TAKS Institute:

II. Transformations: Up to Scale

### Teacher's Comment:

"Using the assessments has helped my students retain the concepts used. Before using the assessments my students would work problems and move on to more concepts losing retention. Now my students have real-world situations they relate the concepts with and retain the concepts used."



The length of XA is calculated as follows:

$$XA = \sqrt{(1-1)^2 + (1-(-5))^2} = \sqrt{0+6^2} = 6 .$$

$$\text{scale factor} = \frac{XE}{XA} = \frac{2}{6} = \frac{1}{3}$$

This scale factor is true for each pair of corresponding sides. Corresponding sides are  $\overline{DA}$  and  $\overline{HE}$ ,  $\overline{DC}$ , and  $\overline{HG}$ ,  $\overline{CB}$ , and  $\overline{GF}$ ,  $\overline{AB}$  and  $\overline{EF}$ .

$$DA = \sqrt{(-2-1)^2 + (-2-(-5))^2} = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18}$$

$$HE = \sqrt{(0-1)^2 + (0-(-1))^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

$$\text{The ratio of the lengths of sides is } \frac{HE}{DA} = \frac{\sqrt{2}}{\sqrt{18}} = \sqrt{\frac{2}{18}} = \sqrt{\frac{1}{9}} = \frac{1}{3} .$$

$$DC = \sqrt{(-2-1)^2 + (-2-4)^2} = \sqrt{(-3)^2 + (-6)^2} = \sqrt{45}$$

$$HG = \sqrt{(0-1)^2 + (0-2)^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$$

$$\text{The ratio of the lengths of sides is } \frac{HG}{DC} = \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}} = \sqrt{\frac{1}{9}} = \frac{1}{3} .$$

$$CB = \sqrt{(-1-7)^2 + (4-1)^2} = \sqrt{(-6)^2 + (3)^2} = \sqrt{45}$$

$$GF = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$$

$$\text{The ratio of the lengths of sides is } \frac{GF}{CB} = \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}} = \sqrt{\frac{1}{9}} = \frac{1}{3} .$$



$$AB = \sqrt{(1-7)^2 + (-5-1)^2} = \sqrt{(-6)^2 + (-6)^2} = \sqrt{72}$$

$$EF = \sqrt{(1-3)^2 + (-1-1)^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8}$$

The ratio of the lengths of sides is  $\frac{EF}{AB} = \frac{\sqrt{8}}{\sqrt{72}} = \sqrt{\frac{8}{72}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$ .

### Extension Question:

- Landscape timbers are sold in 8-foot lengths. Suppose the original project was designed for 850 feet of landscaping timber, at a cost of \$1.39 per timber. Calculate the minimum amount of materials that will need to be ordered for the new bed and the cost of materials before tax. Explain how you arrived at your answers.

The scale factor for the new project is  $\frac{1}{3}$  of the original design plan. If the original project

was designed for 850 feet of landscaping timber, the new project will require  $\frac{1}{3}$  as much.

$$\frac{1}{3} (850 \text{ feet}) = 283.3 \text{ feet.}$$

If the timbers are 8 feet long, divide 283.3 by 8. The result is 35.4125 timbers. Timbers are sold in 8-foot lengths; therefore, a minimum of 36 will be needed.

The cost of the project per landscape bed is calculated by multiplying \$1.39 by 36 timbers.

$$1.39 (36) = 50.04$$

The cost for each landscape bed will be \$50.04.



## Student Work Sample

The problem analysis on the next page was created by a group of students. The students were given the problem, told to work silently on the problem for at least three minutes, and asked to create a group poster of their analysis of the problem.

The work is a good example of:

- Shows a relationship among the elements.

*They demonstrated their understanding of which points corresponded to one another and used the appropriated labeling  $(A, A')$ .*

- Uses appropriate terminology and notation.

*The students used correct language for transformations (image, dilation, scale factor). They wrote and used the formulas for distance between two points.*

The students neglected to give reasons for the steps and to explain why the point  $(1,1)$  is the center of dilation.



$$\frac{A'B'}{AB} =$$

$$\frac{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}$$

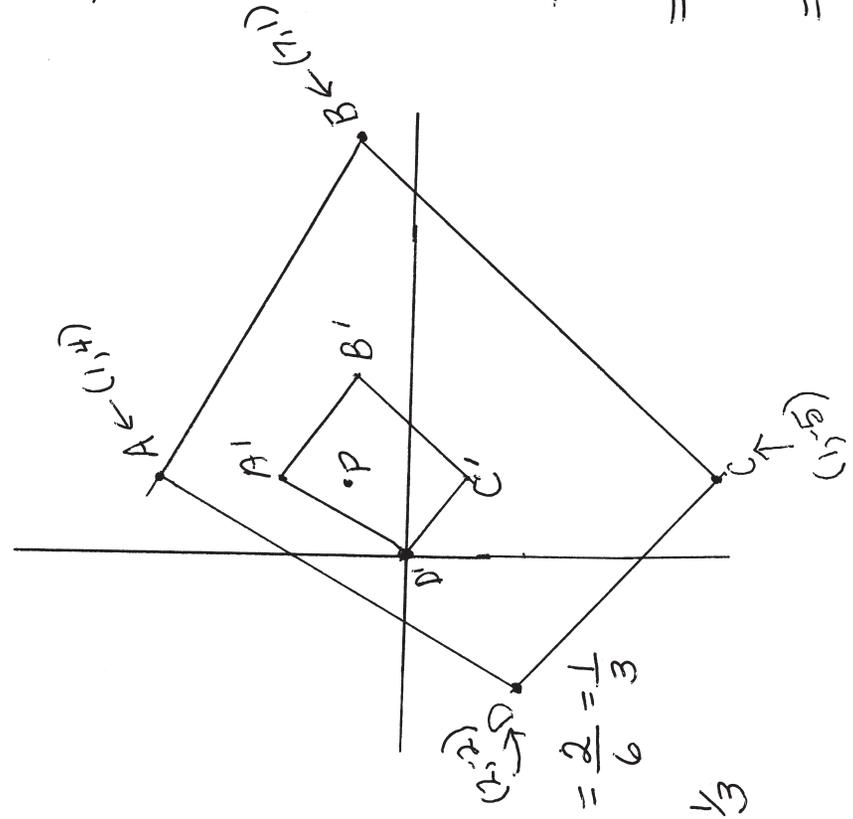
$$= \frac{\sqrt{(2-1)^2 + (1-3)^2}}{\sqrt{(4-1)^2 + (1-7)^2}}$$

$$= \frac{\sqrt{1^2 + (-2)^2}}{\sqrt{3^2 + (-6)^2}}$$

$$= \frac{\sqrt{5}}{\sqrt{45}} = \sqrt{\frac{5}{45}}$$

$$= \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Quadrilateral Quandy



Original Image  
 A (1, 4) B (7, 1)  
 C (1, -5) D (-2, 2)

Revised Image  
 A' (1, 2) B' (3, 1)  
 C' (1, -1) D' (0, 0)

Center of Dilation  
 P (1, 1)

$$\frac{PB'}{PB} = SF = \frac{2}{6} = \frac{1}{3}$$

Scale Factor =  $\frac{1}{3}$

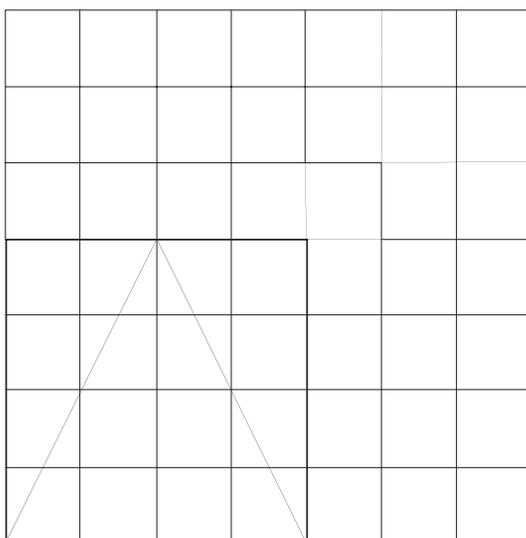




## Quite a Quilt

Ed has decided to enter a national geometry competition. The contest rules state that individuals must submit plans for a 4-inch quilt square design that will produce an octagonal region within the square whose area is no larger than 3 square inches. The entry must clearly explain and illustrate how the design is to be created, and it must prove that the inner octagonal area is within the contest guidelines.

Ed has designed his square so that the midpoint of each side of the square is joined to its two opposite vertices. The figure below shows Ed's beginning sketch.



1. Complete the quilt design, and label the figure representing Ed's octagon.
2. Determine the coordinates of the vertices of the octagon. Justify your answer.
3. Explain how you know whether or not the octagon is a regular octagon.
4. Does his design meet the contest's area criteria as outlined above? Justify your solution using coordinate geometry, and show all your work.



## Teacher Notes

### Materials:

One graphing calculator per student

Dynamic geometry computer program (optional)

### Connections to Geometry

#### TEKS:

#### (b.3) Geometric structure.

The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

(B) constructs and justifies statements about geometric figures and their properties;

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

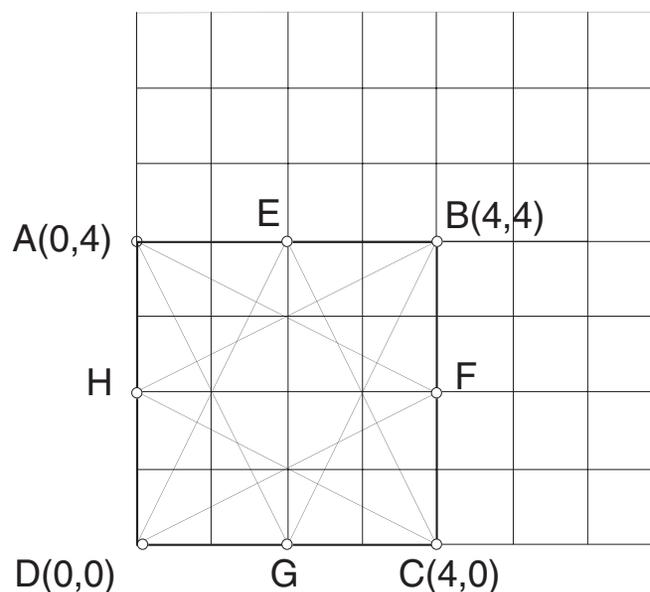
(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

### Scaffolding Questions:

- Describe how to determine the coordinates of each vertex of the square.
- Determine the coordinates of the midpoint of each side of the square.
- Determine the coordinates of the center of the square.
- How can you mathematically determine these points? (Note: Students may investigate this problem using a dynamic geometry computer program.)
- If you connect the vertices of the octagon to the center of the square, what do you know about the 8 triangles that are formed?
- What two properties must be satisfied if the octagon is to be a regular octagon?
- How could you determine the area of one-fourth of the octagon?

### Sample Solutions:

1.



The length of the side of the square is 4 inches. The coordinates of the midpoints will be:



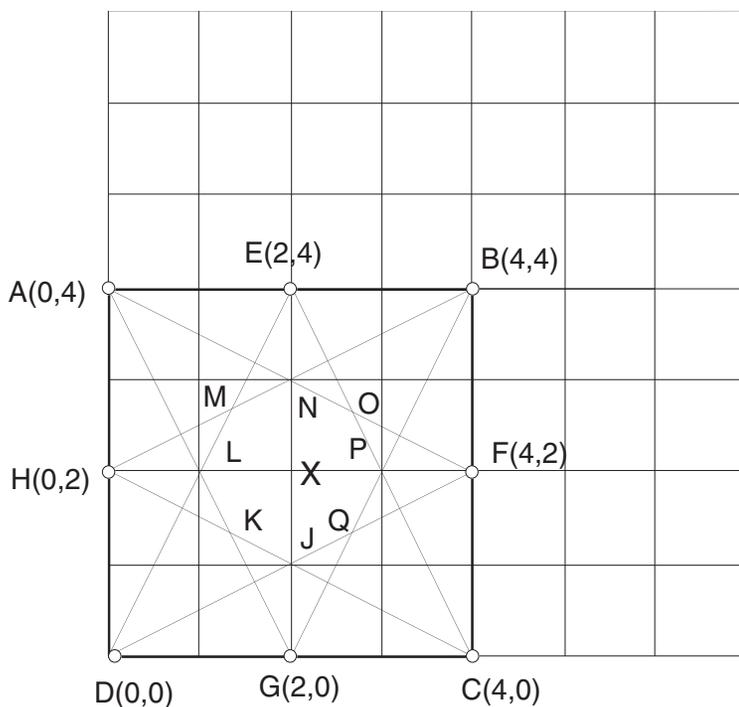
$$G = \left( \frac{0+4}{2}, \frac{0+0}{2} \right) = (2,0)$$

$$F = \left( \frac{4+4}{2}, \frac{0+4}{2} \right) = (4,2)$$

$$E = \left( \frac{0+4}{2}, \frac{4+4}{2} \right) = (2,4)$$

$$H = \left( \frac{0+0}{2}, \frac{0+4}{2} \right) = (0,2)$$

The center of the square will be at (2,2). Label the center point X. The center of the square is also the center of the octagon. Label the vertices of the octagon J, K, L, M, N, O, P, Q.



2. To determine the vertices, the equations of each of the lines can be determined. The slopes are determined using the graph.

(C) develops and uses formulas including distance and midpoint.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures.

### Texas Assessment of Knowledge and Skills:

Objective 2: The student will demonstrate an understanding of properties and attributes of functions.

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

### Connections to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Investigating Quadrilateral Properties

IV. Planar Figures: Student Activity



$\overleftrightarrow{AG}$  :

Slope: -2    Point A(0,4)

$$y - 4 = -2(x - 0)$$

$$y = -2x + 4$$

$\overleftrightarrow{AF}$ :

Slope:  $-\frac{1}{2}$     Point A(0,4)

$$y = -\frac{1}{2}x + 4$$

$\overleftrightarrow{BG}$  :

Slope: 2    Point B(4,4)

$$y - 4 = 2(x - 4)$$

$$y = 2x - 4$$

$\overleftrightarrow{BH}$ :

Slope:  $\frac{1}{2}$     Point B(4,4)

$$y - 4 = \frac{1}{2}(x - 4)$$

$$y = \frac{1}{2}x + 2$$

$\overleftrightarrow{CH}$  :

Slope:  $-\frac{1}{2}$     Point H(2,0)

$$y - 2 = -\frac{1}{2}(x - 0)$$

$$y = -\frac{1}{2}x + 2$$



$\overleftrightarrow{CE}$ :

Slope: -2    Point E(2,4)

$$y - 4 = -2(x - 2)$$

$$y = -2x + 8$$

$\overleftrightarrow{DF}$ :

Slope:  $\frac{1}{2}$     Point D(0,0)

$$y - 0 = \frac{1}{2}(x - 0)$$

$$y = \frac{1}{2}x$$

$\overleftrightarrow{DE}$ :

Slope: 2    Point D(0,0)

$$y - 0 = 2(x - 0)$$

$$y = 2x$$

Using linear combination to solve the systems, the values of  $x$  and  $y$  can be found to determine the intersection of lines  $\overleftrightarrow{HC}$  and  $\overleftrightarrow{DF}$ . The point of intersection will be labeled point J as in the diagram above.

$$y = -\frac{1}{2}x + 2$$

$$y = \frac{1}{2}x$$

Adding these 2 equations together produces the following:

$$2y = 2, \text{ therefore } y = 1.$$

Substitute  $y = 1$  into either equation to solve for  $x$ :

$$1 = \frac{1}{2}x$$

$$2 = x$$

Therefore the point of intersection is J(2,1).



The intersection of line  $\overleftrightarrow{AG}$  and  $\overleftrightarrow{HC}$  is the point K.

Using linear combination on the equation for lines  $\overleftrightarrow{AG}$  and  $\overleftrightarrow{HC}$  will allow the coordinates of point K to be found.

$$y = -2x + 4$$

$$y = -\frac{1}{2}x + 2$$

$$0 = -\frac{3}{2}x + 2$$

$$-2 = -\frac{3}{2}x$$

$$\frac{4}{3} = x$$

Substitute  $x = \frac{4}{3}$  into either equation to find the coordinate of y.

$$y = -2\left(\frac{4}{3}\right) + 4$$

$$y = \frac{4}{3}$$

The coordinates of K are  $\left(\frac{4}{3}, \frac{4}{3}\right)$ .

Point L is the intersection of lines  $\overleftrightarrow{AG}$  and  $\overleftrightarrow{DE}$ .

$$\overleftrightarrow{AG} : y = -2x + 4$$

$$\overleftrightarrow{DE} : y = 2x$$

$$2x = -2x + 4$$

$$4x = 4$$

$$x = 1$$

$$y = 2x = 2(1) = 2$$

The coordinates of L are (1,2).



Point M is the intersection of lines  $\overleftrightarrow{DE}$  and  $\overleftrightarrow{BH}$ .

$$\overleftrightarrow{DE}: y = 2x$$

$$\overleftrightarrow{BH}: y = \frac{1}{2}x + 2$$

$$2x = \frac{1}{2}x + 2$$

$$4x = x + 4$$

$$x = \frac{4}{3}$$

$$y = 2x = 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

The coordinates of point M are  $\left(\frac{4}{3}, \frac{8}{3}\right)$ .

Point N is the intersection of lines  $\overleftrightarrow{AF}$  and  $\overleftrightarrow{BH}$ .

$$\overleftrightarrow{AF}: y = -\frac{1}{2}x + 4$$

$$\overleftrightarrow{BH}: y = \frac{1}{2}x + 2$$

$$-\frac{1}{2}x + 4 = \frac{1}{2}x + 2$$

$$-1x = -2$$

$$x = 2$$

$$y = \frac{1}{2}x + 2 = 1 + 2 = 3$$

The coordinates of N are (2,3).



Point O is the intersection of lines  $\overleftrightarrow{AF}$  and  $\overleftrightarrow{CE}$ .

$$\overleftrightarrow{AF}: y = -\frac{1}{2}x + 4$$

$$\overleftrightarrow{CE}: y = -2x + 8$$

$$-\frac{1}{2}x + 4 = -2x + 8$$

$$-1x + 8 = -4x + 16$$

$$x = \frac{8}{3}$$

$$y = -2x + 8 = -2\left(\frac{8}{3}\right) + 8 = \frac{8}{3}$$

The coordinates of O are  $\left(\frac{8}{3}, \frac{8}{3}\right)$ .

Point P is the intersection of lines  $\overleftrightarrow{CE}$  and  $\overleftrightarrow{BG}$ .

$$\overleftrightarrow{CE} : y = -2x + 8$$

$$\overleftrightarrow{BG} : y = 2x - 4$$

$$2x - 4 = -2x + 8$$

$$4x = 12$$

$$x = 3$$

$$y = 2x - 4 = 2(3) - 4 = 2$$

The coordinates of P are (3,2).



Point Q is the intersection of lines  $\overleftrightarrow{DF}$  and  $\overleftrightarrow{BG}$ .

$$\overleftrightarrow{DF}: y = \frac{1}{2}x$$

$$\overleftrightarrow{BG}: y = 2x - 4$$

$$\frac{1}{2}x = 2x - 4$$

$$x = 4x - 8$$

$$-3x = -8$$

$$x = \frac{8}{3}$$

$$y = \frac{1}{2}x = \frac{1}{2}\left(\frac{8}{3}\right) = \frac{4}{3}$$

The coordinates of Q are  $\left(\frac{8}{3}, \frac{4}{3}\right)$ .

3. A regular octagon has 8 congruent sides and 8 congruent angles.

It can be demonstrated that the octagon is not a regular octagon by determining the distances of LK, KJ, XK, XJ, and XL.

$$LK = \sqrt{\left(1 - \frac{4}{3}\right)^2 + \left(2 - \frac{4}{3}\right)^2} = \sqrt{\left(-\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{5}{9}}$$

$$KJ = \sqrt{\left(2 - \frac{4}{3}\right)^2 + \left(1 - \frac{4}{3}\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2} = \sqrt{\frac{5}{9}}$$

$$XK = \sqrt{\left(2 - \frac{4}{3}\right)^2 + \left(2 - \frac{4}{3}\right)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{8}{9}}$$

$$XL = \sqrt{(2-1)^2 + (2-2)^2} = 1$$

$$XJ = \sqrt{(2-2)^2 + (2-1)^2} = 1$$



$\overline{XK}$  is not equal in length to  $\overline{XL}$  or  $\overline{XJ}$ .

Triangles  $XKJ$  and  $XLK$  are congruent triangles, but they are not isosceles triangles because the sides are of three different lengths.

$$XL = XJ = 1$$

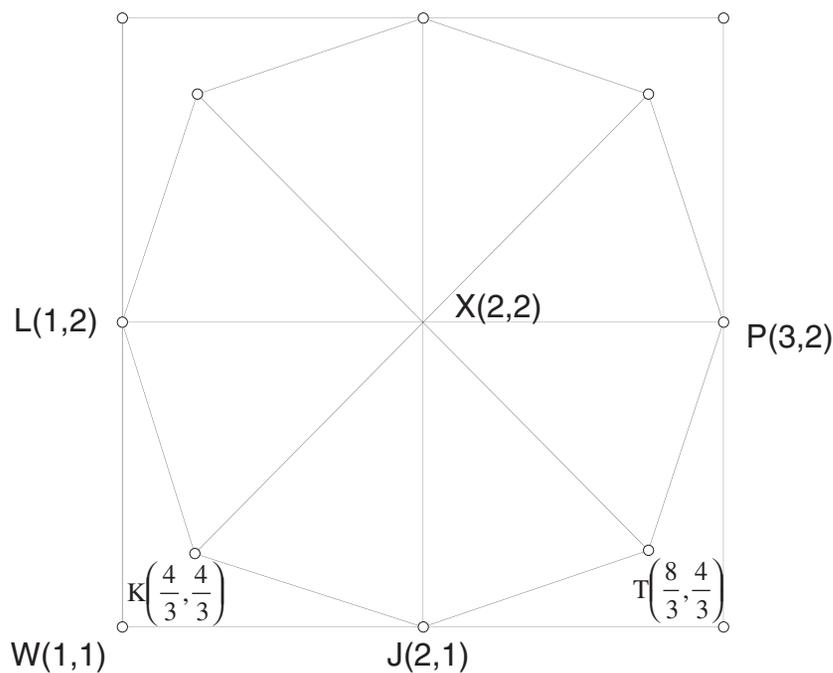
$$LK = KJ = \sqrt{\frac{5}{9}}$$

The length of the common side  $\overline{XK}$  is  $\sqrt{\frac{8}{9}}$ .

Thus, the octagon is not a regular octagon.

4. It is still possible to find the area of the octagon as it can be divided into 4 parts that are congruent. Each part of the bottom half is composed of the two congruent triangles.

For example, one-fourth of the octagon,  $LKJX$ , is composed of the two congruent triangles  $LXK$  and  $JXK$ .



The area of square LXJW is 1 square unit.

Draw  $\overline{WK}$ . Triangle WJK has a base,  $\overline{WJ}$ , that measures one unit.

The height from K to  $\overline{WJ}$  is  $\frac{4}{3} - 1 = \frac{1}{3}$ .

The area of the triangle is  $\frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$ .

Similarly, the area of triangle LWK is  $\frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$ .

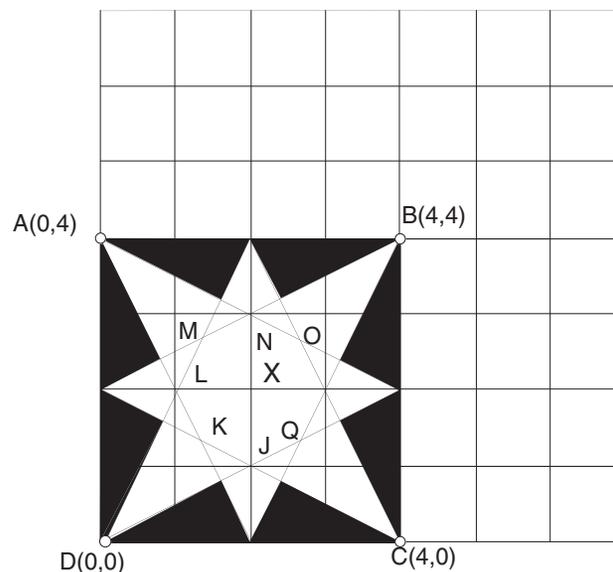
The area of LKJX is the area of LXJW minus the area of the two triangles LWK and WKJ, or  $1 - \frac{1}{6} - \frac{1}{6} = \frac{2}{3}$ .

The area of the octagon is four times the area of LKXJ, or  $4\left(\frac{2}{3}\right) = \frac{8}{3} = 2\frac{2}{3}$  square inches.

Ed's design meets the guidelines because the total area is no larger than 3 square inches.

### Extension Question:

- Determine the area of the white star.



Call the intersection of the lines  $\overleftrightarrow{DF}$  and  $\overleftrightarrow{AG}$  point  $W$ .

$$\overleftrightarrow{DF}: y = \frac{1}{2}x$$

$$\overleftrightarrow{AG}: y = -2x + 4$$

$$\frac{1}{2}x = -2x + 4$$

$$x = -4x + 4$$

$$x = \frac{8}{5}$$

$$y = \frac{1}{2}x = \frac{1}{2}\left(\frac{8}{5}\right) = \frac{4}{5}$$

Triangle  $DWG$  has a base of 2 inches and a height  $\frac{4}{5}$  inches.

The area of triangle  $DWG$  is  $\frac{1}{2} \cdot 2 \cdot \frac{4}{5} = \frac{4}{5}$  square inches.

The area of the star is the area of the large square minus 8 of the triangles that are congruent to triangle  $DWG$ .

$$16 - 8\left(\frac{4}{5}\right) = 16 - \frac{32}{5} = 16 - 6\frac{2}{5} = 9\frac{3}{5} \text{ square inches.}$$





## Student Work Sample

The student work displayed on the next page was completed using geometry computer software.

The criteria of the Geometry Solution Guide that are exemplified in this example are the following:

- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

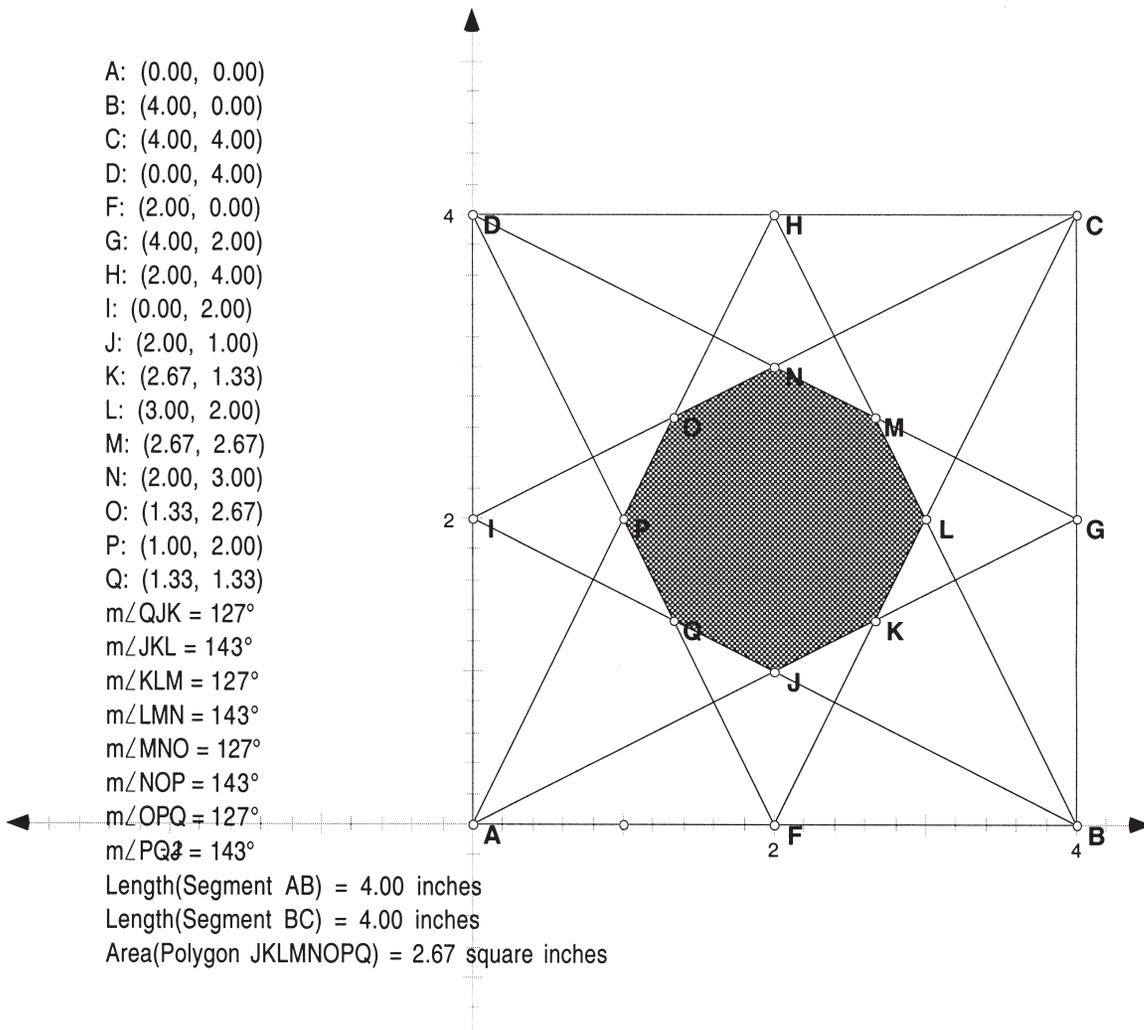
*The student has clearly and correctly labeled points. The measurements have been taken that will allow the student to answer the questions about the quilt design.*

- Uses appropriate tools.

*The problem did not require that the measurements or calculations be done using algebraic methods. The use of the geometric software demonstrates the solution, but note that it does constitute a geometric proof.*



These are my coordinates for each design point. From each point to another point the points are listed. The coordinates are accurate on the graph.



The octagon is not a regular octagon. The angles are not the same.

This quilt matches the guidelines to enter the contest. The quilt is a 4 in quilt and the shape is not larger than 3 square inches.





## Wearable Art

Lorraine's graphics arts class has been assigned a t-shirt design project. Each student is to create a design by drawing any quadrilateral, connecting the midpoints of the sides to form another quadrilateral, and coloring the regions. Lorraine claims that everyone's inner quadrilateral will be a parallelogram. Use coordinate geometry to determine if she is correct. Show all of your work, and explain your reasoning.



## Teacher Notes

### Materials:

One graphing calculator per student

Computer drawing utility (optional)

### Connections to Geometry TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

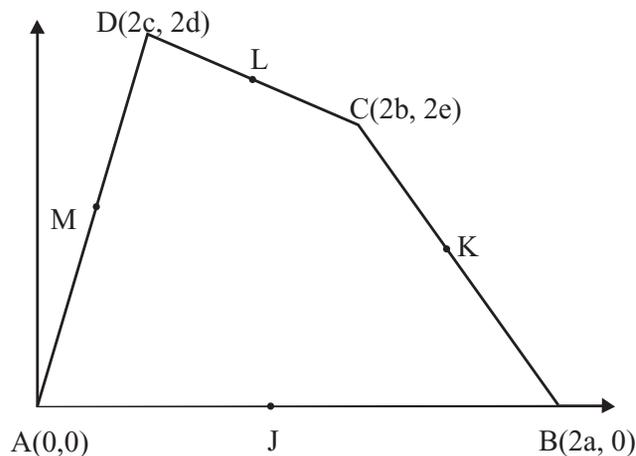
This problem addresses the same mathematical concepts as the problem *Whitebeard's Treasure* that gives the numerical coordinates. In *Wearable Art* the coordinates are given and the student must represent the situation using variable coordinates for the points. The teacher may choose to use one or both of these problems.

### Scaffolding Questions:

- How will using randomly selected coordinates for the vertices of the quadrilateral help prove Lorraine's claim?
- How could you place your quadrilateral on the coordinate plane so that the coordinates of your vertices will be "easy" to work with?
- How can you select coordinate values that will be "friendly" when figuring midpoints?

### Sample Solution:

Draw a quadrilateral on a coordinate plane. Locate one vertex at the origin and one side on the x-axis. Remembering that midpoints will be needed, select coordinates that are multiples of two. Label the vertices of the quadrilateral A, B, C, and D and the midpoints J, K, L, and M.



Find the coordinates of the midpoints as follows:

The midpoint of  $\overline{AB}$ , J, is the point

$$\left(\frac{0+2a}{2}, \frac{0+0}{2}\right) = (a, 0)$$

The midpoint of  $\overline{BC}$ , K, is the point

$$\left(\frac{2a+2b}{2}, \frac{0+2e}{2}\right) = (a+b, e)$$

The midpoint of  $\overline{CD}$ , L, is the point

$$\left(\frac{2b+2c}{2}, \frac{2d+2e}{2}\right) = (b+c, d+e)$$

The midpoint of  $\overline{DA}$ , M, is the point

$$\left(\frac{2c+0}{2}, \frac{2d+0}{2}\right) = (c, d)$$

Draw line segments  $\overline{JK}$ ,  $\overline{KL}$ ,  $\overline{LM}$ , and  $\overline{MJ}$ . These segments form the sides of the inner quadrilateral.

If the inner quadrilateral is a parallelogram, then opposite sides must be parallel. Using the slope formula, it can be shown that the lines are parallel because parallel lines have the same slope.  $\overline{JK}$  must be parallel to  $\overline{LM}$ , and  $\overline{KL}$  must be parallel to  $\overline{MJ}$  if the figure is a parallelogram.

$$\text{The slope of } \overline{JK} = \frac{e-0}{(a+b)-a} = \frac{e}{b}$$

$$\text{The slope of } \overline{LM} = \frac{(d+e)-d}{(b+c)-c} = \frac{e}{b}$$

Segments  $\overline{JK}$  and  $\overline{LM}$  both have the same slope, therefore they are parallel.

$$\text{The slope of } \overline{MJ} = \frac{d-0}{c-a} = \frac{d}{c-a}$$

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

(C) develops and uses formulas including distance and midpoint.

#### Texas Assessment of Knowledge and Skills:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

#### Connections to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Midpoint Quadrilaterals
- IV. Planar Figures: Investigating Quadrilateral Properties



The slope of  $\overline{KL}$  =  $\frac{(d+e)-e}{(b+c)-(a+b)} = \frac{d}{c-a}$ .

Segments  $\overline{MJ}$  and  $\overline{KL}$  both have the same slope, therefore they are parallel.

By definition quadrilateral JKLM is a parallelogram (both pairs of opposite sides are parallel). Lorraine's claim will be true for any quadrilateral that joins the midpoints of the sides of any convex quadrilateral.

### Extension Questions:

- Mark claims that the inner quadrilateral created by joining the midpoints of the sides will be a rhombus. Prove or disprove his conjecture. Be sure to show all of your work.

*A drawing utility can be used to test the conjecture. A rhombus has perpendicular diagonals and 4 congruent sides. In this case  $\overline{MK}$  is not perpendicular to  $\overline{JL}$  because the slopes of the diagonals are not opposite reciprocals of one another.*

The slope of  $\overline{MK}$  is  $\frac{e-d}{a+b-c}$ .

The slope of  $\overline{JL}$  is  $\frac{d+e-0}{b+c-a}$ .

$$-\frac{1}{\frac{d+e-0}{b+c-a}} = -\frac{b+c-a}{d+e-0} \neq \frac{e-d}{a+b-c}$$

*The slopes of the two lines are not opposite reciprocals. Therefore, the quadrilateral is not a rhombus.*

*There is no need to check the 4 congruent sides because the first condition of a rhombus was not met.*

- Can the quadrilateral formed by joining the midpoints be a rectangle? A square? Write your conjectures and test them using a drawing program. Justify your answers.

*If the quadrilateral is a rectangle, then  $\overline{MJ}$  must be perpendicular to  $\overline{JK}$ . That would mean that the slope of  $\overline{MJ}$  is equal to the opposite reciprocal of the slope of  $\overline{JK}$ .*



$$\frac{d}{c-a} = \frac{-1}{e}$$

$$\frac{d}{c-a} = \frac{-b}{e}$$

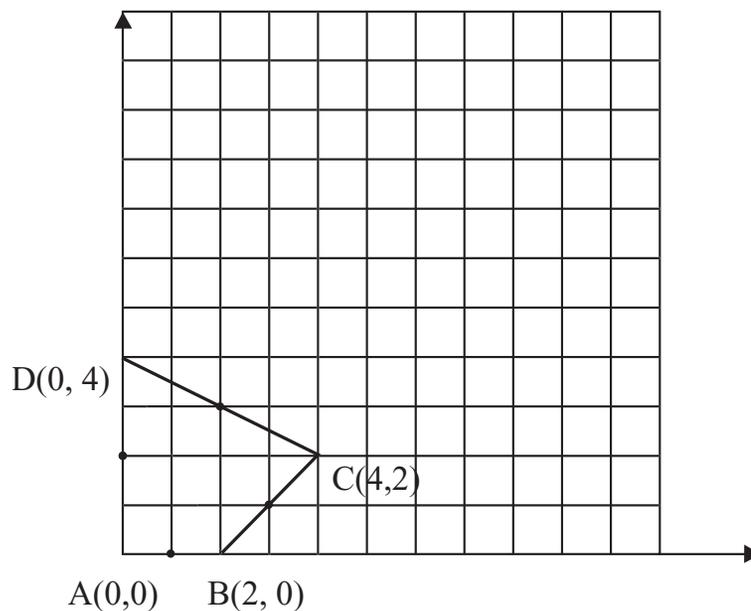
$$de = -bc + ab$$

Choose any values for  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $e$  that make this a true statement. One set of values is  $d = 2$ ,  $e = 1$ ,  $a = 1$ ,  $c = 0$ ,  $b = 2$ .

$$de = -bc + ab$$

$$2(1) = -2(0) + 1(2)$$

This choice results in a rectangle.



The midpoints are  $J(1,0)$ ;  $K(3,1)$ ;  $L(2,3)$ ; and  $M(0,2)$ .

Each side of the rectangle is the hypotenuse of a right triangle with legs measuring 1 unit and 2 units. Therefore, by the Pythagorean Theorem, the length of the hypotenuse is

$$\sqrt{1^2 + 2^2} = \sqrt{5}.$$

The resulting figure,  $JKLM$ , is also a square because it is a rectangle with four congruent sides.





## Sea Quest

The Coastal Marine Institute is going to gather data in Galveston Bay. A new computer device will be submerged just outside the ship channel fairway. In order to get the most accurate readings, the device must be located at a point such that the distance from the bay floor to the device is equal to one-third of the distance between the computer device and the top of the antenna. The depth of the bay at the selected location is 18.5 feet. The height of the antenna will be 21.5 feet above the water at mean tide.

Draw a diagram of the situation, and determine the depth at which the new computer device will be located at mean tide. Justify your solution.



## Teacher Notes

### Materials:

One graphing calculator per student.

### Connections to Geometry

#### TEKS:

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(C) develops and uses formulas including distance and midpoint.

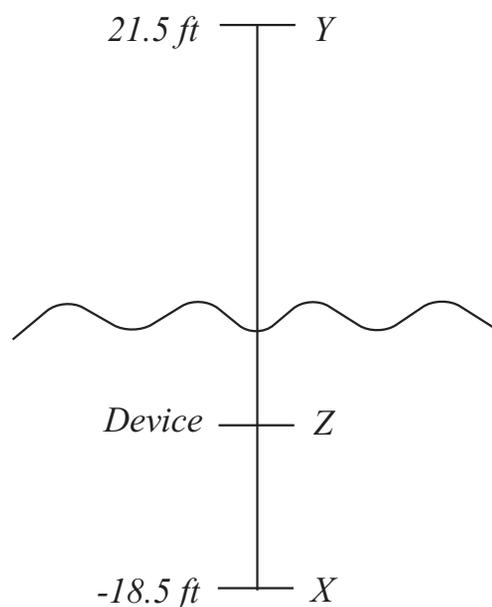
### Scaffolding Questions:

- If you assign numerical values to the objects, where will the zero value be?
- What type of diagram would best help illustrate this situation?
- How can you find the total distance between the floor of the bay and the top of the antenna?

### Sample Solution:

If the depth of the water is 18.5 feet, and the height of the antenna is 21.5 feet, both positive and negative numbers will need to be used. The depth will correspond to  $-18.5$  feet because the distance is below the water. The height of the antenna will correspond to a positive 21.5 feet.

A vertical number line can be used to illustrate this situation.



The distance between X and Z is equal to  $\frac{1}{3}$  the distance between Z and Y. Using segment addition, it can be shown that  $XZ + ZY = XY$ , and we are told that  $XZ = \frac{1}{3}(ZY)$ . The distance between X and Y is  $|-18.5 - 21.5|$  or 40 feet.



To find the values of XZ and ZY, the system of 2 equations may be solved by substitution.

$$XZ + ZY = XY$$

$$XZ = \frac{1}{3}(ZY)$$

$$\frac{1}{3}(ZY) + ZY = 40$$

$$\frac{4}{3}(ZY) = 40$$

$$\frac{3}{4} \cdot \frac{4}{3}(ZY) = \frac{3}{4} \cdot 40$$

$ZY = 30$  and  $XZ = 10$  because  $XZ + ZY = 40$ .

The depth at point Z must be 10 feet from the floor of Galveston Bay. If the depth at the selected location is 18.5 ft, the computer device will be located at a depth of 8.5 feet. ( $-18.5 + 10 = -8.5$ ).

**Texas Assessment of Knowledge and Skills:**

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

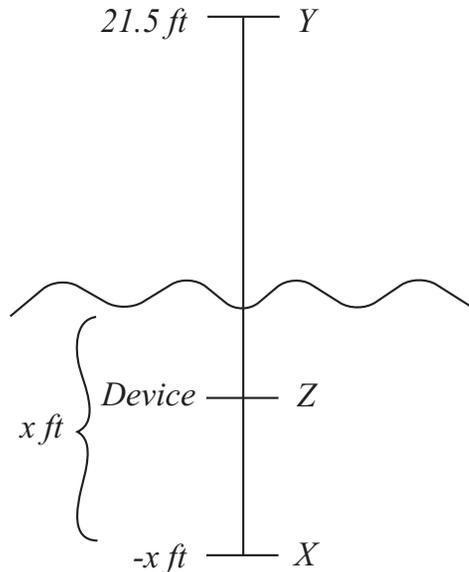
III. Triangles: Pythagorean Theorem



### Extension Questions:

- How deep would the bay have to be in order to locate the computer device at half the depth of the bay? Assume the device still gives the most accurate readings when the distance from the bay floor to the device is one-third the distance from the device to the top of the antenna, 21.5 feet above the water.

Let  $x$  represent the depth of the bay. The coordinate  $-x$  would represent the depth. The location of the device would be represented by  $-\frac{1}{2}x$ .



The distance from the top of the antenna to the device would be presented by  $21.5 - \left(-\frac{1}{2}x\right)$ .

Half of the depth is  $\frac{1}{2}x$  which must be equal to one-third of the distance from the top of the antenna to the device.

$$\frac{1}{2}x = \frac{1}{3}\left(21.5 - \left(-\frac{1}{2}x\right)\right)$$

$$\frac{1}{2}x = \frac{21.5}{3} + \frac{1}{6}x$$

$$3x = 43 + x$$

$$2x = 43$$

$$x = 21.5$$

The depth of the bay would have to be 21.5 feet.







# **Chapter 2:** *Patterns, Conjecture, and Proof*





# Introduction

The assessments in Chapter 2 emphasize geometric thinking and spatial reasoning. “Students use geometric thinking to understand mathematical concepts and the relationships among them.”\* High School Geometry affords one of the first opportunities for students to explore the structure of mathematical systems and to understand what it means to prove mathematically that a conjecture is true. The emphasis of the problems in this chapter is on discovering this structure through investigations, observing patterns and formulating conjectures based on these experiences. Once the conjecture is formed, the students are asked to justify why it is or is not true.

Mad as a Hatter or Hat as a Madder assesses the students’ understanding of logical reasoning. The remainder of the problems are grouped together to emphasize the process of having students investigate, make conjectures, and then justify their conjectures. For example Pizza Delivery Service Regions provides the opportunity for investigation and conjecture while More Pizza Delivery requires the verification of students’ conjectures.

Students use a variety of representations (concrete, pictorial, algebraic, and coordinate), tools, and technology, including, but not limited to, powerful and accessible hand-held calculators and computers with graphing capabilities to solve meaningful problems representing figures, transforming figures, analyzing relationships, and proving things about them. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999.)





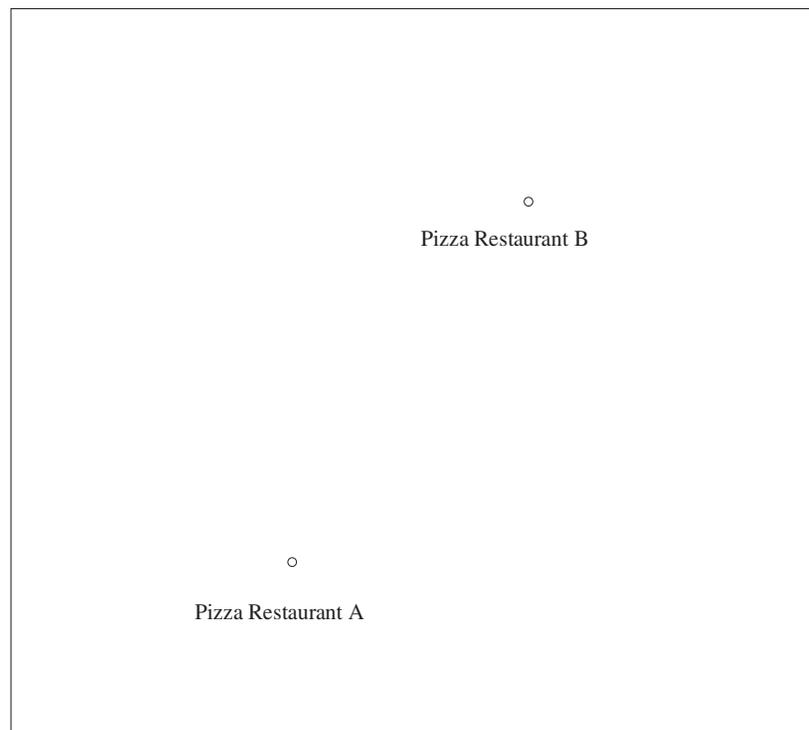
# Pizza Delivery Service Regions

## Problem 1

### Two Restaurants

The rectangle below represents a map of a city, and the two points represent pizza restaurants. Your task is to accurately determine the delivery service region for each of the restaurants. A household is in a restaurant's delivery service region if it is located closer to that restaurant than to the other restaurant.

You will define each restaurant's delivery service region by using compass and straightedge to accurately construct the boundary between the two regions.



- a) Write a few sentences explaining how you determined where to locate the boundary.



b) Based on your observations and construction, what do you think must be true about a household located on the boundary between service regions?

Complete the following conjectures: If a household is on the boundary between two service regions, then the household \_\_\_\_\_

\_\_\_\_\_.

If a point is on a line segment's perpendicular bisector, then the point

\_\_\_\_\_

\_\_\_\_\_.



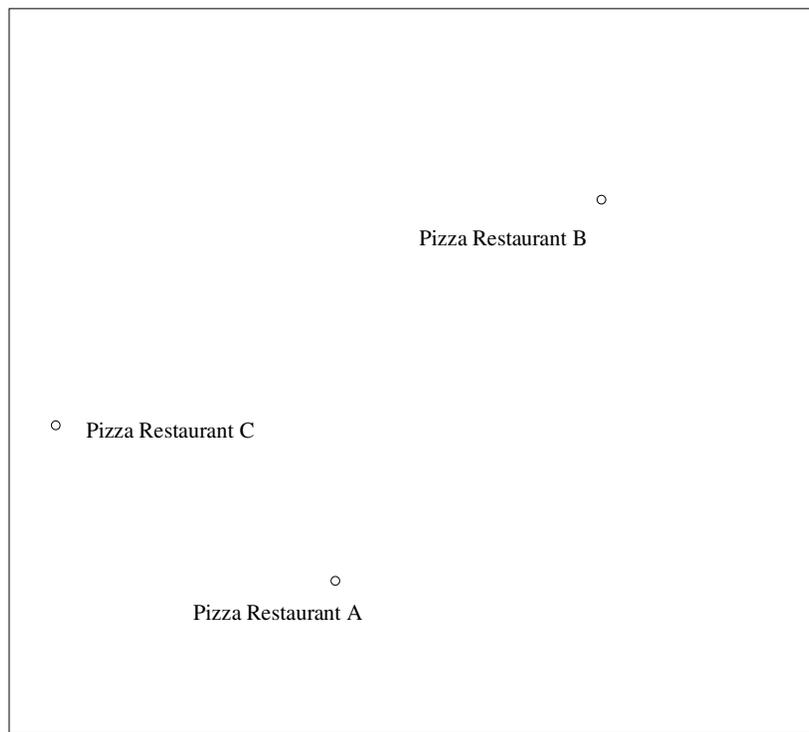
## Problem 2

### Three Restaurants

Determine the delivery service regions for pizza restaurants A, B, and C.

As in the previous task, a household is located in a delivery service region if it is closer to that restaurant than to either of the other two restaurants.

As before, you can define each restaurant's service region by using compass and straightedge to accurately construct the boundaries between each of the three regions.



- a) Based on your observations and construction, what do you think must be true about a household located on all the boundaries between three service regions?

Complete the following conjectures: If a household is on all the boundaries between three service regions, then the household

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The perpendicular bisectors of the three segments that form a triangle intersect at a point which is \_\_\_\_\_  
\_\_\_\_\_.

### Problem 3

#### Your Conjecture

Using your conjecture from Problem 1, write a few sentences explaining why your conjecture from Problem 2 must be true. In other words, if your conjecture about households on the boundary between two service regions is true, then use this fact to explain why your conjecture about households on the boundary between three service regions must, therefore, be true.





## Teacher Notes

### Materials:

One compass and straightedge per student.

### Connections to Geometry TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(D) uses inductive reasoning to formulate a conjecture.

### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

This set of problems is intended to determine if a student can make conjectures based on what they observe (use inductive reasoning). The next problem, More Pizza Delivery, will determine if the students can use deductive reasoning to prove the statement.

### Student Response to Problem 1, Question 2:

“It means you can order service from either one, because the two are equal distance from you. I myself would buy pizza from the one that delivers faster and has better pizza.”

### Scaffolding Questions:

#### Problem 1

##### Two Restaurants

- If you mark a point at random on the map to represent a household, how can you use your compass to determine to which restaurant the household is closer?
- If you mark a point at random on the map to represent a household, how can you use your compass to determine the locations of other households at that same distance from either of the two restaurants?
- Use your compass to locate at least three households that are the same distance from both restaurants. Is it possible to find all the households that are the same distance from both restaurants?
- If you have not already done so, construct the perpendicular bisector of the line segment joining restaurant A and restaurant B. What appears to be true about all the points on the perpendicular bisector of the line segment joining restaurant A and restaurant B?



## Problem 2

### Three Restaurants

- Which construction could you use to draw the boundary between the service regions of restaurant A and restaurant C?
- Which construction could you use to draw the boundary between the service regions of restaurant C and restaurant B?
- What appears to be true about the three boundary lines you have constructed?

## Problem 3

### Your Conjecture

- What do you think is true about a household on the boundary line between restaurants A and B?
- What appears to be true about a point located on the perpendicular bisector of the segment joining restaurants A and B?
- What do you think is true about a household on the boundary line between restaurants A and C?
- What seems to be true about a point located on the perpendicular bisector of the segment joining restaurants A and C?
- If your conjecture is proven to be true, what must be true about a household located on both boundary lines?
- If your conjecture is proven to be true, what must be true about a household located on both perpendicular bisectors?

### Sample Solutions:

#### Problem 1

- a) Construct the perpendicular bisector of the segment joining restaurant A and restaurant B. The compass setting represents the distance between a household and a restaurant. A circle with its center at one of

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

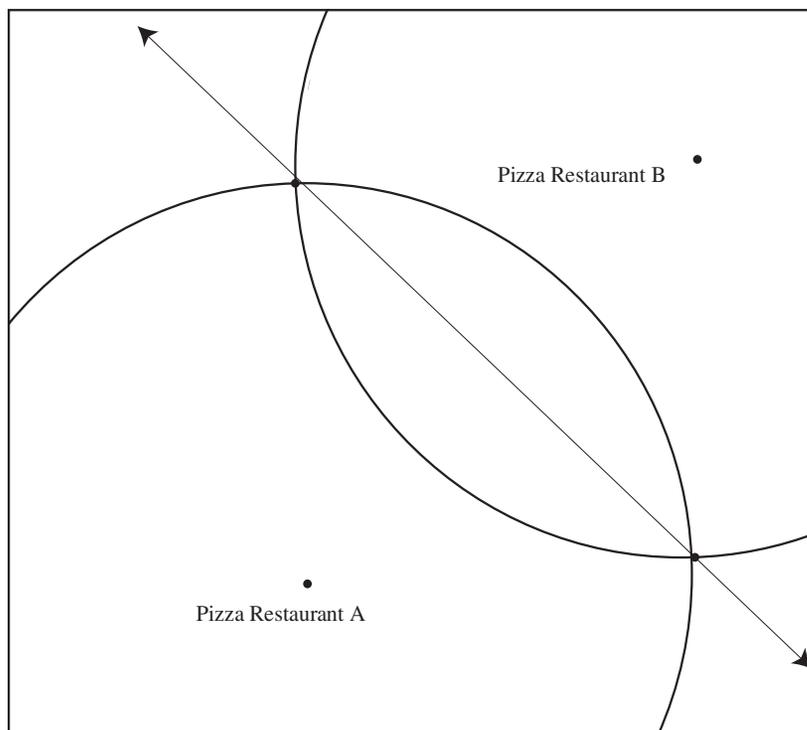
- I. Structure: Bayou City Geometry

#### Teacher's Comment:

"To introduce this problem I explained they had all just been hired to work for the pizza company to decide on delivery areas and to be cost effective."



the restaurants, and a fixed compass setting as a radius, represents all households at that fixed distance from the restaurant. A household is the same distance from both restaurants only if its location is at the intersection of two circles with the same radii and centers, respectively, at each of the restaurants.



- b) Complete the following conjectures: If a household is on the boundary between two service regions, then the household is the same distance from restaurant A as it is from restaurant B.

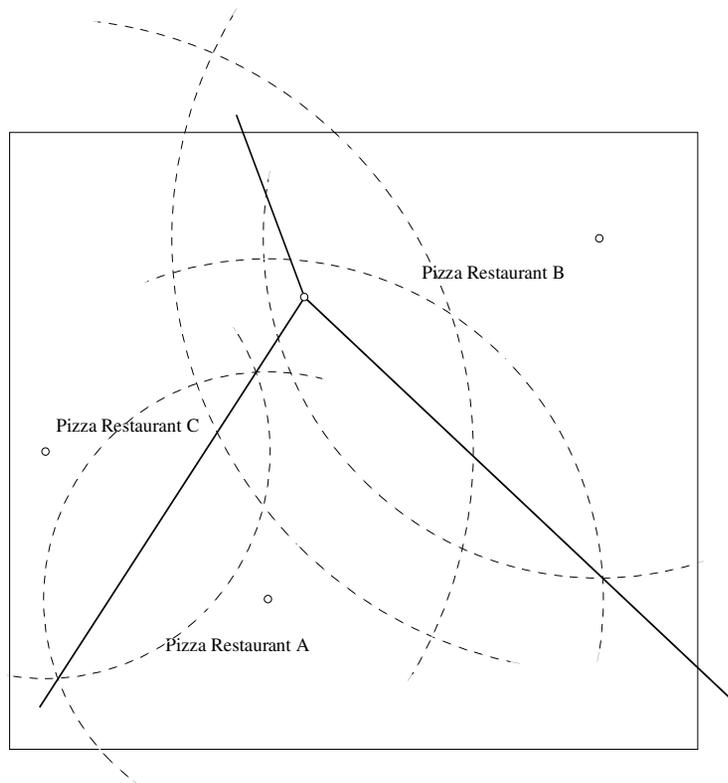
If a point is on a line segment's perpendicular bisector, then the point is the same distance from each of the line segment's endpoints.

### Problem 2

Construct the three perpendicular bisectors of the line segments joining restaurants A, B, and C. Repeat the construction from problem 1 to find the service region boundaries between restaurants A and C and between restaurants B and C.

These boundary lines intersect at a single point. This point is the circumcenter of the triangle formed by joining the points representing the three restaurants.





- a) Complete the following conjectures: If a household is on the intersection point that is the boundary between three service regions, then the household is the same distance from all three restaurants.

The perpendicular bisectors of the three segments that form a triangle intersect at a point which is the same distance from all three of the triangles' vertices.

### Problem 3

Note: Students can be required to answer this question using both the terms specific to the problem's context (households, service regions, boundaries) as well as the more formal language of geometric abstraction (points, line segments, perpendicular bisectors).

A household located at the intersection of the three service region boundaries must be the same distance from all three restaurants because of the following:

- This household is on the boundary between restaurant A's and restaurant B's service regions, and because of the conjecture from Problem 1, it is, therefore, the same distance from restaurant A as it is from restaurant B.
- This household is also on the boundary between restaurant A's and restaurant C's service regions (or alternately, restaurant B's and restaurant C's service regions), and because of the conjecture from Problem 1, it is, therefore, the same distance from restaurant A as it is from restaurant C (or alternately, restaurant B from restaurant C).



This makes the household the same distance from all three restaurants.

A point located at the intersection of the perpendicular bisectors of a triangle must be the same distance from all three of the triangle's vertices because of the following:

This point is on the perpendicular bisector of the line segment joining vertices A and B, and because of the conjecture from Problem 1, it is, therefore, the same distance from vertex A as it is from vertex B.

This point is also on the perpendicular bisector of the line segment joining vertices A and C (or alternately, vertices B and C), and because of the conjecture from Problem 1, it is, therefore, the same distance from vertex A as it is from vertex C (or alternately, vertex B and vertex C).

### Extension Questions:

- Suppose that the three restaurants are located on a coordinate grid at the points A (13,17), B (4,11), and C (3,21). Determine the equations of the boundary lines for the service regions and the circumcenter of the triangle ABC.

To determine the equation of the perpendicular bisector of  $\overline{AB}$ , find the midpoint of the segment, and determine the opposite reciprocal of the slope of  $\overleftrightarrow{AB}$ .

The midpoint of  $\overline{AB}$  is  $\left(\frac{13+4}{2}, \frac{17+11}{2}\right)$  or (8.5,14).

The slope of  $\overleftrightarrow{AB}$  is  $\frac{17-11}{13-4} = \frac{6}{9} = \frac{2}{3}$ .

The slope of a perpendicular line is the opposite reciprocal,  $-\frac{3}{2}$ .

The equation of the perpendicular bisector is

$$y - 14 = -\frac{3}{2}(x - 8.5)$$

$$y = -\frac{3}{2}x + 26.75$$

To determine the equation of the perpendicular bisector of  $\overline{AC}$ , find the midpoint of the segment, and determine the opposite reciprocal of the slope of  $\overleftrightarrow{AC}$ .

The midpoint of  $\overline{AC}$  is  $\left(\frac{13+3}{2}, \frac{17+21}{2}\right)$  or (8,19).

The slope of  $\overleftrightarrow{AC}$  is  $\frac{17-21}{13-3} = \frac{-4}{10} = -\frac{2}{5}$ .



The slope of a perpendicular line is the opposite reciprocal,  $\frac{5}{2}$ .

The equation of the perpendicular bisector is

$$y - 19 = \frac{5}{2}(x - 8)$$

$$y = \frac{5}{2}x - 1$$

To determine the equation of the perpendicular bisector of  $\overline{BC}$ , find the midpoint of the segment and determine the opposite reciprocal of the slope of  $\overline{BC}$ .

The midpoint of  $\overline{BC}$  is  $\left(\frac{3+4}{2}, \frac{21+11}{2}\right)$  or  $(3.5, 16)$ .

The slope of  $\overline{BC}$  is  $\frac{21-11}{3-4} = \frac{10}{-1} = -10$ .

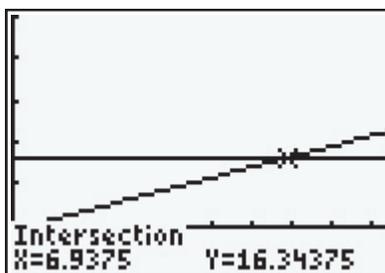
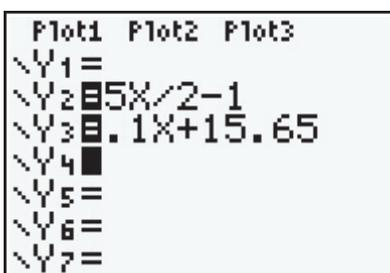
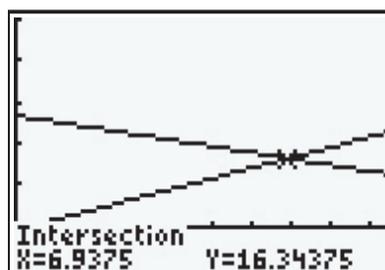
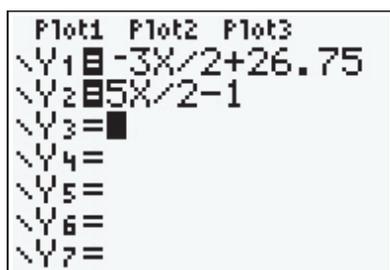
The slope of a perpendicular line is the opposite reciprocal,  $\frac{1}{10}$ .

The equation of the perpendicular bisector is

$$y - 16 = \frac{1}{10}(x - 3.5)$$

$$y = \frac{1}{10}x + 15.65$$

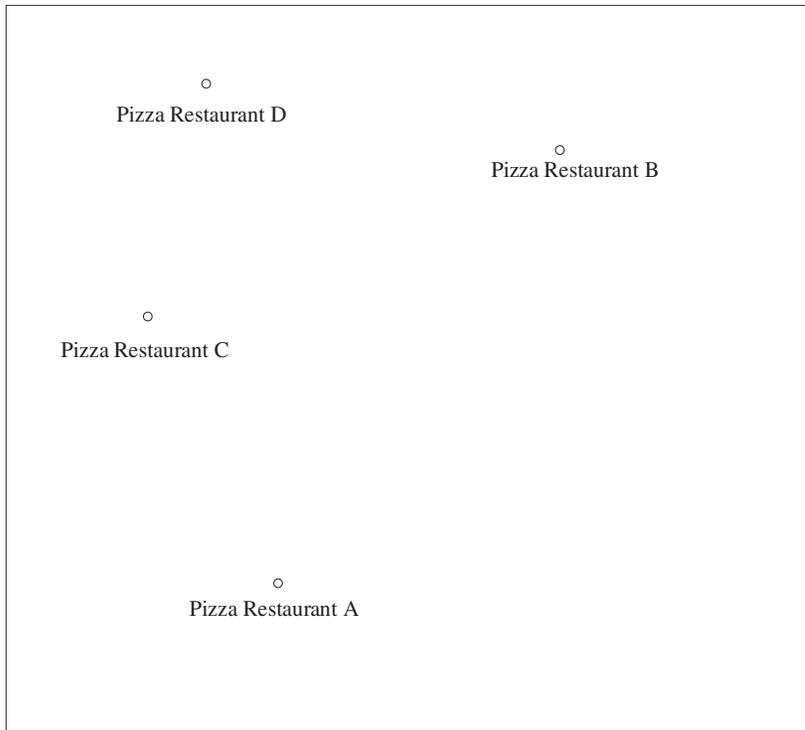
The intersection point of the three lines may be found by using a graphing calculator.

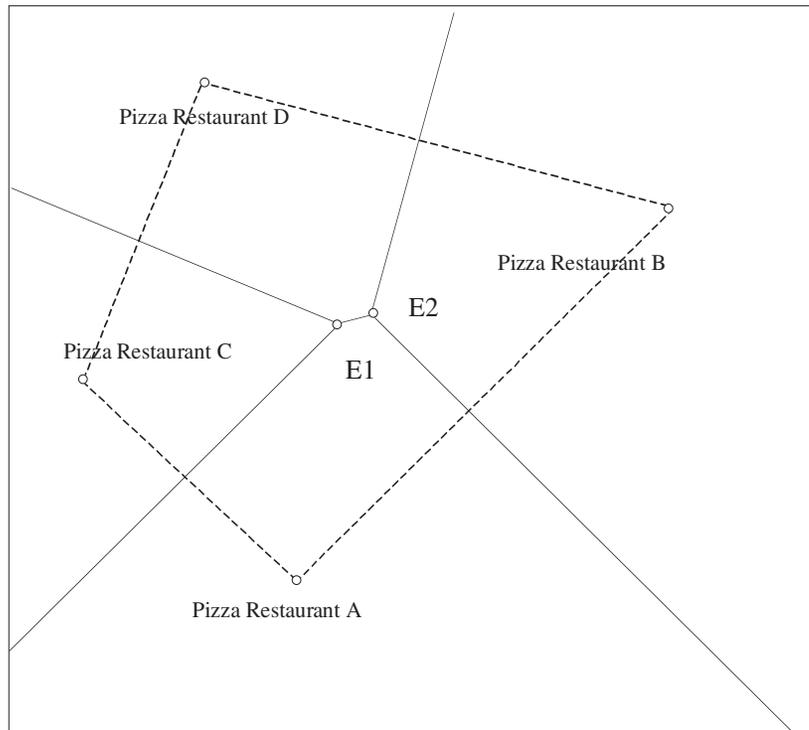


The intersection point is  $(6.9375, 16.34375)$ .



- Determine the pizza service boundary regions for restaurants A, B, C, D.

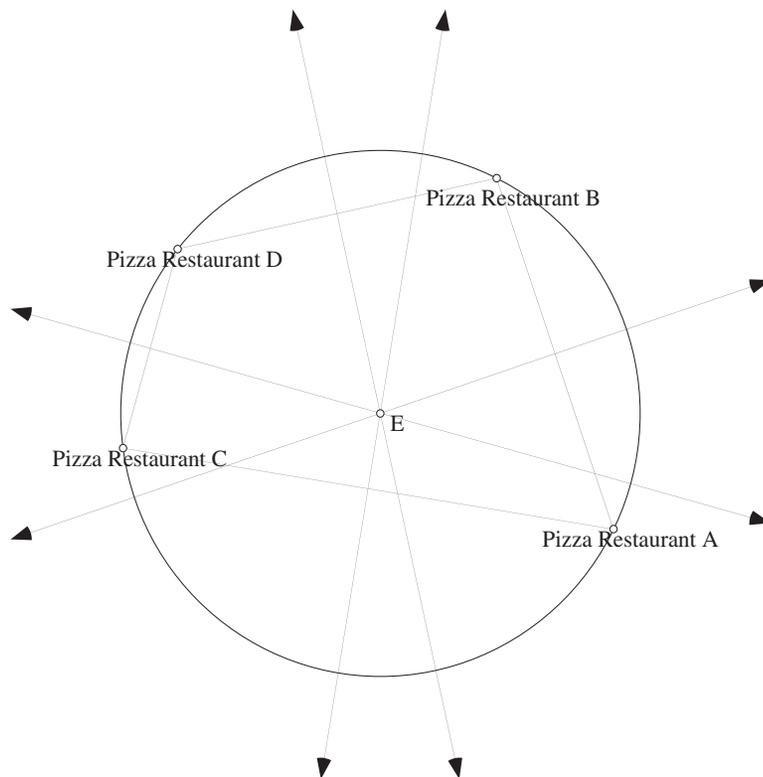




*Point E1 is the circumcenter of the triangle formed by restaurants A, C, and D.  
 Point E2 is the circumcenter of the triangle formed by restaurants A, B, and D.*

- Position the four restaurants so that there is a single household that is the same distance from all four restaurants. (Challenge: you must accomplish this task without positioning the restaurants so that they form the vertices of a square or a rectangle.)





*The four restaurants must be located on a circle. The household represented by point E is equidistant from all four restaurants since it is the center of the circle. The four restaurants must be the vertices of a cyclic quadrilateral.*



## More Pizza Delivery

In the assessment, Pizza Delivery Service Regions, you used constructions to define the boundary between the delivery regions for two pizza restaurants.

You then used your construction to make the following conjecture:

If a point is on a line segment's perpendicular bisector, then the point is the same distance from each of the line segment's endpoints.

### Problem 1

Choose one of the following methods to verify the conjecture:

#### Axiomatic Approach

Using the definitions, postulates, and theorems of your geometry course, write a deductive proof of the conjecture.

#### Coordinate Geometry Approach

Given pizza restaurant A, with coordinates  $A(2,-7)$ , and pizza restaurant B, with coordinates  $B(8,11)$ :

- Find the equation of the perpendicular bisector of  $\overline{AB}$ .
- Use the equation you found to find the coordinates of a point on the perpendicular bisector of  $\overline{AB}$ .
- Verify that the point you found satisfies the conditions of the conjecture.

### Problem 2

#### Transformational Approach

Use transformational geometry to verify the converse of the conjecture:

If a point is equidistant from the two endpoints of a line segment, then the point is on the segment's perpendicular bisector.

Given pizza restaurant A, with coordinates  $A(2,-7)$ , and pizza restaurant B,



with coordinates  $B(8,11)$ . Also, given point  $D$ , with coordinates  $D(8,1)$ , and point  $E$ , with coordinates  $E(2,3)$ :

- a) Use coordinate geometry formulas to verify:  $AD = BD$  and  $AE = BE$ .
- b) Find the equation of the line containing points  $D$  and  $E$ .
- c) Use transformations to show that this line is the perpendicular bisector of  $\overline{AB}$ .

Recall that a reflection takes a pre-image point and moves it across a *mirror line*, so that the *mirror line* is the perpendicular bisector of the segment connecting the point and its image.

Verify that point  $B$  is the reflection image of point  $A$  across line  $\overleftrightarrow{DE}$ . Thus,  $\overleftrightarrow{DE}$  is the perpendicular bisector of  $\overline{AB}$ .





# Teacher Notes

## Materials:

Graph paper

## Connections to Geometry

### TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons; and

## Scaffolding Questions:

### Problem 1

#### Axiomatic Approach

- How would you set up a “given,” a “prove,” and a diagram that represents the conjecture?
- Which proof style would you like to use to write the proof: 2 column, flow chart, or paragraph?
- Depending on proficiency level of students, provide appropriate amounts of set up and/or proof. (See sample solution.)

#### Coordinate Geometry Approach

- What is the slope of  $\overline{AB}$ ?
- What must be true about the slope of the perpendicular bisector of  $\overline{AB}$ ?
- What are the coordinates of the midpoint of  $\overline{AB}$ ?
- Use the slope and midpoint information to write the equation of the perpendicular bisector of  $\overline{AB}$ .
- What must be true about the coordinates of a point if it lies on this perpendicular bisector?
- What coordinate geometry formula could you use to show that the point you found is the same distance from A as from B?

### Problem 2

#### Transformational Approach

- What coordinate geometry formula could you use to verify that these distances are the same?
- Find the slope of  $\overleftrightarrow{DE}$ .
- Use this information to write the equation of  $\overleftrightarrow{DE}$  in slope-intercept form.



- What is the equation of the line that is perpendicular to  $\overleftrightarrow{DE}$  and passes through point A?
- What are the coordinates of point F, the intersection point of the two perpendicular lines?
- What is the distance from point A to this intersection point?
- What are the coordinates of a point at distance AF, from point F, along  $\overleftrightarrow{AF}$ ?

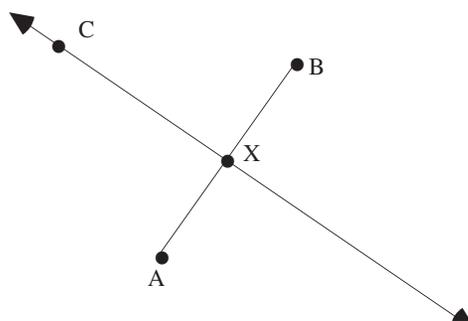
### Sample Solution:

#### Problem 1

Axiomatic Approach

Given: Point C on  $\overleftrightarrow{CX}$ , the perpendicular bisector of  $\overline{AB}$ .

Prove:  $CA = CB$



Proof:

$AX = XB$	(Definition of perpendicular bisector)
$m\angle AXC = m\angle BXC = 90^\circ$	(Definition of perpendicular bisector)
$CX = CX$	(Reflexive property)
$\triangle AXC \cong \triangle BXC$	(SAS)
Therefore $CA = CB$	(Corresponding parts of congruent triangles are congruent)

Coordinate Geometry Approach

a) The midpoint of  $\overline{AB}$  is (5, 2).

The line passing through the points A(2, -7) and B(8, 11) will have slope  $\frac{11 - (-7)}{8 - 2} = \frac{18}{6} = 3$ .

(C) develops and uses formulas including distance and midpoint.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines.

#### Texas Assessment of Knowledge and Skills:

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Additional objectives are addressed by the Extension Questions that follow the sample solution:

Objective 4: The student will formulate and use linear equations and inequalities.

#### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Geometric Structure
- II. Transformations: Reflections
- III. Triangles: Pythagorean Theorem



The slope of a perpendicular line is the opposite reciprocal, or  $-\frac{1}{3}$ .

The perpendicular bisector of line  $\overleftrightarrow{AB}$  must have slope  $-\frac{1}{3}$  and pass through (5, 2). Using point slope form, the equation of the perpendicular bisector is:

$$y - 2 = -\frac{1}{3}(x - 5)$$

or

$$y = -\frac{1}{3}x + \frac{11}{3}$$

b) Choose a value for  $x$ , and solve for  $y$ .

$$x = 2, y = -\frac{1}{3}(2) + \frac{11}{3} = 3$$

Point D(2, 3) is a point on the perpendicular bisector of  $\overline{AB}$ .

c) Using the distance formula:

$$DA = \sqrt{(2-2)^2 + (3-(-7))^2} = 10$$

$$DB = \sqrt{(2-8)^2 + (3-11)^2} = 10$$

Therefore  $DA = DB$ , and point D satisfies the conditions of the conjecture.

Note: Showing that one point works does not prove the statement in general. To prove in general select any point  $E(x,y)$  on the line.

$$y = -\frac{1}{3}x + \frac{11}{3}$$

$$EA = \sqrt{(2-x)^2 + (-7 - (-\frac{1}{3}x + \frac{11}{3}))^2} =$$

$$\sqrt{(2-x)^2 + (-7 + \frac{1}{3}x - \frac{11}{3})^2} =$$

$$\sqrt{(2-x)^2 + (\frac{1}{3}x - \frac{32}{3})^2} =$$

$$\sqrt{4 - 4x + x^2 + \frac{1}{9}x^2 - \frac{64}{9}x + \frac{1024}{9}} =$$

$$\sqrt{\frac{10}{9}x^2 - \frac{100}{9}x + \frac{1060}{9}}$$



$$\begin{aligned}
 EB &= \sqrt{(8-x)^2 + \left(11 - \left(-\frac{1}{3}x + \frac{11}{3}\right)\right)^2} = \\
 &= \sqrt{(8-x)^2 + \left(11 + \frac{1}{3}x - \frac{11}{3}\right)^2} = \\
 &= \sqrt{(8-x)^2 + \left(\frac{1}{3}x + \frac{22}{3}\right)^2} = \\
 &= \sqrt{64 - 16x + x^2 + \frac{1}{9}x^2 + \frac{44}{9}x + \frac{484}{9}} = \\
 &= \sqrt{\frac{10}{9}x^2 - \frac{100}{9}x + \frac{1060}{9}}
 \end{aligned}$$

EA = EB for any point E on  $\overleftrightarrow{CX}$ .



## Problem 2

Transformational Approach

a) Using the distance formula:

$$AD = \sqrt{(2-8)^2 + (-7-1)^2} = 10$$

$$BD = \sqrt{(8-8)^2 + (11-1)^2} = 10$$

$$AE = \sqrt{(2-2)^2 + (3-(-7))^2} = 10$$

$$BE = \sqrt{(2-8)^2 + (3-11)^2} = 10$$

Therefore  $AD = BD$ , and  $AE = BE$ . This makes D and E both equidistant from A and B.

b) The slope of line  $\overleftrightarrow{DE}$  is  $\frac{3-1}{2-8} = \frac{2}{-6} = -\frac{1}{3}$ .

Using point slope form, the equation of line  $\overleftrightarrow{DE}$  is:

$$y-1 = -\frac{1}{3}(x-8)$$

or

$$y = -\frac{1}{3}x + \frac{11}{3}$$

c) The line that is perpendicular to line  $\overleftrightarrow{DE}$  and passes through  $A(2,-7)$  must have a slope that is the opposite reciprocal of  $-\frac{1}{3}$ . Using point slope form, the equation of that line is:

$$y - (-7) = 3(x - 2). \text{ In slope intercept form: } y = 3x - 13.$$

The point of intersection of the two lines can be found by algebraically solving both equations for both variables:

$$-\frac{1}{3}x + \frac{11}{3} = 3x - 13$$

$$x = 5 \text{ and } y = 2$$

The intersection point has coordinates  $F(5, 2)$ .

$$\text{The distance } AF = \sqrt{(5-2)^2 + (2-(-7))^2} = \sqrt{90} = 3\sqrt{10}.$$

We want the coordinates of the reflection image of point A. This point is  $3\sqrt{10}$

units from point F along  $\overleftrightarrow{AF}$ . To find the coordinates of the point  $A(x,y)$ , use the distance



formula and the equation of  $\overleftrightarrow{AF}$ .

$$3\sqrt{10} = \sqrt{(x-5)^2 + (y-2)^2}$$

$$\text{Since } y = 3x - 13, 3\sqrt{10} = \sqrt{(x-5)^2 + (3x-13-2)^2}$$

$$\text{Solve this equation for } x: 3\sqrt{10} = \sqrt{x^2 - 10x + 25 + 9x^2 - 90x + 225}$$

$$90 = 10x^2 - 100x + 250$$

$$0 = 10x^2 - 100x + 160$$

$$0 = 10(x^2 - 10x + 16)$$

$$0 = 10(x - 8)(x - 2)$$

$$x = 8 \text{ or } x = 2$$

When  $x = 2$ ,  $y = -7$ . These are the coordinates of point A.

When  $x = 8$ ,  $y = 11$ . These are the coordinates of point A', the reflection of point A across  $\overleftrightarrow{DE}$ .

But these are precisely the coordinates of point B. Point B is, therefore, the reflection image of point A across  $\overleftrightarrow{DE}$ . This makes  $\overleftrightarrow{DE}$  the perpendicular bisector of  $\overline{AB}$ .

### Extension Questions:

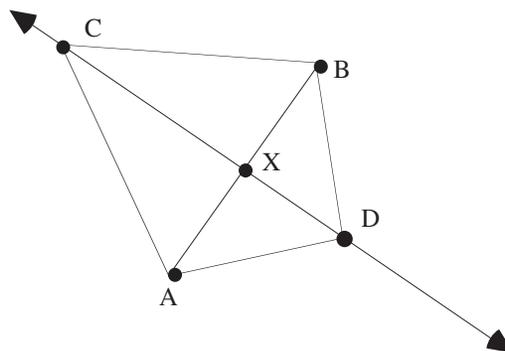
- Verify the converse of the conjecture using an Axiomatic Approach.

*Converse of Conjecture: If a point is equidistant from the two endpoints of a line segment, then the point is on the segment's perpendicular bisector.*

Given:  $CB = CA$

$$DB = DA$$

Prove: Line  $\overleftrightarrow{CD}$  is the perpendicular bisector of segment  $\overline{AB}$ .



**Proof:**

$CA = CB$  and  $DA = DB$  (Given)  
This makes quadrilateral  $ACBD$  a kite (Definition of a kite)  
Then,  $\overleftrightarrow{CX} \perp \overline{AB}$  (Diagonals of a kite are perpendicular)  
This makes  $\text{right } \triangle ACX \cong \text{right } \triangle BCX$  (HL)  
and  $AX = XB$  (CPCTC)  
Therefore,  $\overleftrightarrow{CX}$  is the perpendicular bisector of segment  $\overline{AB}$ . (Definition of perpendicular bisector)

Another approach that does not use the properties of kites, involves proving that  $\triangle CAD \cong \triangle CBD$  (SSS).

Then  $\angle ACX \cong \angle BCX$  (CPCTC).

Now,  $\triangle ACX \cong \triangle BCX$  (HA or SAS).

Then establish  $AX = XB$  and  $m\angle AXC = m\angle BXC = 90^\circ$ .

Line  $\overleftrightarrow{CX}$  is the perpendicular bisector of segment  $\overline{AB}$ . (Definition of perpendicular bisector)



## Conjecture as Discovery and Proof as Explanation

### Problem 1

#### Triangle Midsegment Conjecture

Use paper, pencil, construction and measuring tools  
or  
appropriate geometry technology to complete this problem.

1. Sketch and label a  $\triangle ABC$ .
2. Find and label point D (the midpoint of side  $\overline{AB}$ ) and point E (the midpoint of side  $\overline{AC}$ ).
3. Draw midsegment  $\overline{DE}$ .
4. Take and record the following measurements in centimeters and degrees.

DE = \_\_\_\_\_

BC = \_\_\_\_\_

$\angle ADE =$  \_\_\_\_\_

$\angle AED =$  \_\_\_\_\_

$\angle DBC =$  \_\_\_\_\_

$\angle ECB =$  \_\_\_\_\_

5. Repeat steps 1 – 4 to complete the sketch, and take measurements on at least two more triangles that are different from your original triangle.

If you are working with a group, you may compare your triangle measurements with the other group members.

If you are using geometry technology, you may drag the vertices of the original triangle to generate new triangles and sets of measurements.

6. Based on your drawings and observations, complete the following conjecture:



The midsegment of a triangle is \_\_\_\_\_ to one side of the triangle, and it measures \_\_\_\_\_ of that side.

## Problem 2

Why is it true?

If you completed Problem 1, you discovered two important characteristics of a triangle's midsegment:

- a) The midsegment is parallel to a side of the triangle.
- b) The midsegment is  $\frac{1}{2}$  the length of the side of the triangle it is parallel to.

You might have already known about these properties from previous lessons, or you might have even guessed what they were without drawing or measuring. But can you explain why they are true?

In order for your explanation to be fully convincing from a mathematical standpoint, it must satisfy three requirements. First, it must be logical. Second, it must consist of facts, definitions, postulates, or theorems that have been previously proven or accepted as true. Third, it must apply to all cases.

Write an explanation of why the first property of triangle midsegments is true. Your explanation must satisfy all of the above requirements. Use your knowledge of postulates and theorems about parallel lines and angle relationships to help with your explanation.



### Problem 3

#### A Different Look

Mathematicians call explanations similar to the one you wrote in the previous problem “proofs.” In order to prove the second property about triangle midsegments, it is helpful to represent the situation in a different mathematical context.

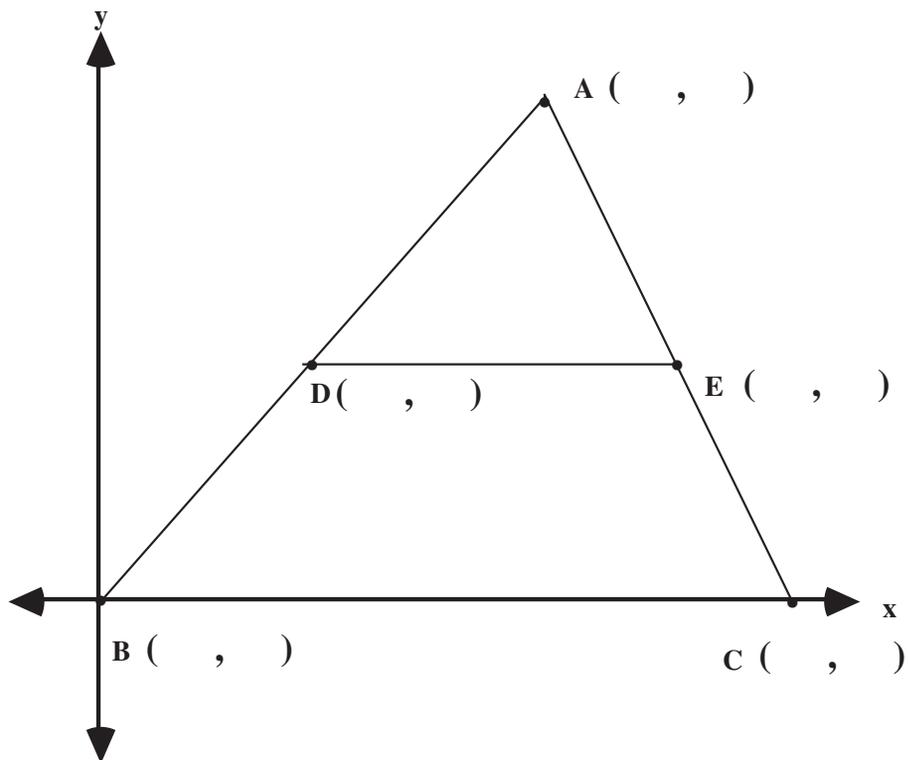
1. Draw and label triangle a  $\triangle ABC$  on graph paper. Label and record the numerical coordinates of points A, B, and C.
2. Use the midpoint formula to find and label the following: point D (the midpoint of segment  $\overline{AB}$ ) and point E (the midpoint of segment  $\overline{AC}$ ).
3. Use the distance formula to calculate the length of  $\overline{DE}$  and to calculate the length of  $\overline{BC}$ .
4. What is the relationship between the length of  $\overline{DE}$  and the length of  $\overline{BC}$  ?
5. Do the diagram and calculations above constitute a proof of the second property of triangle midsegments? Write a few sentences discussing why or why not.



## Problem 4

### The General Situation

Any triangle may be rotated and translated so that one vertex is at the origin and another vertex is on the positive  $x$ -axis.



1. Fill in the coordinates of points A, B, and C in the above diagram using variables to represent any triangle.
2. Use the midpoint formula to calculate and fill in the coordinates of point D (the midpoint of  $\overline{AB}$ ) and point E (the midpoint of  $\overline{AC}$ ).
3. Use the distance formula to calculate the length of  $\overline{DE}$  and to calculate the length of  $\overline{BC}$ . Show all work and calculations.
4. What is the relationship between the length of  $\overline{DE}$  and the length of  $\overline{BC}$ ?
5. Do the diagram and calculations above constitute a proof of the second property of triangle midsegments? Write a few sentences discussing why or why not.





**Materials:**

Graph paper

One pencil, ruler, protractor per student and/or

Appropriate geometry technology

**Connections to Geometry****TEKS:**

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

## Teacher Notes

This problem asks the students to investigate and then prove important characteristics about a triangle's midsegment. The problem, Extending the Triangle Midsegment Conjecture, investigates the midsegment for other polygons.

**Scaffolding Questions:****Problem 1**

- Step 1:  $\triangle ABC$  can be any type of triangle. If students are working in groups, encourage different group members to draw different types of triangles.
- Step 3: What is the definition of a triangle's midsegment?
- Steps 5 and 6: What types of triangles seem to provide more convincing evidence in support of the conjecture? Why?

**Problem 2**

- What type of angles are  $\angle ADE$  and  $\angle DBC$  ?
- What type of angles are  $\angle AED$  and  $\angle ECB$  ?

**Problem 3**

- Steps 1 – 4: Students will orient their triangles on the graph paper many different ways and will soon find out that different triangle placements result in “easier” or “harder” numbers to work with.
- Is it all right to place your triangle on the graph paper in the most convenient way for the calculations to work out?
- Step 4: Can you write the relationship as a mathematical statement?
- Step 5: It can be helpful to point out that in fact the first two requirements of a mathematical proof have been fulfilled in this case. The formulas used are accepted



as true, and the diagram as well as the sequence of calculations constitute the manipulation of symbols according to logical rules. Many students do not realize this.

#### Problem 4

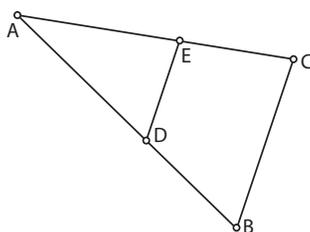
- After problem 3, students will have realized that their numerical calculations do not constitute a proof.
- What can we replace the numerical coordinates with so that the calculations will apply to any triangle?
- Why does this particular placement of the triangle make it easier to fill in the coordinates than other possible placements?
- What are the fewest number of variables that can be used to accurately label all the required coordinates?
- Step 2: What could we do to the coordinates of A, B, and C to make the calculations easier?
- What effect does multiplying a coordinate variable by a constant have on subsequent calculations? Is it mathematically all right to do this?
- Encourage students to simplify the complicated-looking expressions as much as possible. Review of notation and symbol manipulation under the radical symbol may be necessary.

#### Sample Solution:

##### Problem 1

1 – 5 This is one possible solution.

Length (Segment AD) = 3.26 cm  
 Length (Segment DB) = 3.26 cm  
 Length (Segment AE) = 2.83 cm  
 Length (Segment EC) = 2.83 cm  
 Angle (ADE) =  $59^\circ$   
 Angle (AED) =  $81^\circ$   
 Angle (DBC) =  $59^\circ$   
 Angle (ECB) =  $81^\circ$



6. The midsegment of a triangle is parallel to one side of the triangle and measures one-half the length of that side.



The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures; and

(C) develops and uses formulas including distance and midpoint.

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(B) justifies and applies triangle congruence relationships.

#### Texas Assessment of Knowledge and Skills:

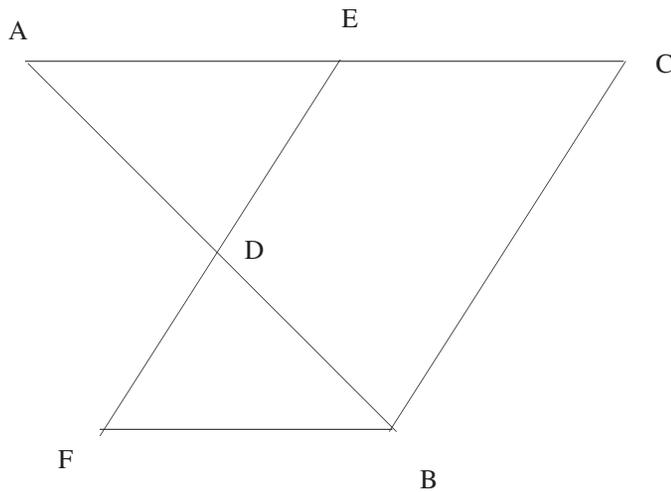
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

#### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

I. Structure: Midpoint Triangle

## Problem 2

A possible axiomatic proof:



Given: D is the midpoint of  $\overline{AB}$ , E is the midpoint of  $\overline{AC}$ .

Prove:  $DE = \frac{1}{2}BC$  and  $\overline{DE}$  is parallel to  $\overline{BC}$ .

Extend  $\overline{ED}$  to point F such that  $\overline{ED} \cong \overline{DF}$ .

$\overline{AE} \cong \overline{EC}$ ,  $\overline{AD} \cong \overline{DB}$  by definition of midpoint

$\angle ADE \cong \angle BDF$  because vertical angles are congruent.

$\triangle ADE \cong \triangle BDF$  because two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle.

$\angle DBF \cong \angle A$  and  $\overline{AE} \cong \overline{FB}$  because corresponding parts of congruent triangles are congruent.

$\overline{AC}$  is parallel to  $\overline{BF}$  because two lines are parallel if the alternate interior angles are congruent.

$\overline{EC} \cong \overline{FB}$  by the transitive property of congruence.

BCEF is a parallelogram because the two sides of the quadrilateral are parallel and congruent.

$EF = BC$  because the opposite sides of a parallelogram are equal in length.

$FD + DE = BC$  by segment addition

$2 DE = BC$  by substitution



$$DE = \frac{1}{2} BC \text{ by division}$$

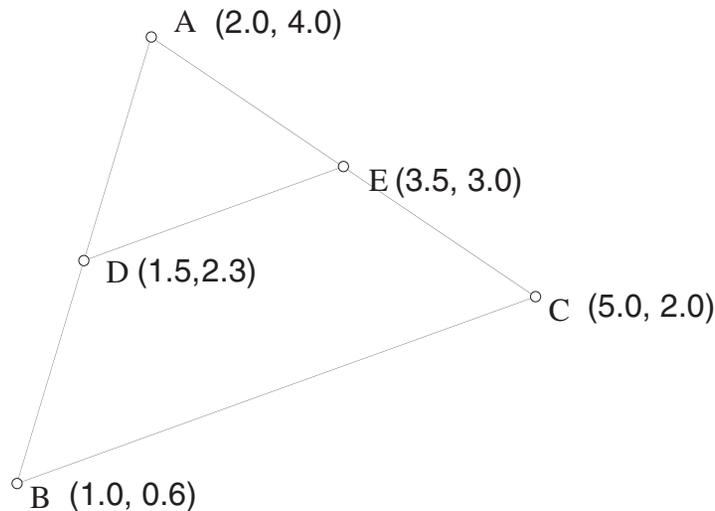
$\overline{DE}$  is parallel to  $\overline{BC}$  because opposite sides of a parallelogram are parallel.

This argument applies to the midsegment of any triangle.

### Problem 3

One possible coordinate representation:

1 and 2.



$$3. DE = \sqrt{(3.5-1.5)^2 + (3.0-2.3)^2} = \sqrt{4+0.49} \approx 2.12$$

$$BC = \sqrt{(5.0-1.0)^2 + (2.0-0.6)^2} = \sqrt{16+3.7636} \approx 4.44$$

$$4. DE = \frac{1}{2} BC$$

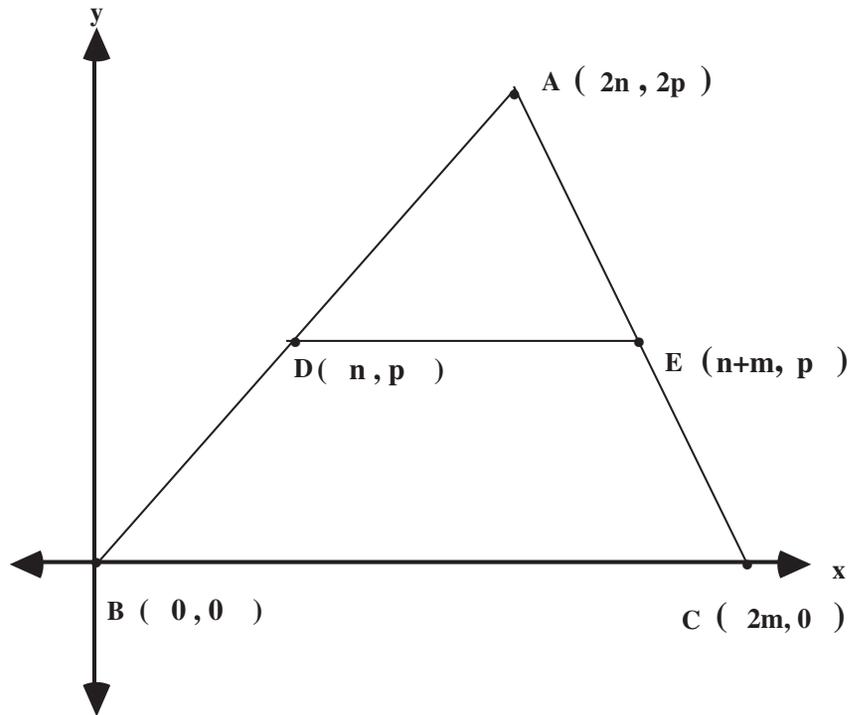
5. The diagram and calculations establish that the midsegment is one-half the length of the side it is parallel to only for a particular triangle with specific coordinates. They do not establish that this property is true for the midsegments of all triangles.



### Problem 4

For example:

1 and 2.



3.  $DE = \sqrt{(n+m-n)^2 + (p-p)^2} = \sqrt{m^2} = |m| = m$  because  $m > 0$ .

$BC = \sqrt{(2m-0)^2 + (0-0)^2} = \sqrt{4m^2} = |2m| = 2m$  because  $m > 0$ .

4.  $DE = \frac{1}{2} BC$

5. The diagram and the calculations do constitute a proof of the second property of triangle midsegments. The formulas used are accepted as true, and the diagram, as well as the sequence of calculations, constitute the manipulation of symbols according to logical rules. Finally, using variables instead of specific numbers for coordinates makes the relationship true for all triangle midsegments.



### Extension Questions:

- Write a few sentences detailing why the postulates and theorems about parallel lines and angle relationships **cannot** be used to explain why the second property of triangle midsegments described in Problem 2 (b) is true.

*The postulates and theorems about parallel lines and angle relationships establish that lines are parallel given that certain angle relationships are true, or that certain angle relationships must be true given that parallel lines are cut by a transversal. The second property of triangle midsegments deals with the relationship between the lengths of the parallel segments. These postulates and theorems don't provide any information about the lengths of parallel lines.*

- Using the diagram and the coordinates from problem 4, prove the first property of the triangle midsegments: the midsegment of a triangle is parallel to a side of the triangle.

*The lines are parallel since it can be shown that they have the same slope.*

$$\text{Slope of } DE = \frac{p-p}{n+m-n} = 0.$$

$$\text{Slope of } AB = \frac{0-0}{2m-0} = 0.$$



## Student Work Sample

The work on the next two pages shows a student's approach to problems 1, 2 and 3.

This work is a good example of the criteria

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

*In problems 2 and 3, this student explains his reasons for the statements he makes. (Since line  $a$  is the midsegment of triangle  $B$ , it bisects  $AB$  and  $AC$ ....).*

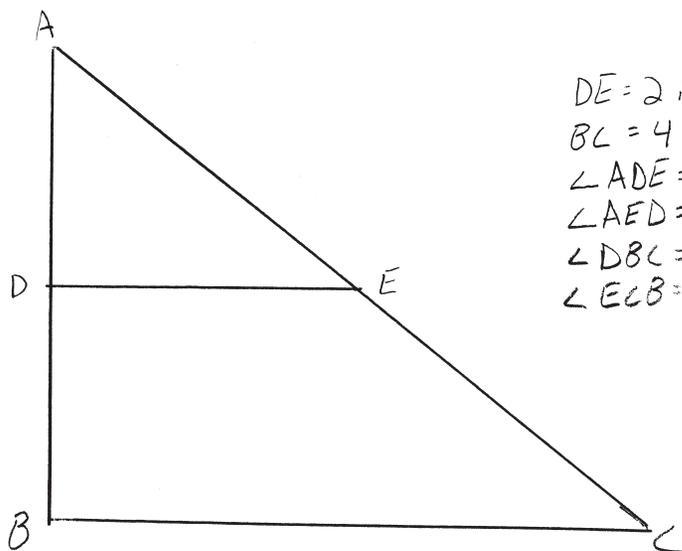
- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

Note that the student does not explain how he got the measurements.

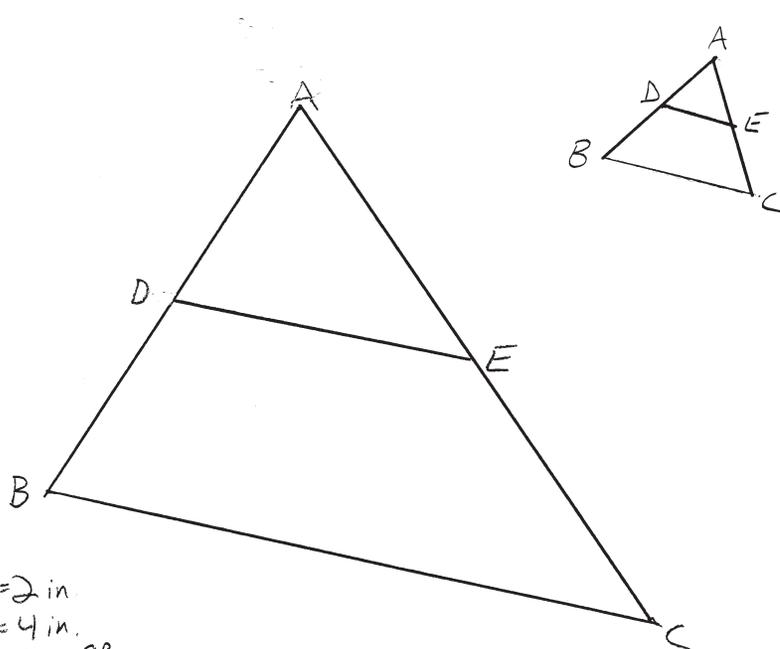


Conjecture as Discovery and Proof as Explanation

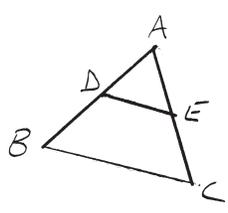
Problem 1.



$DE = 2$  in.  
 $BC = 4$  in.  
 $\angle ADE = 90^\circ$   
 $\angle AED = 32^\circ$   
 $\angle DBC = 90^\circ$   
 $\angle ECB = 32^\circ$



$DE = 2$  in.  
 $BC = 4$  in.  
 $\angle ADE = 69^\circ$   
 $\angle AED = 44^\circ$   
 $\angle DBC = 69^\circ$   
 $\angle ECB = 44^\circ$

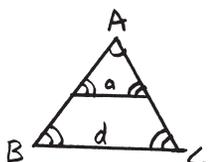
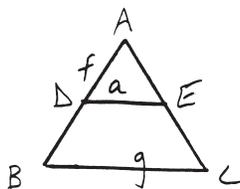


$DE = \frac{1}{2}$  in.  
 $BC = 1$  in.  
 $\angle ADE = 50^\circ$   
 $\angle AED = 58^\circ$   
 $\angle DBC = 50^\circ$   
 $\angle ECB = 58^\circ$

The midsegment of a triangle is proportional to one side of the triangle and measures  $\frac{1}{2}$  of that side.



Problem 2.



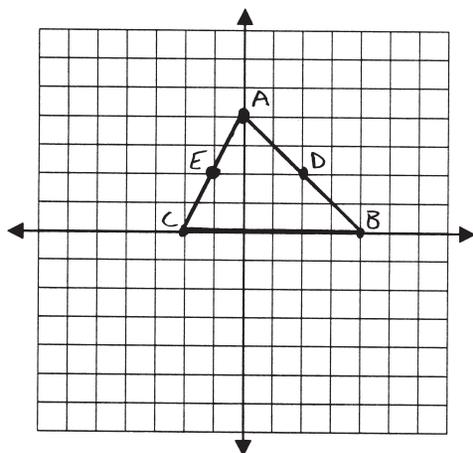
Since line  $a$  is the mid-segment of triangle  $B$ , it bisects  $AB$  and  $AC$ . Therefore,  $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{2}$ .

$\triangle ABC$  and  $\triangle ADE$  share  $\angle A$ , so by the SAS theorem,  $\triangle ABC \sim \triangle ADE$

Since they are  $\sim$ , their angles are  $\cong$

If a line intersects 2 other lines and corresponding angles are  $\cong$ , then the lines are  $\parallel$ , so line  $a$  and line  $d$  are  $\parallel$ .

3.



$$\begin{aligned} A &= (0,4) \\ B &= (4,0) \\ C &= (-2,0) \\ D &= (2,2) \\ E &= (-1,2) \end{aligned}$$

$$\overline{DE} = \sqrt{(2 - (-1))^2 + (2 - 2)^2} = \sqrt{9} = 3$$

$$\overline{CB} = \sqrt{(4 - (-2))^2 + (0 - 0)^2} = \sqrt{36} = 6$$

$$\overline{DE} = \frac{1}{2} \overline{CB}$$

The diagram does not constitute a proof because it is only one situation. For it to be a theorem, it must work in all situations.



## Extending the Triangle Midsegment Conjecture

These problems investigate the relationship between a polygon's diagonals and the polygon formed by connecting the midpoints of its sides. We begin the investigation by considering a quadrilateral.

### Problem 1

Use paper, pencil, construction and measuring tools  
or  
appropriate geometry technology to complete this problem.

1. Sketch and label Quadrilateral ABCD.
2. Draw all possible diagonals for Quadrilateral ABCD. In this case there are 2 diagonals,  $\overline{AC}$  and  $\overline{BD}$ .
3. Create a polygon (in this case a quadrilateral) by connecting the midpoints of the sides of quadrilateral ABCD.
4. Take appropriate measurements, and write a conjecture that relates the perimeter of the midpoint quadrilateral to the diagonals of the original quadrilateral.
5. Repeat steps 1 – 4 to complete the sketch and take measurements on at least two more quadrilaterals that are different from your original quadrilateral.

If you are working with a group, you may compare your quadrilateral measurements with the other group members.

If you are using geometry technology, you may drag the vertices of the original quadrilateral to generate new quadrilaterals and sets of measurements.



## Problem 2

In the Assessment, Conjecture as Proof and Proof as Explanation, you discovered and proved the Triangle Midsegment Conjecture:

The midsegment of a triangle is parallel to one side of the triangle and measures half the length of that side.

Since you have proved this conjecture, you can use it to help prove and explain why other conjectures may or may not be true.

Use the Triangle Midsegment Conjecture to write an explanation of why your quadrilateral conjecture is true. Provide a diagram with your explanation.

## Problem 3

Do you think your conjecture will be true for other polygons besides quadrilaterals? Why or why not?

Sketch, measure, and investigate the relationship between a polygon's diagonals and the polygon formed by connecting the midpoints of its sides. Start with a pentagon, then a hexagon, etc.

## Problem 4

Modify your original conjecture (or write new conjectures) to take into account other polygons besides quadrilaterals.





## Teacher Notes

### Materials:

Paper and graph paper

One pencil, ruler, and protractor per student, and/or

Appropriate geometry technology

### Connections to Geometry

#### TEKS:

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(C) develops and uses formulas including distance and midpoint.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The problem, Conjecture as Discovery and Proof as Explanation, asks the students to investigate and then prove important characteristics about a triangle's midsegment. This problem investigates the midsegment for other polygons. The next problem, Why Doesn't My Conjecture Always Work?, asks students to explain the results of their investigations for the different polygons.

### Scaffolding Questions:

#### Problem 1

- If students need additional structure during the investigation, tell them to restrict their measurements to the lengths of segments and to ignore angle measures.

#### Problem 2

- This explanation does not need to be a formal, 2-column proof. Encourage students to examine a diagram of the triangle midsegment theorem and to see how this diagram applies to their quadrilateral diagram.

#### Problem 3

- Many students will assume that since other polygons have diagonals, the conjecture should “work the same way” no matter how many sides the polygon has. These investigations are a good way to get students to “listen to the math,” and to see how the geometry changes as the number of sides of the polygon increase.

#### Problem 4

- Students should see that the initial conjecture doesn't hold for pentagons; moreover, the conjecture for pentagons doesn't hold for hexagons. Encourage students to write separate conjectures for quadrilaterals, pentagons, and polygons with more than 5 sides.

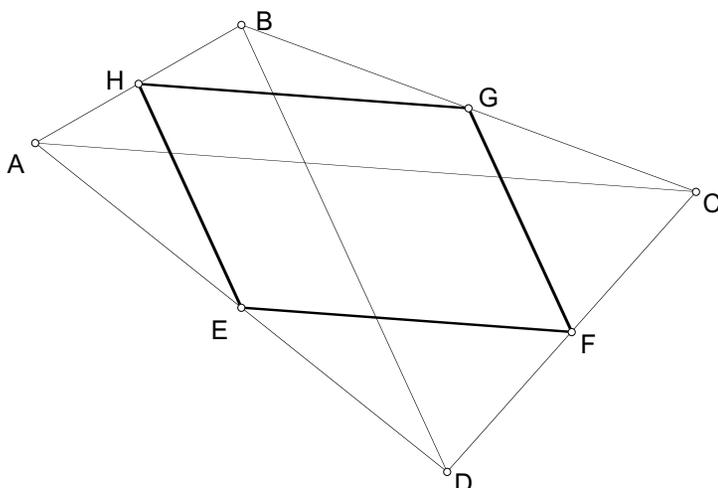


## Sample Solution:

### Problem 1

Perimeter  $p_1 = 21.86$  cm  
 $n = 9.26$  cm  
 $o = 12.59$  cm

$$n+o = 21.86 \text{ cm}$$



Conjecture: The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

### Problem 2

By the Triangle Midsegment Conjecture:

$$GF = \frac{1}{2} BD \quad HE = \frac{1}{2} BD \quad HG = \frac{1}{2} AC$$

$$EF = \frac{1}{2} AC,$$

so

$$GF + HE + HG + EF = \frac{1}{2} BD + \frac{1}{2} BD + \frac{1}{2} AC + \frac{1}{2} AC$$

$$GF + HE + HG + EF = BD + AC.$$

Therefore the perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

(2) uses ratios to solve problems involving similar figures.

### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Geometric  
Structure: Midpoint  
Quadrilateral



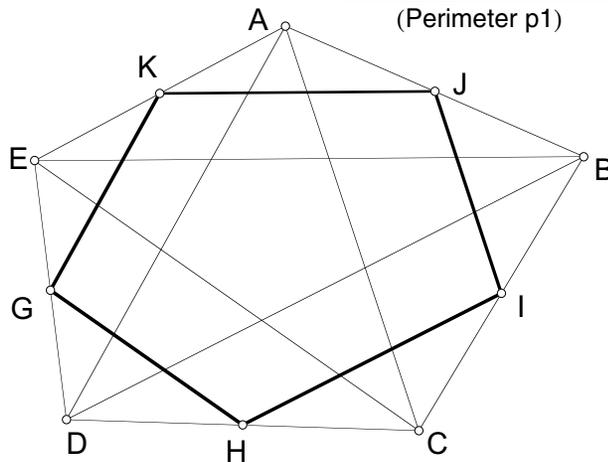
### Problem 3

For a pentagon:

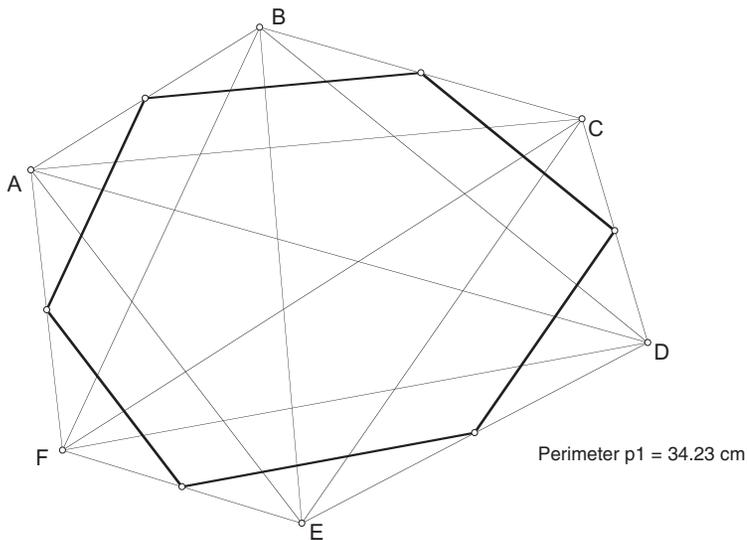
$$\text{Perimeter } p_1 = 20.24 \text{ cm}$$

$$EB+BD+DA+AC+CE = 40.47 \text{ cm}$$

$$\frac{(EB+BD+DA+AC+CE)}{(\text{Perimeter } p_1)} = 2.00$$



For a hexagon, and for polygons with more than six sides:



$$\text{Perimeter } p_1 = 34.23 \text{ cm}$$

$$AC+AD+AE+BD+BE+BF+CF+CE+DF=107.80 \text{ cm}$$

There is no relationship; the ratio between perimeter of midpoint polygon and sum of diagonals changes with different hexagons.



### Problem 4

Conjecture: The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

Conjecture: The perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

Conjecture: There does not appear to be any relationship between the perimeter of the midpoint polygon and the sums of the lengths of the diagonals for polygons with more than 5 sides.

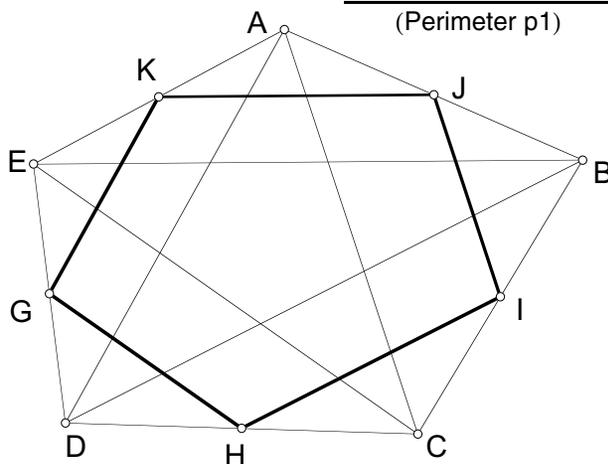
### Extension Questions:

- Use the Triangle Midsegment Conjecture to explain why your conjecture for pentagons must be true.

$$\text{Perimeter } p_1 = 20.24 \text{ cm}$$

$$EB+BD+DA+AC+CE = 40.47 \text{ cm}$$

$$\frac{(EB+BD+DA+AC+CE)}{(\text{Perimeter } p_1)} = 2.00$$



By the Triangle Midsegment Conjecture:

$$KJ = \frac{1}{2} EB \quad JI = \frac{1}{2} AC \quad IH = \frac{1}{2} BD \quad GH = \frac{1}{2} EC \quad GK = \frac{1}{2} AD,$$

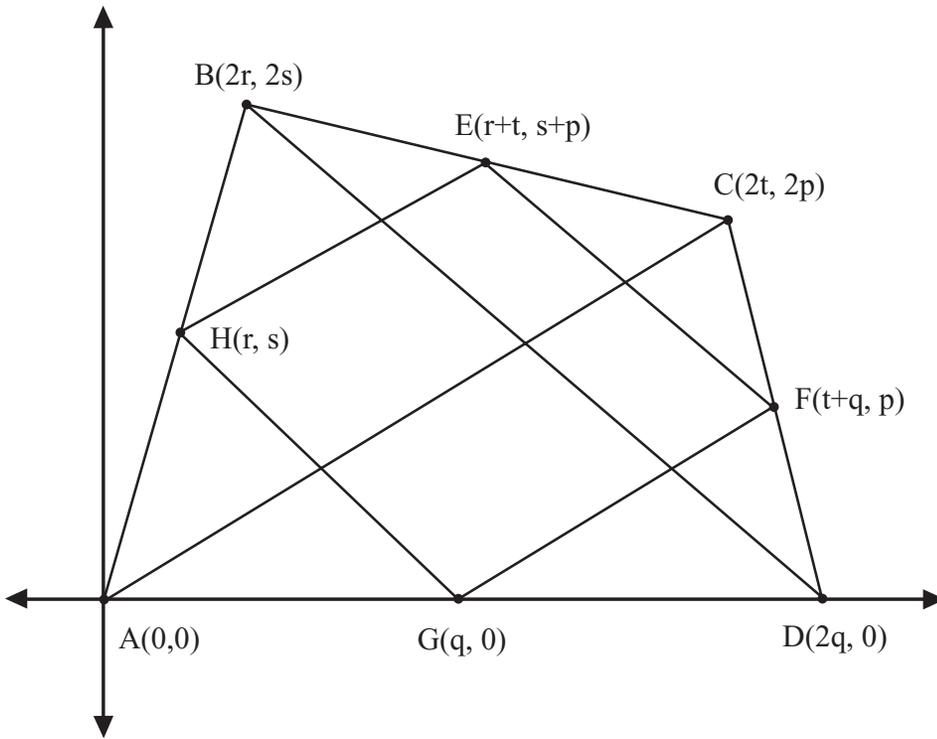
so

$$KJ + JI + IH + GH + GK = \frac{1}{2} EB + AC + BD + EC + AD.$$

Therefore the perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

- Use Coordinate Geometry to prove the quadrilateral conjecture.





$$HE = \sqrt{(r+t-r)^2 + (s+p-s)^2} = \sqrt{t^2 + p^2}$$

$$FG = \sqrt{(t+q-q)^2 + p^2} = \sqrt{t^2 + p^2}$$

$$EF = \sqrt{(r+t-t-q)^2 + (s+p-p)^2} = \sqrt{(r-q)^2 + s^2}$$

$$GH = \sqrt{(r-q)^2 + s^2}$$

$$\begin{aligned} BD &= \sqrt{(2r-2q)^2 + (2s)^2} = \sqrt{4(r-q)^2 + 4s^2} = \sqrt{4((r-q)^2 + s^2)} \\ &= 2\sqrt{(r-q)^2 + s^2} \end{aligned}$$

$$AC = \sqrt{(2t)^2 + (2p)^2} = \sqrt{4t^2 + 4p^2} = \sqrt{4(t^2 + p^2)} = 2\sqrt{t^2 + p^2}$$

$$HE + GF + EF + GH = 2\sqrt{t^2 + p^2} + 2\sqrt{(r-q)^2 + s^2}$$

$$BD + AC = 2\sqrt{t^2 + p^2} + 2\sqrt{(r-q)^2 + s^2}$$

Therefore, the perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.



## Why Doesn't My Conjecture Always Work?

In the assessment, Extending the Triangle Midsegment Theorem, you investigated the relationship between a polygon's diagonals and the perimeter of the polygon formed by connecting the midpoints of its sides.

You discovered that there is no single mathematical connection between these two things that is true for all polygons; in fact, the relationship changes as the number of the polygon's sides increases.

### For quadrilaterals:

The perimeter of the midpoint quadrilateral is equal to the sum of the lengths of the diagonals.

### For pentagons:

The perimeter of the midpoint pentagon is one-half the sum of the lengths of the diagonals.

### For polygons with more than 5 sides:

There does not appear to be any relationship between the perimeter of the midpoint polygon and the sums of the lengths of the diagonals for polygons with more than 5 sides.

You also used the previously proven Triangle Midsegment Conjecture to explain why the conjectures for quadrilaterals and for pentagons must be true.

### Problem 1

Use what you have learned in previous investigations to explain why the relationship between diagonals and midpoint polygon is different for pentagons than it is for quadrilaterals.

Provide a written explanation with diagrams.

### Problem 2

Use what you have learned in previous investigations to explain why there is no relationship between diagonals and midpoint polygons for shapes with more than 5 sides.

Provide a written explanation with a diagram(s).



# Teacher Notes

## Materials:

One pencil, ruler, and protractor per student and/or appropriate geometry technology

## Connections to Geometry

### TEKS:

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

## Scaffolding Questions:

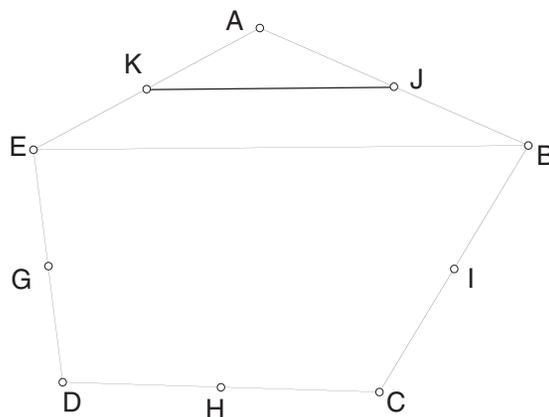
- What determines the number of times you can apply the Triangle Midsegment Conjecture to a shape?
- What is the relationship between the segment drawn between the midpoints of consecutive sides and the diagonal drawn between two endpoints of consecutive sides?
- Complete the chart:

number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4		
5		
6		

## Sample Solution:

### Key Concepts:

Every time a segment is drawn between the midpoints of consecutive sides of a polygon, it creates the midsegment of a triangle. This triangle's two sides are the consecutive sides of the polygon, and its third side is the diagonal drawn between the endpoints of the two consecutive sides.



For example, segment  $\overline{KJ}$  is the midsegment of the triangle formed by consecutive sides  $\overline{AE}$  and  $\overline{AB}$  and diagonal  $\overline{EB}$ .

The Triangle Midsegment Conjecture applies to the geometry of this situation, since it states that  $KJ = \frac{1}{2} EB$ .

When a midpoint polygon is formed, each of its sides is the midsegment of a triangle, and its length is therefore one-half the length of one of the original polygon's diagonals.

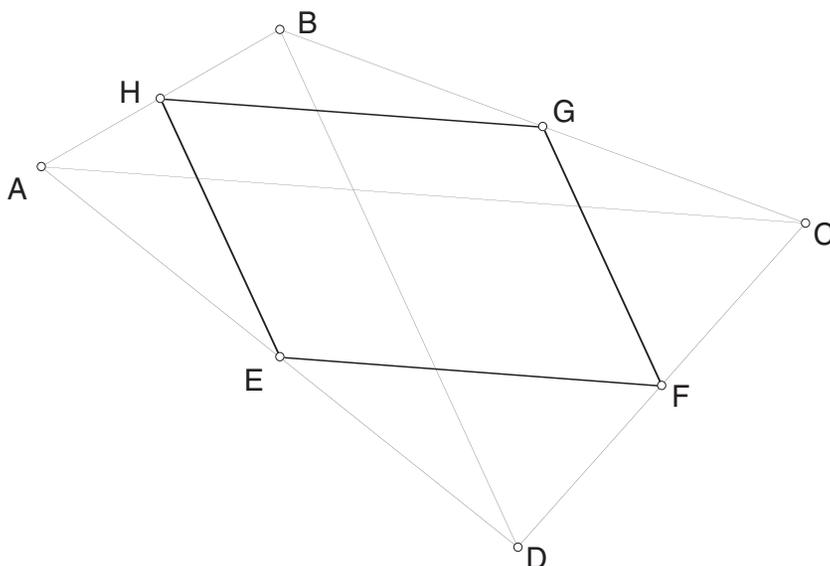
The reason that the relationship between the perimeter of a midpoint polygon and length of diagonals does not stay constant is that the number of times a diagonal is matched to a midsegment does not stay constant as the number of sides of the polygons increases. For some polygons there are more diagonals than sides to the midpoint polygon.

### Problem 1

In a quadrilateral:

Perimeter HGFE = 21.86 cm  
 $m \overline{BD} = 9.26$  cm  
 $m \overline{AC} = 12.59$  cm

$$(m \overline{BD}) + (m \overline{AC}) = 21.86 \text{ cm}$$



There are two diagonals and four triangle midsegments. So each diagonal gets matched with two different triangle midsegments. For example, diagonal  $\overline{AC}$  is matched with midsegment  $\overline{HG}$  and midsegment  $\overline{EF}$ .

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines; and

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts.

### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Midpoint Quadrilateral
- II. Structure: Midpoint Triangle



In a quadrilateral, the Triangle Midsegment Conjecture is applied to 4 different triangles corresponding to the lengths of only 2 different diagonals. Each diagonal, therefore, is counted twice.

Thus, in a quadrilateral:

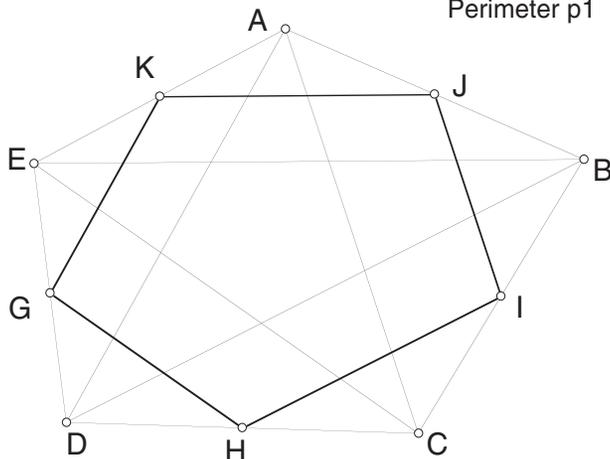
$$\text{perimeter of the midpoint polygon} = 2\left(\frac{1}{2}\right) (\text{sum of diagonal lengths}).$$

In a pentagon, there are five diagonals and exactly five triangle midsegments. Each midsegment gets matched with one and only one diagonal.

$$\text{Perimeter } p_1 = 20.24 \text{ cm}$$

$$EB + BD + DA + AC + CE = 40.47 \text{ cm}$$

$$\frac{EB + BD + DA + AC + CE}{\text{Perimeter } p_1} = 2.00$$



Therefore:

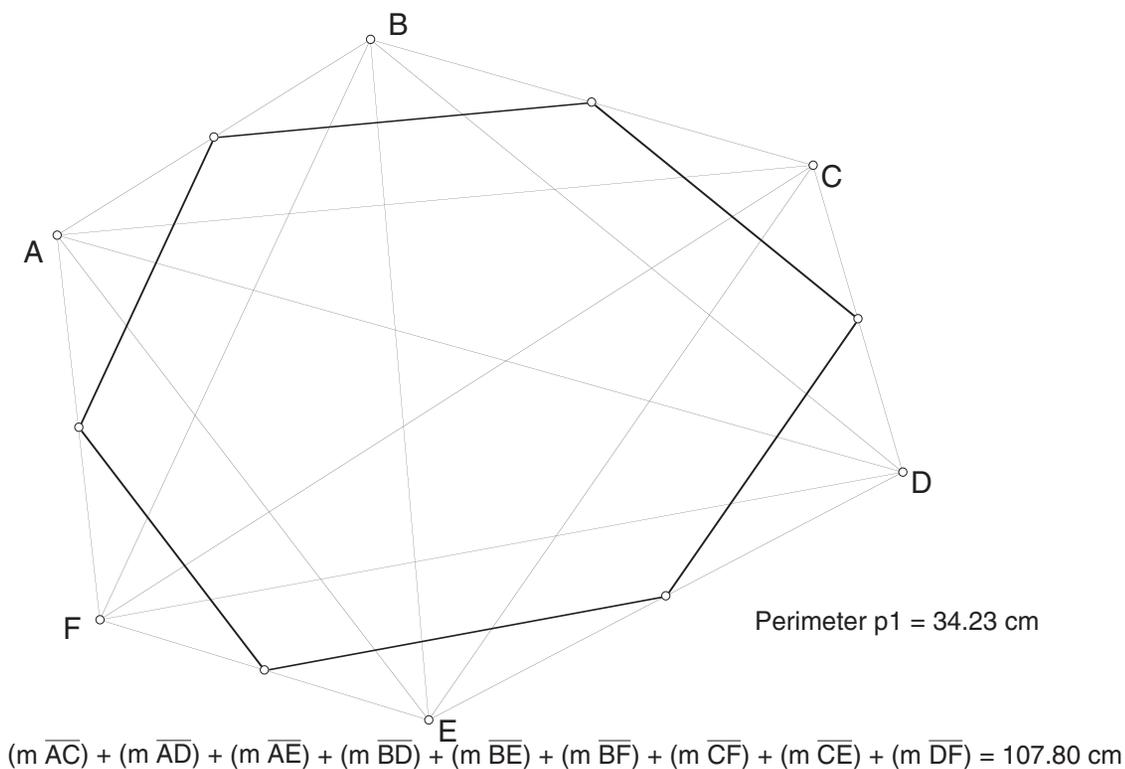
$$\text{perimeter of the midpoint polygon } (p_1) = \frac{1}{2} (\text{sum of diagonal lengths.})$$



## Problem 2

For hexagons and polygons with more than 5 sides:

Since the number of diagonals is more than the number of sides of the midpoint polygon, some diagonals get matched with a triangle midsegment, and some don't.



In the hexagon above, diagonals  $\overline{AC}$ ,  $\overline{BD}$ ,  $\overline{CE}$ ,  $\overline{DF}$ , and  $\overline{AE}$  are matched to triangle midsegments. The remaining three diagonals,  $\overline{AD}$ ,  $\overline{BE}$  and  $\overline{CF}$  are not matched to a diagonal.

By the Midsegment Conjecture:

$$\text{perimeter of the midpoint polygon} = \frac{1}{2} (AC + BD + CE + DF + AE)$$

The conjecture, however, has nothing to say about the remaining four diagonals. Therefore, it cannot be used to establish a relationship between the perimeter of the midpoint polygon and the sum of the length of the polygon's diagonals.

The situation is the same for polygons with more sides than hexagons.



### Extension Questions:

- Modify your conjecture about the relationship between the perimeter of a midpoint polygon and the sum of the length of the polygon's diagonals so that it applies to polygons with 4 or more sides.

*For polygons with 5 or more sides:*

*The perimeter of the midpoint polygon is one-half the length of the sum of the lengths of the diagonals formed by connecting the endpoints of consecutive sides of the original polygon.*

- Prove your conjecture for problem 1.

*By the Triangle Midsegment Conjecture, each segment connecting two midpoints is one-half the length of the diagonal connecting the endpoints of 2 consecutive sides. Since there are the same number of midpoint segments as there are diagonals:*

*perimeter of the midpoint polygon =  $\frac{1}{2}$  (sum of lengths of consecutive sides diagonals.)*

- Complete the chart to find a formula for the number of diagonals of a polygon in terms of  $n$ , the number of sides.

number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4		
5		
6		
$n$		



number of sides of polygon	number of sides of midpoint polygon	number of diagonals
4	4	2
5	5	5
6	6	9
$n$		

$$\text{number of diagonals} = \frac{n(n-1)}{2} - n$$





## Steiner's Point

A developer wants to locate sites for three campgrounds and to build a system of roads to connect them. To accomplish this economically, he needs to find the shortest path between the three campgrounds.

A mathematician named Jakob Steiner investigated problems like this during the nineteenth century. He found that the minimum path connecting three nonlinear points is a line segment from each of the vertices of the triangle formed by the three points. These three line segments meet at a single point in the interior of the triangle. Steiner proved that each segment of the minimum path forms a  $120^\circ$  angle with the other segments it intersects.

### The Situation

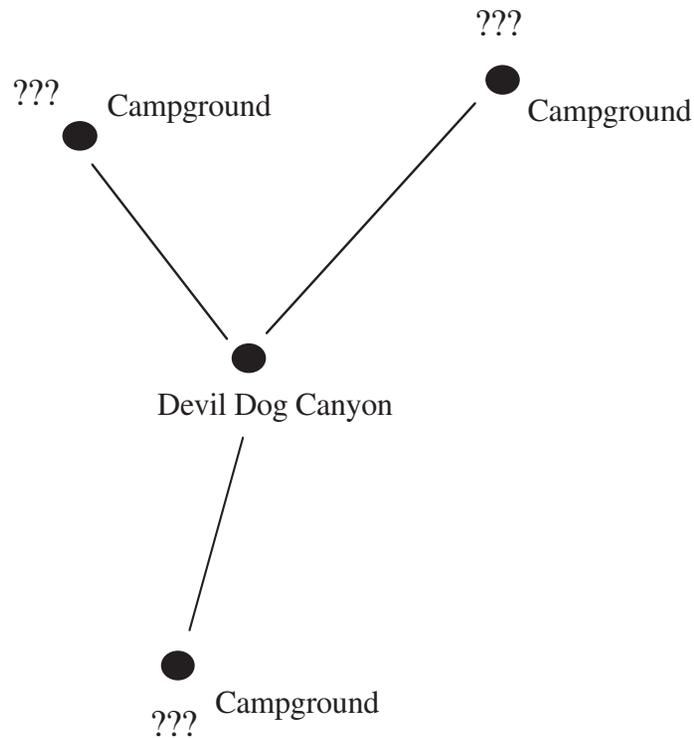
A developer wants to locate sites for three campgrounds so that campers have access to Devil Dog Canyon, a major tourist attraction, by the shortest possible route. Bill Steiner, the developer, has a budget which will allow him to build 18 miles of roads connecting the campgrounds to the canyon. Campground B, his high-end luxury camping resort, will have trams shuttling tourists to Devil Dog Canyon, so Steiner wants the road leading to it to be the shortest. Campground A, on the other hand, is a primitive camping area with plenty of privacy, so the road to this campground should be the longest. Finally, budgeting, payroll, and scheduling are all much simpler if the roads from Devil Dog Canyon to the campgrounds are each an integer number of miles in length.



## Problem 1

### Ruler and Protractor

Where should Steiner locate the campgrounds so that all the conditions are satisfied?



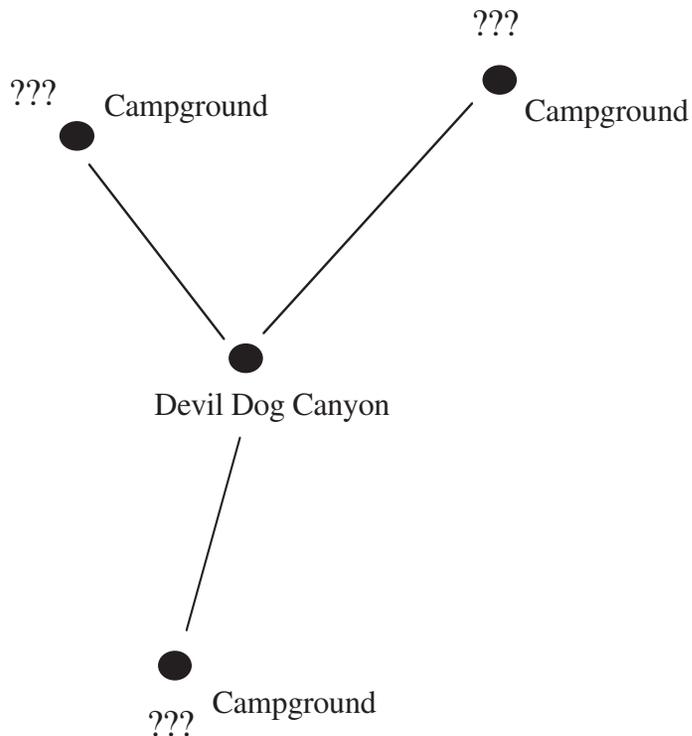
- Use a ruler and protractor to develop a possible model to represent this situation.
- Mark the position of Devil Dog Canyon and of the three campgrounds on your model.
- Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.



## Problem 2

### Compass and Straightedge

Where should Steiner locate the campgrounds so that all the conditions are satisfied?



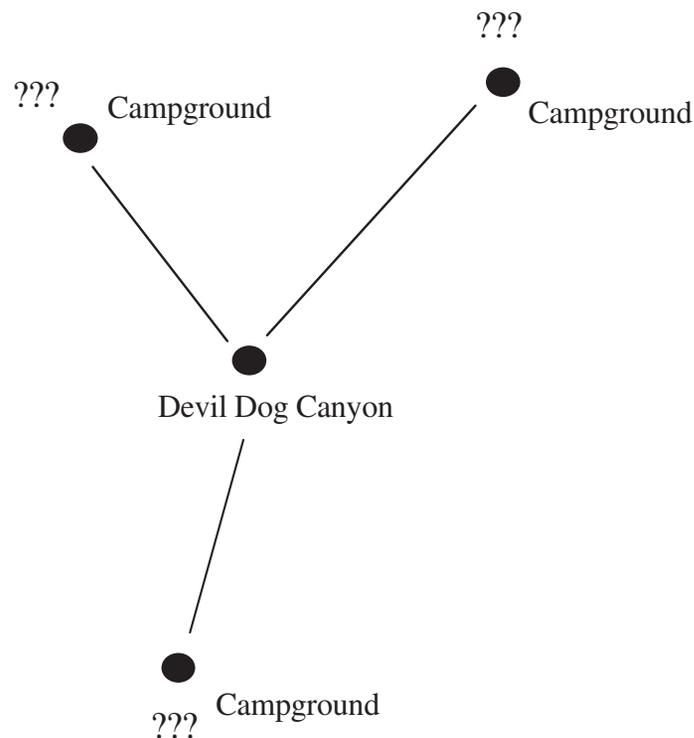
- Use a compass and straightedge to develop a possible model to represent this situation
- Mark the positions of Devil Dog Canyon and of the three campgrounds on your model.
- Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.



### Problem 3

#### Graph Paper and a Coordinate System

Where should Steiner locate the campgrounds so that all the conditions are satisfied?



- Develop a model of this situation using graph paper and an  $x,y$  coordinate system.
- Locate the positions, and label the numerical coordinates of Devil Dog Canyon and of the three campgrounds on your model. (Note: where appropriate, express coordinates as exact values.)
- Provide a written explanation of how your model represents the situation, and explain how your campground sites fulfill the requirements given in the problem.





**Materials:**

Paper and graph paper

One ruler, protractor, compass, and straightedge per student

**Connections to Geometry TEKS:**

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

## Teacher Notes

Jakob Steiner (1796-1863) was a well-known geometer. He was born in Bern, Switzerland, but spent most of his life in Germany and was a professor at the University of Berlin. He is most famous for his contributions to projective geometry. Another one of his famous theorems shows that only one given circle and a straight edge are required for Euclidean constructions. It is said that Steiner did not like algebra and analysis and felt that calculation replaces thinking while geometry stimulates thinking.<sup>1</sup>

**Scaffolding Questions:****Problem 1**

- What are some possible integer lengths of the roadways leading from the canyon to each of the campgrounds?
- Why does  $AD + BD + DC$  represent the minimum path between the three campgrounds?

**Problem 2**

- What construction(s) could you use to make three  $120^\circ$  angles at point D?
- What construction(s) could you use to mark off distances from the canyon (point D) to the three campgrounds so that BD is the shortest and DA is the longest?
- In marking off these distances to meet the problem's requirements, which segment should you use as your unit length? Why?
- Why does  $AD + BD + DC$  represent the minimum path between the three campgrounds?

**Problem 3**

- Which point in the model would be the best choice to locate at the origin? Why?

1. University of St. Andrews, St. Andrews Scotland. (1996, December). Retrieved August 8, 2002 from <http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Steiner.html>.

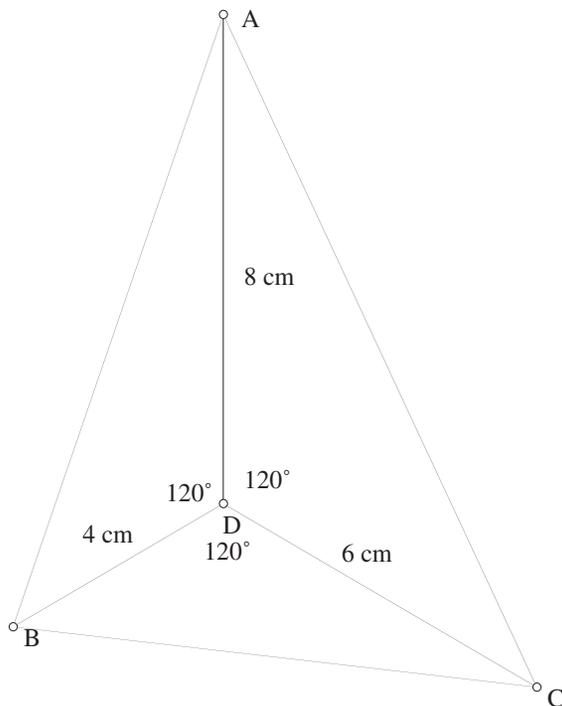


- Is it possible to orient the vertices of your triangle so that the distance from one of the campgrounds to the canyon lies along one of the axes?
- (If above is accomplished) What are the angle measures between the  $x$  axis and the segments from the canyon to each of the two other canyons?
- How can you use the properties of 30-60-90 right triangles to determine the coordinates of points B and C?
- For campgrounds B and C, which coordinate,  $x$  or  $y$ , corresponds to the short leg of the 30-60-90 right triangle? Which coordinate,  $x$  or  $y$ , corresponds to the long leg of the 30-60-90 right triangle?
- Why does  $AD + BD + DC$  represent the minimum path between the three campgrounds?

### Sample Solutions:

#### Problem 1

There are many possible solutions.



(Note: Distances from D to triangle vertices can be any three integers that sum to 18, such that  $DB < DC < DA$ .)



(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures.

#### Texas Assessment of Knowledge and Skills:

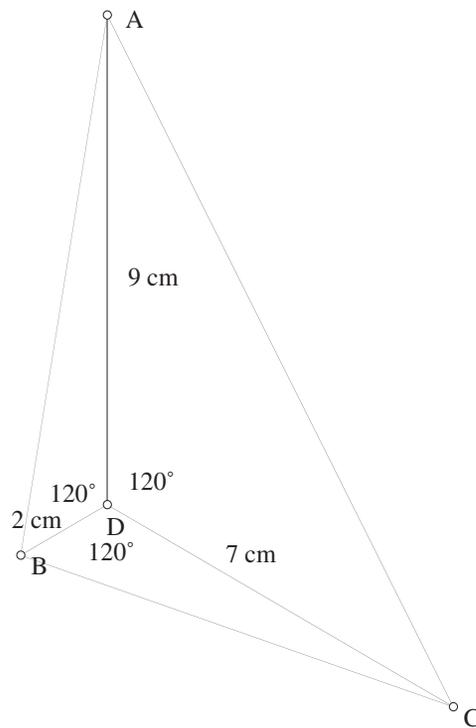
Objective 7: The student will demonstrate an understanding of two- and three-dimensional representation of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

#### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- II. Transformations: Rotations
- III. Triangles: Pythagorean Theorem

Another possible solution:



c) Here is one possible explanation for this last example:

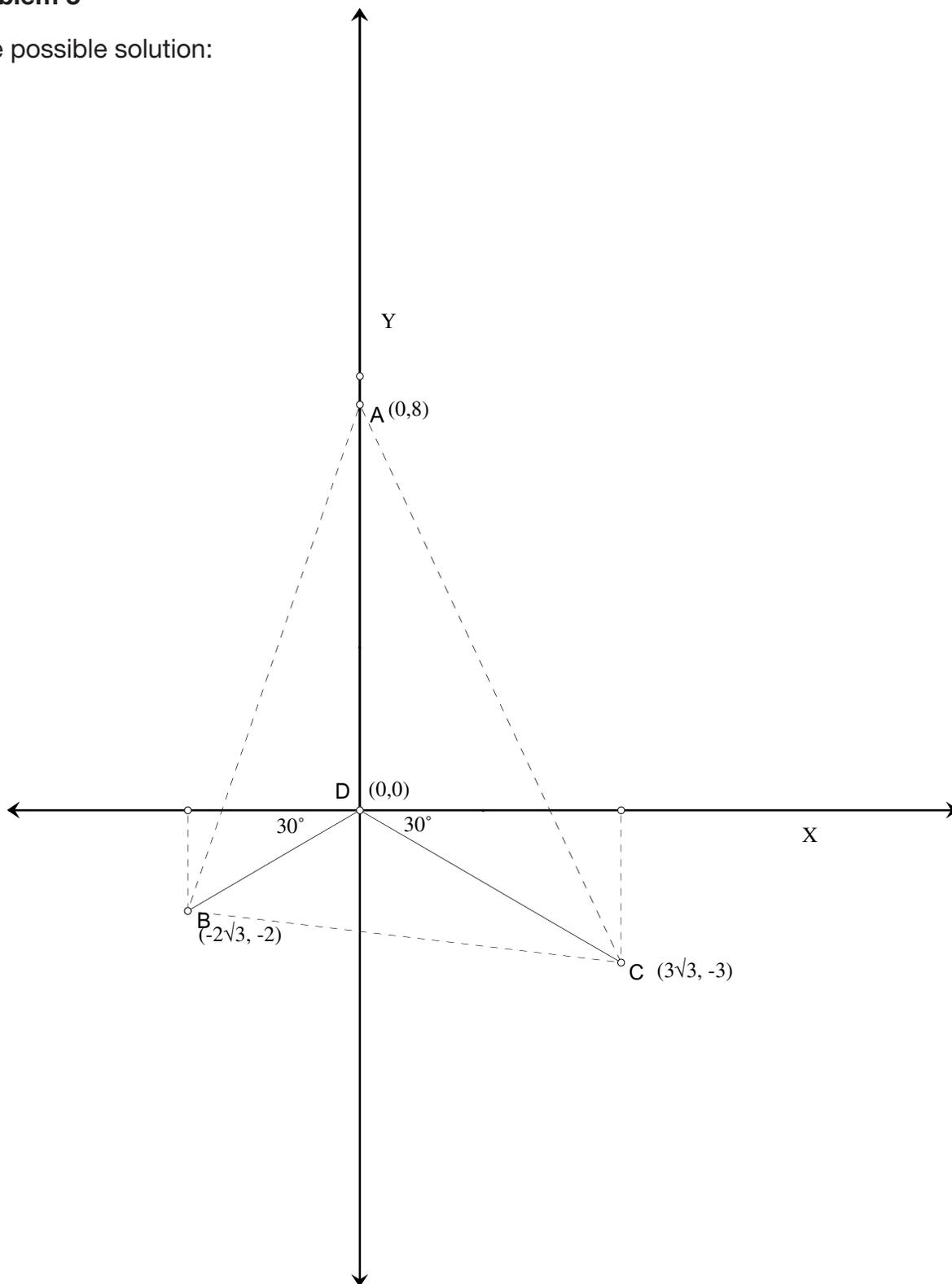
$\overline{DA}$  was drawn with length 9 centimeters. At point D the two 120-degree angles were measured. The third angle will also be 120 degrees.  $\overline{DB}$  was measured 2 centimeters, and  $\overline{DC}$  was measured 7 centimeters. The sum of the segments is  $9 + 2 + 7$  or 18 centimeters. Point D meets the requirements for Steiner's minimum path because each path forms a 120 degree angle at D.





### Problem 3

One possible solution:

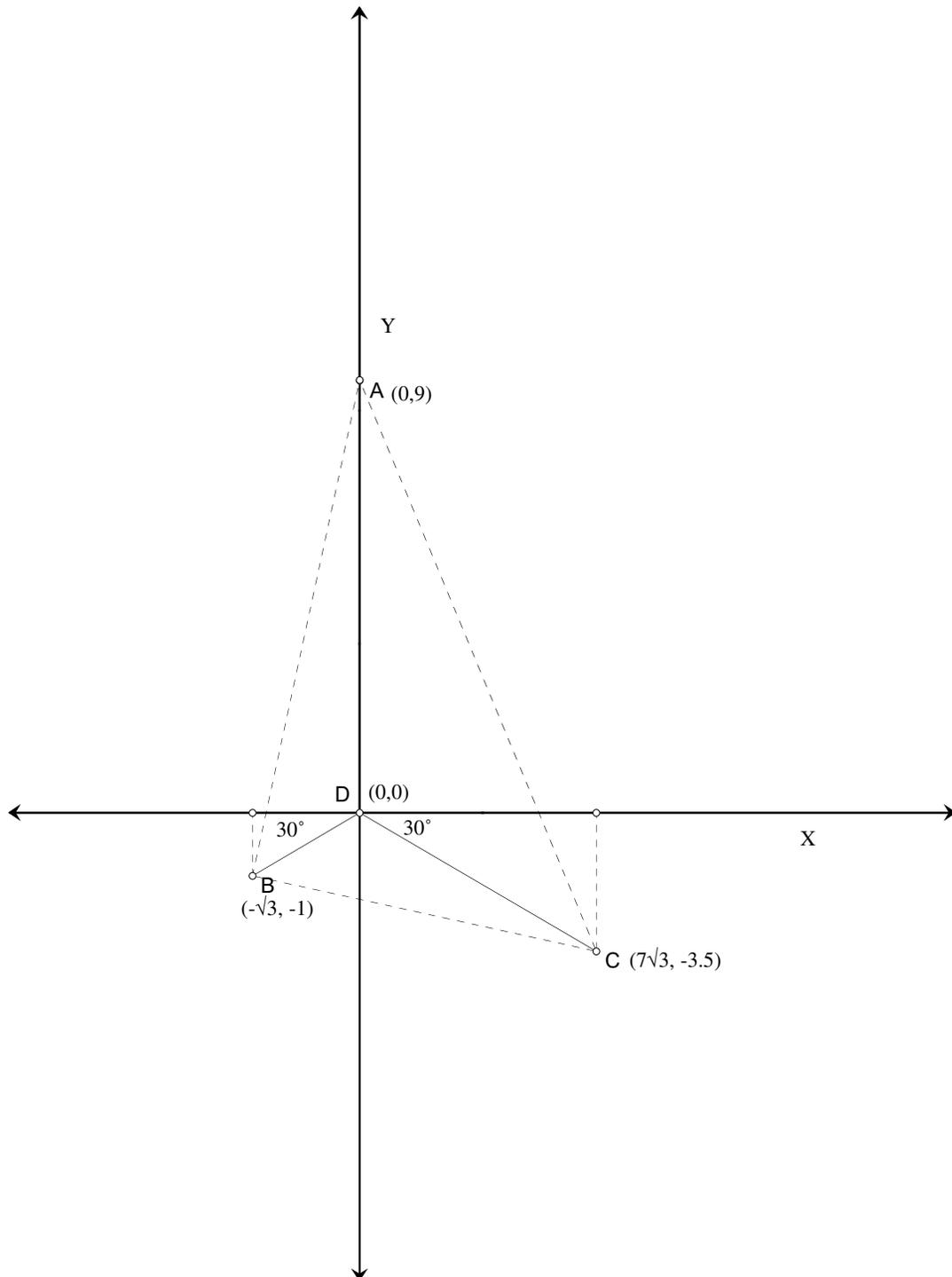


$$(-2\sqrt{3}, -2) \approx (-3.5, -2)$$

$$(3\sqrt{3}, -3) \approx (5.2, -3)$$



Another possible solution:



c) Consider the first solution.

Decimal approximations may be used to graph the points  $A(0,8)$ ,  $B(-3.5,-2)$ , and  $C(5.2,-3)$ . The measure of angle  $ADB$  is  $(90 + 30)$  or  $120$  degrees. The measure of angle  $ADC$  is  $(90 + 30)$  or  $120$  degrees. The length of  $\overline{DB}$  is  $2(2)$  or  $4$ . The length of  $\overline{DC}$  is  $2(3)$  or  $6$ . The sum of the distances  $BD + CD + AD$  is  $4 + 6 + 8$  or  $18$  units. The points  $A$ ,  $B$ , and  $C$  meet the requirements of Steiner's minimum path.

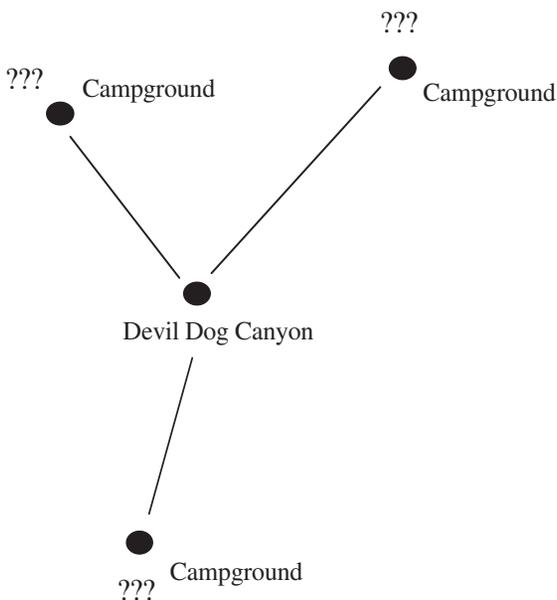




## Steiner's Point Revisited

In your work on the Steiner's Point assessment, you developed geometric models for locating campground sites. There you used ruler and protractor, compass and straightedge, and coordinate geometry.

Where should Steiner locate the campgrounds so that all the conditions are satisfied?



The mathematician George Steiner proved that the minimum path connecting three points is a line segment from each of the vertices of the triangle formed by the three points. In addition, these three line segments form a  $120^\circ$  angle with each other at their point of intersection in the triangle's interior.

You may want to experiment by having the three segments intersect at angles other than  $120^\circ$  (representing the campground problem using geometry software is useful for this). After some trial and error, it should be evident that the 18-unit path with lines intersecting at  $120^\circ$  is indeed the minimum path between the three campgrounds.

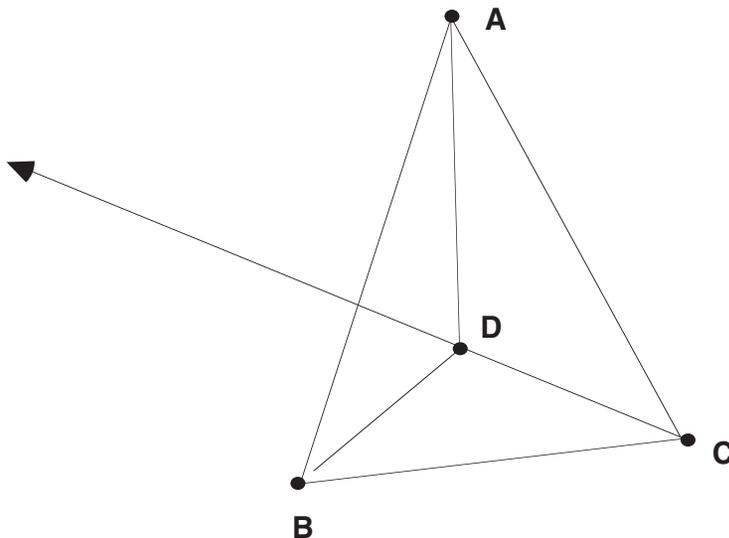
Your task is to use your models to explain why.



## The Setup

Use your Ruler and Protractor Model and your Coordinate Geometry Model.

Draw a ray from one of the vertex points of the triangle, making the ray intersect the point in the triangle's interior, which represents the canyon, and then continuing through the triangle's exterior.



## Problem 1

Ruler and Protractor Model

- a) Use transformations to locate and label point F on the ray so that it satisfies the following requirement:

The distance from F to the ray's endpoint = Distance from Campground A to canyon + Distance from Campground B to canyon + Distance from Campground C to canyon.

(Note:  $FC = 18$  units. How can you use transformations to get the images of  $\overline{BD}$  and  $\overline{AD}$  to lie adjacent to each other on the ray?)

- b) Provide a written explanation of how you used transformations to locate point F.



## Problem 2

### Explain Your Thinking

Write a paragraph explaining how the path between the campgrounds and the canyon must be the minimum path between these points. Use your findings from the previous problem as support for your argument.

## Problem 3

### Coordinate Geometry Model

- a) Use your knowledge of special right triangles to locate and label the exact numerical coordinates of point F on the ray so that it satisfies the following requirement:

The distance from F to the ray's endpoint = Distance from Campground A to canyon + Distance from Campground B to canyon + Distance from Campground C to canyon.

- b) Provide a written explanation of how you used special right triangles to find the coordinates of point F.



# Teacher Notes

## Materials:

Graph paper

One pencil, ruler, protractor, and compass per student

## Connections to Geometry TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

## Scaffolding Questions:

### Problem 1

- If we know that  $FC = 18$  cm, how is it possible to use transformations to get images of segments  $\overline{BD}$  and  $\overline{AD}$  to lie adjacent to each other on segment  $\overline{FC}$ ?
- Why would it be advantageous to do this?
- What shape contains both segment  $\overline{BD}$  and segment  $\overline{AD}$ ?
- How many degrees, and around what point, would  $\triangle ADB$  have to rotate in order for the image of segment  $\overline{AD}$  to lie on ray  $\overline{CD}$ ?
- Why is  $\triangle BDD'$  an equilateral triangle?

### Problem 2

- What is the shortest path between points F and C?
- Express  $\overline{FC}$  as the sum of the lengths of the three segments that compose it.
- Which 2 of the 3 segments composing  $\overline{FC}$  are transformed images of segments representing roads from the canyon to the campgrounds?
- Express  $\overline{FC}$  as the sum of the lengths of the three segments leading from the canyon to the campgrounds.

### Problem 3

- What must the length of segment  $\overline{FD}$  be?
- Explain how to determine the measure of angle FDM.
- For point F, what coordinate,  $x$  or  $y$ , corresponds to the short leg of 30-60-90 right  $\triangle FDM$ ? What coordinate,  $x$  or  $y$ , corresponds to the long leg of 30-60-90 right  $\triangle FDM$ ?

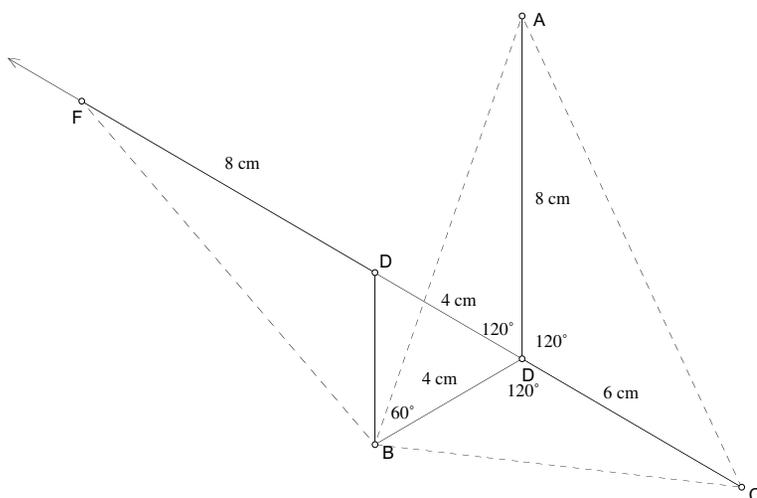


### Sample Solution:

#### Problem 1

This is one possible solution.

a)

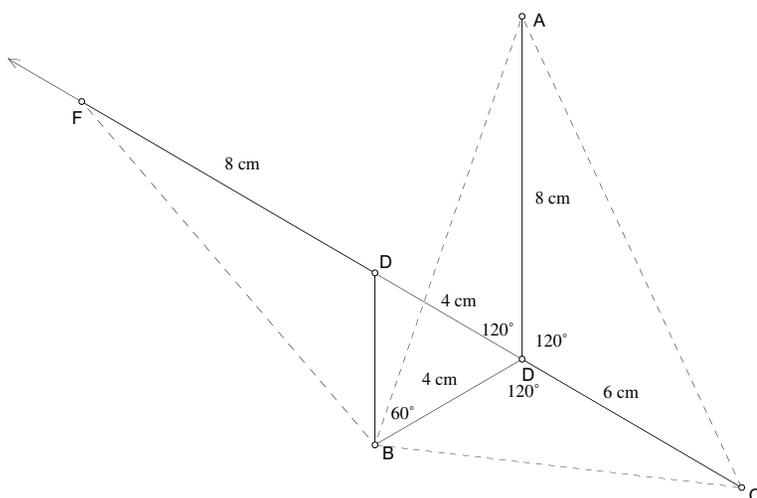


b) Rotate  $\triangle ADB$   $60^\circ$  around point B.  $FD' = AD = 8$  cm  
 $BD = BD' = 4$  cm This makes  $\triangle BDD'$  equilateral.  
Therefore  $DD' = 4$  cm.

$$FC = AD + DB + DC = 18 \text{ cm.}$$

#### Problem 2

For example, using the Ruler and Protractor Model, the minimum distance path between points F and C is the straight line segment FC.



#### Texas Assessment of Knowledge and Skills:

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

#### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

II. Transformations: Rotations

$$FC = FD' + D'D + DC.$$

But by preservation of length under transformations,

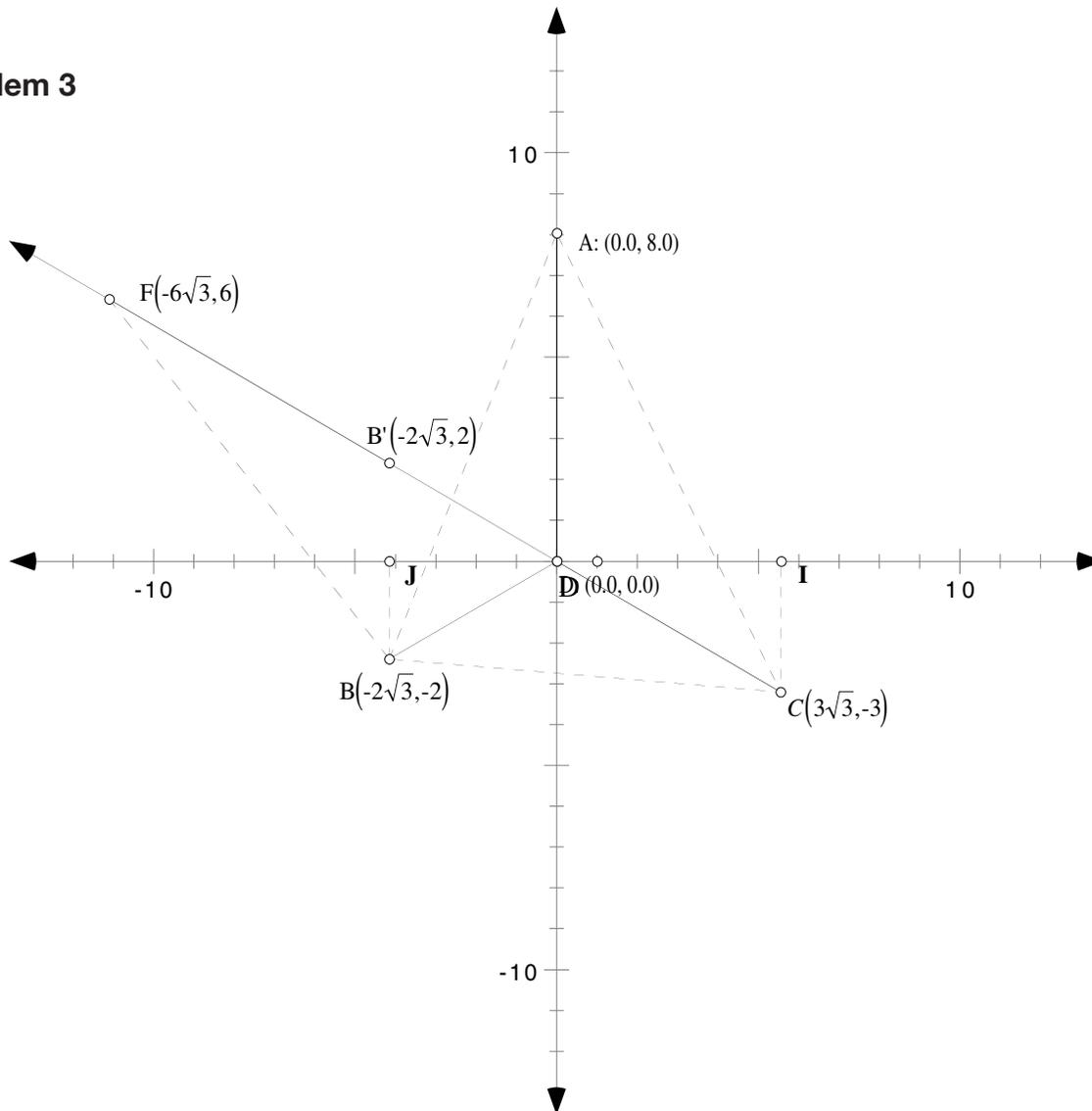
$FD' = AD$  and  $D'D = BD$ , so by substitution,

$$FC = AD + BD + DC.$$

Since  $FC$  is the shortest distance between  $F$  and  $C$ , it therefore, must be the minimum path between points  $A$ ,  $B$ , and  $C$ .

### Problem 3

a)



- b)  $FD = FC - DC = 18 - 6 = 12$ .  $\overline{FD}$  is the hypotenuse of 30-60-90 right triangle  $DMF$ . The short leg of this triangle corresponds to the  $y$  coordinate of point  $F$  and the long leg corresponds to the  $x$  coordinate of point  $F$ . Since the hypotenuse is 12, the short leg is 6, and the long leg is  $6\sqrt{3}$ . Therefore the coordinate of point  $F$  is  $F(-6\sqrt{3}, 6)$ .





b) Rotate point  $A$  about point  $D$  until the rotated point intersects the ray. Label the intersection  $A'$ .  $DA = DA'$ .

Construct a line parallel to ray  $\overrightarrow{DC}$  through point  $B$ . Translate point  $D$  distance  $DA'$  along segment  $\overline{DA'}$ . Translate point  $B$  distance  $DA'$  the parallel line through point  $B$ .  $B'D' = BD$ .

Rotate point  $B'$  about point  $A'$  until the rotated point intersects the ray.

Mark the intersection point  $F$ .  $FA' = B'D' = BD$ .

$FC = AD + BD + DC$ .

- In the solution to the first Extension Question, how is it possible to get the transformed image of segment  $BD$  to lie adjacent to segment  $DA'$  without constructing a line parallel to ray  $\overrightarrow{CD}$  through point  $B$ ?

Rotate point  $B$  about point  $D$  until  $B'$ , the image point, intersects ray  $\overrightarrow{CD}$ . Then translate  $B'$  distance  $DA'$  along ray  $\overrightarrow{CD}$ .



## Walking the Archimedean Walk

Most geometry students know where the value of  $\pi$  comes from—their calculators. Hopefully, most geometry students also realize that the number their calculator gives them is really an approximation of the value of  $\pi$ —the constant ratio between a circle’s circumference and its diameter:

$$\pi = \frac{C}{d}$$

During the course of human history, diverse cultures throughout the world were aware of this constant ratio. The attempt to fix its exact value has been a vexing problem that has occupied many mathematical minds over the centuries. (See, for example, *A History of Pi* by Petra Beckman, St. Martin’s Press, 1971.)

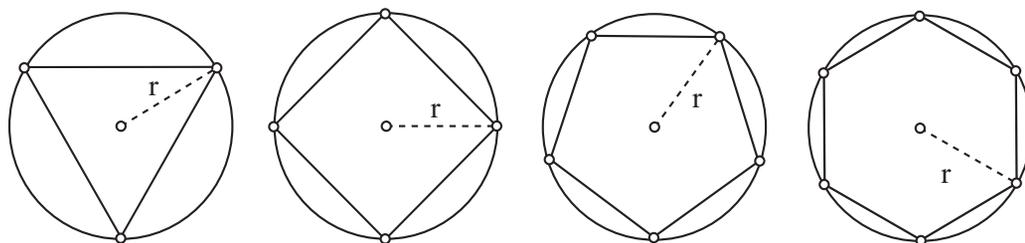
In our Western cultural tradition, the historical record tells us that Archimedes was the first person to provide a mathematically rigorous method for determining the value of  $\pi$ .

In this assessment you will have to retrace his footsteps in order to demonstrate a solid understanding of where that number comes from when you push the “ $\pi$ ” button on your calculator.

A logical starting place for determining  $\pi$  is to measure the circumferences and diameters of many circles and then calculate the ratio  $C/d$  based upon those measurements. You may have done a measurement activity like this in your geometry class. Archimedes realized that this method would always be limited by the precision of the people doing the measuring and by the accuracy of the measuring devices they were using. He sought a way to fix the value of  $\pi$  that was based upon direct calculation rather than upon measurement.



Archimedes' approach involved inscribing regular polygons in circles. He then considered what " $\pi$ " would be for each of the inscribed regular polygons. You will model his approach by examining the figures below in the problems for this assessment.



### Problem 1

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe an equilateral triangle in a circle with a radius of 2 units.
2. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed equilateral triangle.

### Problem 2

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe a square in a circle with a radius of 2 units.
2. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed square.

### Problem 3

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to inscribe a regular hexagon in a circle with a radius of 2 units.
2. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed regular hexagon.



### Problem 4

1. Complete the table below, and summarize your findings.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter	Diameter	Approximation for $\pi$
3			4	
4			4	
6			4	

2. Does this method overestimate or underestimate the value of  $\pi$ ? Will this method result in an exact value for  $\pi$ ?
3. Write a few sentences and provide a diagram to answer question 2.

### Problem 5

Write a few sentences explaining the basics of the method Archimedes used.



## Teacher Notes

### Materials:

Construction tools, circular geoboards, circular dot paper, or geometry software

### Connections to Geometry TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(B) through the historical development of geometric systems, recognizes that mathematics is developed for a variety of purposes.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

The assessment, Talk the Archimedean Talk, is an extension of this problem. It requires the student to repeat this activity for a dodecagon.

### Scaffolding Questions:

#### Problems 1, 2, and 3

- What special right triangle is formed by the radius and the apothem of the inscribed polygon?

#### Problem 4

- Will the perimeter of the inscribed polygon be greater than, less than, or equal to the circumference of the circle?
- Will the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  be greater than, less than, or equal to the ratio  $\frac{\text{circumference}}{\text{diameter}}$  ?
- Will the perimeter of the inscribed polygon ever be equal to the circumference of the circle?

#### Problem 5

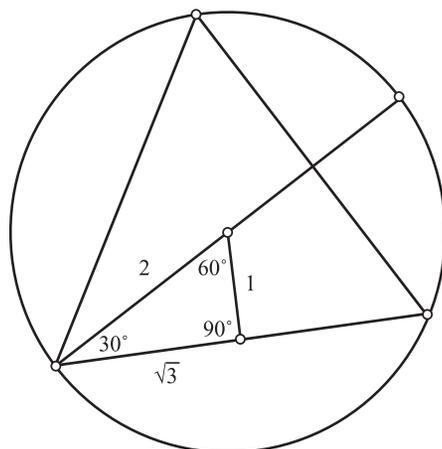
- If you consider the inscribed regular polygon to be a “primitive circle,” what measure in the polygon corresponds to the circumference of the circle?
- If you consider the inscribed regular polygon to be a “primitive circle,” what measure in the polygon corresponds to the diameter of the circle?
- What will happen to the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  as the number of sides of the polygon increases?



## Sample Solutions:

### Problem 1

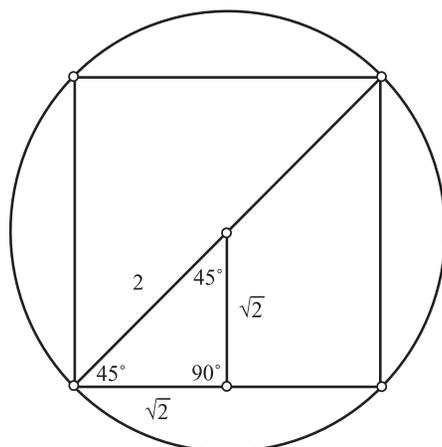
1.



$$2. \frac{\text{perimeter}}{\text{diameter}} = \frac{6\sqrt{3}}{4} = \frac{3\sqrt{3}}{2} \approx 2.5981$$

### Problem 2

1.



$$2. \frac{\text{perimeter}}{\text{diameter}} = \frac{8\sqrt{2}}{4} = 2\sqrt{2} \approx 2.8284$$

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

#### Teacher's Comment:

"I introduced it to my students as a Gold Medal problem (which they knew required a written, detailed response). They worked on it independently for 30 minutes, and then they had an opportunity to work in a group setting.

Many students extended the problem naturally to see how close they could come to  $\pi$ .

I used geoboard circular dot paper and that worked well. The students saw the natural progression to a dodecagon."

#### Student's Comment:

"This problem looked very complex before I started, but when I began working, it helped create itself and turned out to be relatively simple with the proper procedures. I learned to think differently and used sub-problems to meet my conclusions. I don't believe that I've ever done a similar problem, but I know that if I do, I will know how to start. The sub-problems really made it easy for me to follow where the problem led."

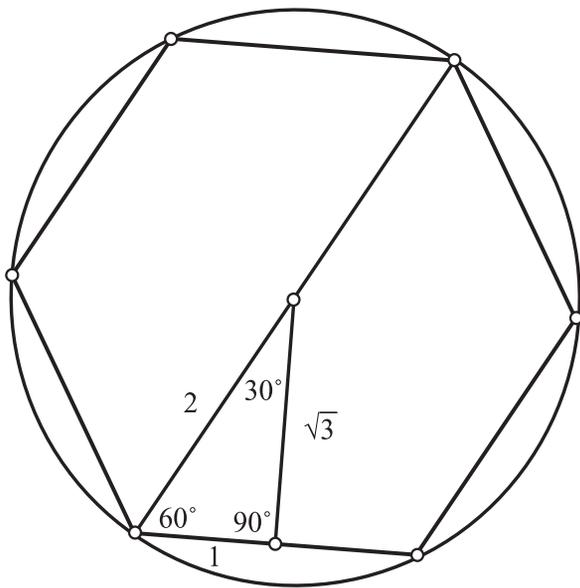
#### Student's Comment:

"Archimedes must have been a really smart man and must have had a lot of time on his hands to figure out this problem, especially without calculators."



### Problem 3

1.



2.  $\frac{\text{perimeter}}{\text{diameter}} = \frac{12}{4} = 3$

### Problem 4

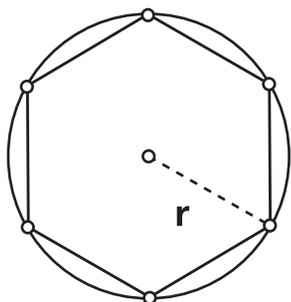
1.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter	Diameter	Approximation for $\pi$
3	$120^\circ$	$6\sqrt{3}$	4	2.5981
4	$90^\circ$	$8\sqrt{2}$	4	2.8284
6	$60^\circ$	12	4	3.000

2. This method underestimates the value of  $\pi$ . The perimeter of the inscribed regular polygon is always less than the circumference of the circle it is inscribed within.



3.

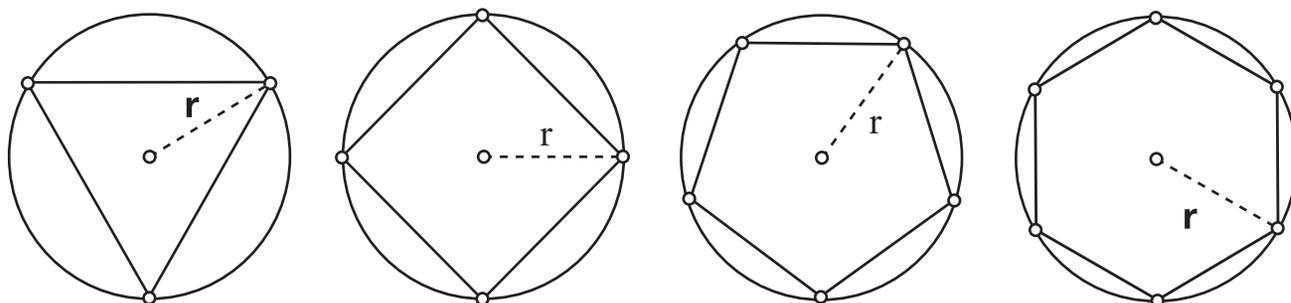


The perimeter of the polygon is always less than the circumference of the circle, but the diameters for both polygon and circle are equal. This means that

the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  is less than the ratio  $\frac{\text{circumference}}{\text{diameter}}$ .

As the number of sides of the polygon increases, the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  will get closer and closer to the value of  $\pi$ . It will never result in the exact value for  $\pi$  because no matter how many sides the inscribed polygon has, its perimeter will always be less than the circumference of the circle with the same center and radius.

### Problem 5



Each inscribed regular polygon can be considered an approximation of the circle it is inscribed within. Consider the diameter of a polygon to be twice the radius of the inscribed circle. Therefore, the ratio of the polygon's perimeter to its "diameter" can be considered an approximation of the ratio of the circle's circumference to its diameter,  $\pi$ . As the number of sides in the regular polygon increases, the ratio; perimeter/diameter, gets closer and closer to the value of  $\pi$ . The method yields successively more accurate approximations of  $\pi$ .



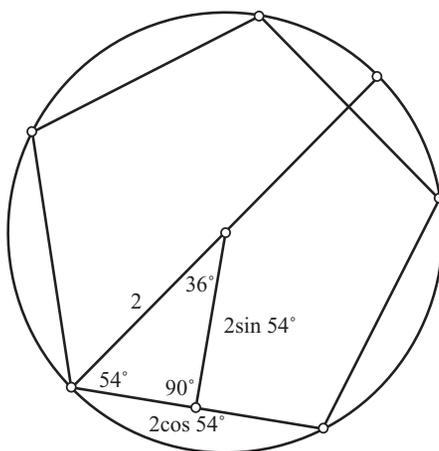
### Extension Questions:

- Based on the chart in Problem 4, between what two values should an estimate of  $\pi$ , based on a pentagon with radius 2, fall?

*The value should be between the values for a quadrilateral and a hexagon, or between 2.8284 and 3.000.*

- Use construction tools or geometry software to construct a regular pentagon inscribed in a circle of radius 2. Then calculate an estimate for  $\pi$  based on the inscribed pentagon.

*Geometry software was used to construct the figure.*



*Use trigonometry to solve this problem.*

$$\frac{\text{perimeter}}{\text{diameter}} = \frac{20 \cos 54}{4} = 5 \cos 54 \approx 2.9398$$

*The approximation for  $\pi$  is 2.9398.*





## Student Work Sample

The students in this class were allowed to report their findings in a variety of ways. This student's work satisfies many of the criteria on the solution guide.

For example:

- Uses geometric and other mathematical principles to justify the reasoning and analyze the problem.

*He uses his prior knowledge to explain the procedures he uses. "because the hypotenuse is the leg times  $\sqrt{2}$ ".*

- Communicates a clear, detailed, and organized solution strategy.

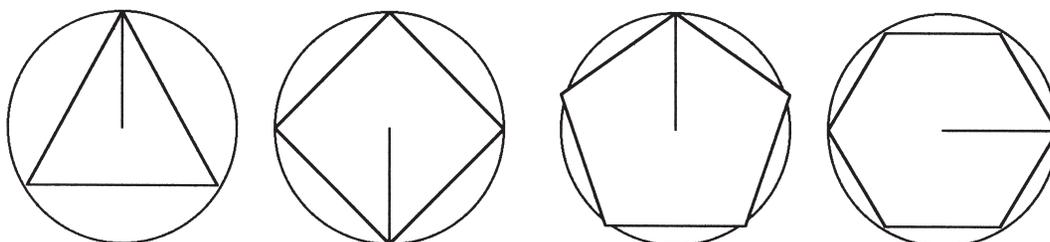
*The student's narrative explains his strategy for determining the perimeter of each triangle.*

The student's work includes appropriate and accurate representations of the problem with correct diagrams that are not labeled. However, labeling may not be necessary because his narrative explains the process and the figures.

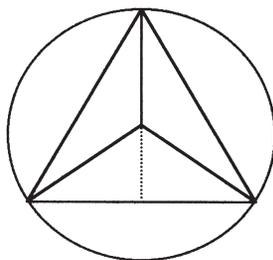


## Archimedean Walk

One way to find what  $\pi$  (pi) equals is by measuring the circumference of a circle and dividing it by the diameter. Archimedes realized that this would always differ by the accuracy of that person. He wanted to find away to fix the value of  $\pi$  that was based on calculations and not on measurement. He decided to put regular polygons inside circles, figure out their perimeters, and find  $\pi$  according to that shape. Archimedes decided to find  $\pi$  this way because the more sides the polygon had the closer he would get to finding the actual value of  $\pi$ .

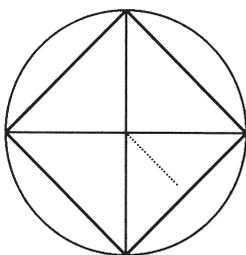


This diagram shows that the closer you get to the circumference of a circle, when you divide by the diameter, which is two, you can get closer and closer to the value for  $\pi$ . To answer the question about finding the approximation for the first, second, and fourth shape, I used trigonometry equations and special triangles. When you dissect the figures as follows you can use special triangles to solve.

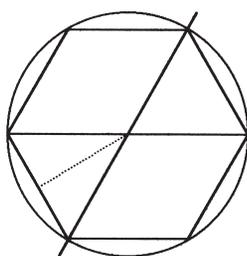


When you divide the equilateral triangle in half (dotted line) you get a right triangle and a hypotenuse of two, also the radius of the circle. When we divided the triangle in half that made the central angle, which we found out using the equation to find the value for each angle in a regular polygon, divide in half also to equal  $60^\circ$ . With this knowledge you can conclude that the last angle is  $30^\circ$  and that the triangle is a 30:60:90 triangle. Since this is a special triangle we can figure out the lengths of the two legs. Knowing that the hypotenuse in two, we know that the short leg is half of that or one, and since the short leg is one, the long leg is  $1\sqrt{3}$  or just  $\sqrt{3}$ . That leaves us with half of the original triangle, and with this information we know the whole side of the triangle is  $2\sqrt{3}$ . To find the perimeter of the triangle we just multiply the value for the side times 3 to get  $6\sqrt{3}$  or  $\approx 10.39$  units. We then take the perimeter and divide it by the diameter of the circle to find the approximation of  $\pi$ , which is  $\approx 2.598$  units. This is the way in which to solve the other polygons hereafter.





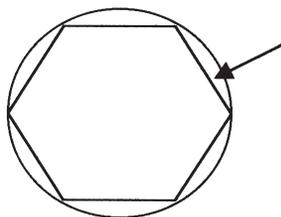
Although the triangle made a 30:60:90 triangle, the square makes a 45:45:90 triangle because when the central angle is divided by two the angle becomes a  $45^\circ$ , which makes the other a  $45^\circ$  angle also. With a 45:45:90 triangle we know that the legs are going to be the same. To find the legs we are going to divide the hypotenuse by  $\sqrt{2}$  because the hypotenuse is a leg times the  $\sqrt{2}$ , but since we already have the hypotenuse, we can reverse the equation. Using the equation,  $2/\sqrt{2}$ , we know that the two legs of the triangle are  $\approx 1.41$  units. This means that one side of the square equals  $\approx 2.83$  units. With this information we know that the perimeter of the square is  $\approx 11.31$  and that divided by 4, the diameter, equals, for the approximation of  $\pi$ ,  $\approx 2.83$  units.



The last shape, a hexagon, makes a 30:60:90 triangle when dissected. Since the angles and hypotenuse are the same as the triangles, we know that the short leg is one. That means that the length of one complete side is two units. When we multiply that by six for the perimeter, we get twelve. Then twelve divided by the diameter, four, equals three units.

# of sides of the inscribed shape	Measure of the central angle	Perimeter	Diameter	Approximation for $\pi$
3	$120^\circ$	$\approx 10.39$ units	4	2.598
4	$90^\circ$	$\approx 11.31$ units	4	2.83
6	$60^\circ$	12 units	4	3

In conclusion the more sides you add the closer the shape get to the circumference of the circle and the closer you get to the circle, when you divide by the diameter, the closer you get to the real value of  $\pi$ . This method will always be an underestimate for the value of  $\pi$ . Since you can never make an exact circle, you can not overestimate the answer. No matter how many sides you add, you can never make a circle, on reason because the sides curve in a circle, but it can come very close, yet never an exact number.



Lines can not curve, so it can never be exact.



## Talking the Archimedean Talk

In the assessment, Walking the Archimedean Walk, you modeled Archimedes method of inscribed polygons to arrive at increasingly accurate approximations for the value of  $\pi$ .

The table shows the results of that investigation.

Number of sides of the inscribed polygon	Measure of the central angle	Perimeter	Diameter	Approximation for $\pi$
3	$120^\circ$	$6\sqrt{3}$	4	2.5981
4	$90^\circ$	$8\sqrt{2}$	4	2.8284
6	$60^\circ$	12	4	3.000

In reality, Archimedes was versed well enough in basic geometry to know that he could start his method for approximating  $\pi$  with an inscribed hexagon. From there, he refined his approximation by doubling the sides of the hexagon and calculating the ratio  $\frac{\text{perimeter}}{\text{diameter}}$  for the resulting dodecagon.

1. Use construction tools, circular geoboards, circular dot paper, or appropriate geometry technology to construct a regular hexagon inscribed in a circle of radius 2 units.
2. Using this construction, double the number of sides to construct a regular dodecagon inscribed in a circle of radius 2 units.
3. Make appropriate calculations (no measuring allowed!) in order to determine the value of  $\pi$  based upon an inscribed regular dodecagon.



## Teacher Notes

### Materials:

Construction tools, circular geoboards, circular dot paper, or geometry software

### Connections to Geometry

#### TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(B) through the historical development of geometric systems, recognizes that mathematics is developed for a variety of purposes.

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

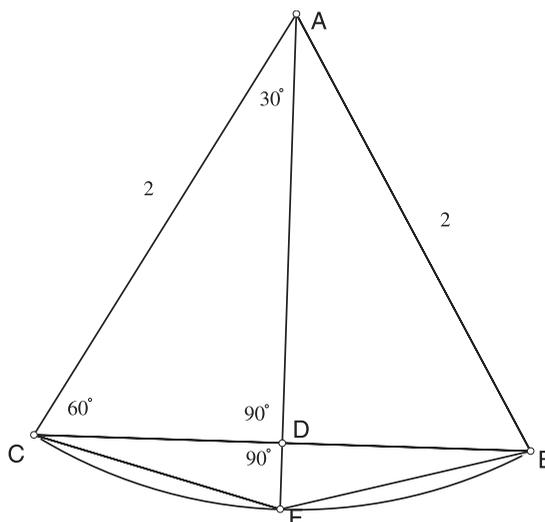
### Scaffolding Questions:

#### Problems 1 and 2

- What is the measure of a central angle of a dodecagon?
- What must the measure of the arc be that is intercepted by the dodecagon's central angle?
- What segment can you extend to intersect the circle in order to create a  $30^\circ$  arc?

#### Problem 3

Note: If necessary have students copy the picture below.



- What are the measurements of the segments?
- What is the length of segment  $\overline{AF}$ ? What is the length of segment  $\overline{AD}$ ?
- What is the length of segment  $\overline{DF}$ ?

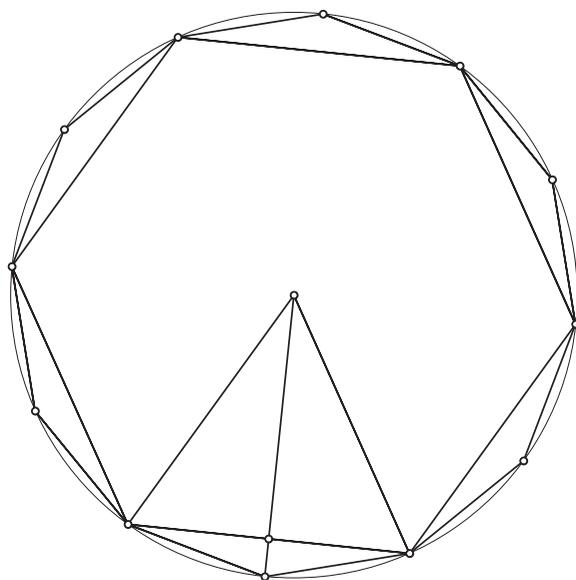
Note : If students use decimal approximations for their square roots, make sure they maintain about 4 decimal places of accuracy to ensure a good approximation of  $\pi$  at the final calculation.



## Sample Solutions:

### Problems 1 and 2

The hexagon was created by drawing a circle. With the radius of the circle as the compass width, place the compass point on the circle and strike an arc. Place the point on this mark, and strike another arc. The radius of the circle is the side of the hexagon. Connect these points to form the hexagon.



To double the sides of the hexagon, extend the hexagon's apothem until it intersects the circle. Then draw segments from the endpoints of the hexagon's sides to the point where the apothem intersects the circle. Every side of the hexagon results in 2 sides of the dodecagon. (Notice that this does **not** mean that the length of the dodecagon's sides is half the length of the hexagon's sides.)

Repeat this process as you go around the hexagon to create the inscribed dodecagon.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

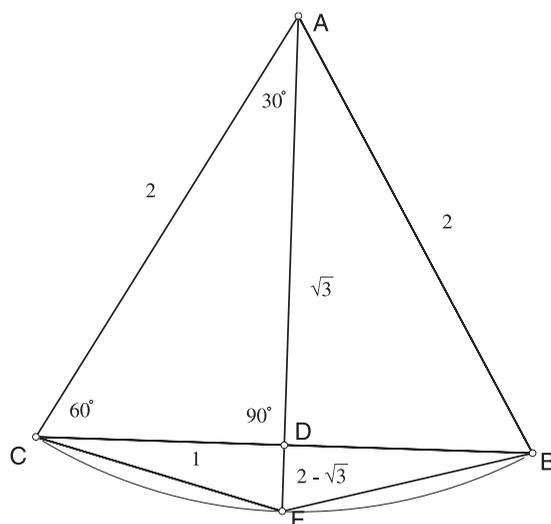
Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles



### Problem 3



CF, the length of the side of the dodecagon, needs to be found in order to complete the calculation for  $\pi$ .

In right  $\triangle ADC$ :

$AC = 2$ ;  $CD$  is one-half of  $AC$ , or 1 unit, and  $AD$  is  $\sqrt{3}$ .

The length of  $\overline{AF}$  is 2, so  $DF = AF - AD$  or  $2 - \sqrt{3}$ .

Use the Pythagorean theorem on right  $\triangle DCF$ .

Use  $\sqrt{3} \approx 1.7321$

$$1^2 + (2 - 1.7321)^2 \approx CF^2$$

$$1 + (0.2679)^2 \approx CF^2$$

$$1 + 0.0718 \approx CF^2$$

$$\sqrt{1.0718} \approx CF$$

$$1.0353 \approx CF$$

The perimeter of the regular dodecagon is approximately  $12(1.0353)$  or 12.4236 units.

So for the dodecagon, ratio  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{12.4236}{4} \approx 3.1059$ .



Using exact values:

$$1^2 + (2 - \sqrt{3})^2 = CF^2$$

$$1 + 4 - 4\sqrt{3} + 3 = CF^2$$

$$8 - 4\sqrt{3} = CF^2$$

$$4(2 - \sqrt{3}) = CF^2$$

$$CF = \sqrt{4(2 - \sqrt{3})} = 2\sqrt{(2 - \sqrt{3})}$$

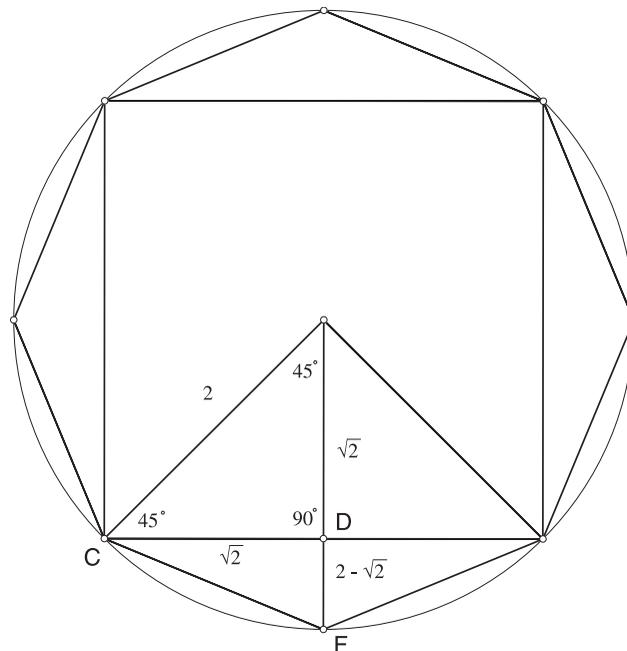
So for the dodecagon, ratio  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{24\sqrt{2 - \sqrt{3}}}{4} = 6\sqrt{2 - \sqrt{3}} \approx 3.1058$ .

### Extension Problems:

- What two values should an estimate of  $\pi$  based on an octagon fall between?

*The value should be between the values for a hexagon and a dodecagon, or between 3.000 and 3.1058.*

- Calculate an estimate for  $\pi$  based on an inscribed octagon.



*Construct a square and extend the apothem to intersect the circle at point F. Connect this point to the adjacent vertices of the square. Repeat this process for each apothem. The segment  $\overline{DF}$  measures  $2 - \sqrt{2}$ .*



In right  $\triangle CDF$ :

Using  $\sqrt{2} \approx 1.4142$

$$1.4142^2 + (2 - 1.4142)^2 \approx CF^2$$

$$2 + (0.5858)^2 \approx CF^2$$

$$2 + 0.3432 \approx CF^2$$

$$\sqrt{2.3432} \approx CF$$

$$1.5308 \approx CF$$

So for the octagon,  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{12.2464}{4} \approx 3.0616$ .

This value is between the values for a hexagon and a dodecagon.

Using exact values:

$$(\sqrt{2})^2 + (2 - \sqrt{2})^2 = CF^2$$

$$2 + 4 - 4\sqrt{2} + 2 = CF^2$$

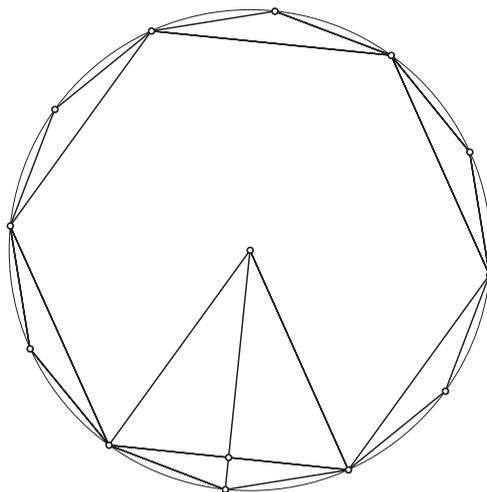
$$8 - 4\sqrt{2} = CF^2$$

$$4(2 - \sqrt{2}) = CF^2$$

$$CF = \sqrt{4(2 - \sqrt{2})} = 2\sqrt{(2 - \sqrt{2})}$$

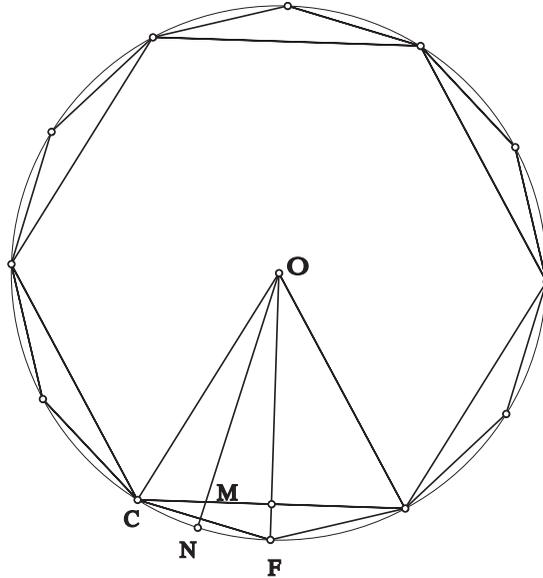
So for the octagon, ratio  $\frac{\text{perimeter}}{\text{diameter}} \approx \frac{16\sqrt{2 - \sqrt{2}}}{4} = 4\sqrt{2 - \sqrt{2}} \approx 3.0615$ .

- Double the sides of a dodecagon. Then calculate an estimate for  $\pi$  based on an inscribed 24-gon. (Circle with radius 2)



To double the sides of the hexagon, extend the hexagon's apothem until it intersects the circle. Then draw segments from the endpoints of the hexagon's sides to the point where the apothem intersects the circle. Every side of the hexagon results in two sides of the dodecagon. (Notice that this does **not** mean that the length of the dodecagon's sides is half the length of the hexagon's sides.)

Repeat this process as you go around the hexagon to create the inscribed dodecagon.



$$CF = 1.0353$$

$$CM = \frac{1}{2}(1.0353)$$

$$CO = 2$$

$$OM^2 + \left(\frac{1}{2}(1.0353)\right)^2 = 2^2$$

$$OM \approx 1.932$$

$$MN = 2 - OM \approx 0.06815$$

$$CN^2 = MN^2 + CM^2$$

$$CN^2 = (0.06815)^2 + \left(\frac{1}{2}(1.0353)\right)^2$$

$$CN = 0.522$$

$$\text{Perimeter} = 24(0.522) = 12.528$$

$$\frac{\text{perimeter}}{\text{diameter}} \approx \frac{12.528}{4} \approx 3.132.$$

Note: Archimedes extended his calculations to an estimate based on an inscribed 96-gon.





## Mad as a Hatter or Hat as a Madder

“Then you should say what you mean,” the March Hare went on.

“I do,” Alice hastily replied, “at least—at least I mean what I say—that’s the same thing, you know.”

“Not the same thing a bit!” said the Hatter. “Why, you might just as well say that ‘I see what I eat’ is the same thing as ‘I eat what I see!’”

Lewis Carroll, *Alice in Wonderland*

### Problem Set 1

For each of the following conditional statements:

- a) Determine if the original statement is true. Explain in writing the reasoning for each of your choices.
  - b) Write the converse of the statement.
  - c) Determine if the converse of the statement is true or false. Explain in writing the reasoning for each of your choices.
1. If a number is divisible by 4, then it is divisible by 2.
  2. If it is raining in Las Vegas, then it is sunny in Nevada.
  3. If it is sunny in Nevada, then it is sunny in Las Vegas.
  4. If a number is greater than -500, then the number is greater than 500.
  5. If a person is a teenager, then that person is between 13 and 19 years old.
  6. If a musician plays classical guitar, then the musician is a girl.
  7. If a girl plays classical guitar, then she is a musician.
  8. If a person reads a lot, then that person is smart.
  9. If a number is even, then it is divisible by 2.
  10. If a car is an SUV, then it has four wheels.



11. If a number squared is greater than 1, then the number itself must be greater than 1.
12. If point B is the midpoint of  $\overline{AC}$ , then the distance from point A to point B is the same as the distance from point B to point C.
13. If a polygon has exactly 8 sides, then it is an octagon.
14. If a parallelogram has a right angle, then it is a square.
15. If a quadrilateral has exactly one pair of parallel sides, then it is a trapezoid.

### Problem Set 2

- Write:
1. a non-mathematical statement
- and
2. a mathematical statement

that fulfill the following requirements.

1. A true conditional statement with a converse that is also true.
2. A true conditional statement with a converse that is false.
3. A false conditional statement with a converse that is true.
4. A false conditional statement with a converse that is also false.

### Problem Set 3

1. Present your statements to fellow classmates. Be prepared to explain and justify why you consider your statements either true or false.
2. Write a few sentences discussing which types of statements were easiest to agree on and which types provoked the most disagreement and discussion.
3. Which statements in Problem Set 1 and Problem Set 2 would you consider to be definitions? Why?





**Materials:**

One pencil or pen for each student.

**Connections to Geometry****TEKS:**

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

(C) demonstrates what it means to prove mathematically that statements are true;

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

## Teacher Notes

**Scaffolding Questions:****Problem Set 1**

- In writing the converse of a conditional statement: “What is the hypothesis?” and “What is the conclusion?”
- Guide students as necessary toward composing correct grammatical sentences rather than simply mechanically switching the hypothesis with the conclusion.
- Guide students as necessary toward providing specific, concrete counterexamples in order to determine the converse or to establish that statements are false.

For the converse of the statement in Problem Set 1, question 1: “Can you think of a number that is divisible by 2, but that is not even?”

In Problem Set 1, question 1: student response: “If it is divisible by two, then a number is divisible by 4.”

Guiding question: “Can you rewrite your sentence so that the subject (a number) is part of the hypothesis?”

For Problem Set 1, question 10: students will justify that the converse of this statement is false with a general statement like: “Not all cars are SUVs.” An appropriate question would then be: “Can you think of a specific car that has four wheels but is not an SUV?”

**Problem Sets 1, 2, and 3**

- Student examples often provoke debate among classmates about the relative merit and/or truth of counterexamples. They might not always agree that a statement is a valid counterexample.
- Students are asked to address this issue in Problem Set 3. Guide students as necessary toward making the distinction between non-mathematical statements, whose truth can depend on people’s viewpoints or opinions, and mathematical statements, whose truth depends on logical reasoning.



## Sample Solution:

### Problem Set 1

1. **True.** Since 4 is divisible by 2, then numbers divisible by 4 must also be divisible by 2.

**Converse:** If a number is divisible by 2, then it is divisible by 4.

**False:** Numbers such as 2, 6, and 10 are divisible by 2, but not by 4.

2. **False.** If it is raining in Las Vegas, then it is not sunny throughout the entire state of Nevada.

**Converse:** If it is sunny in Nevada, then it is raining in Las Vegas.

**False:** Las Vegas would be sunny if it were sunny in Nevada; therefore it can't be raining in Las Vegas.

3. **True.** Since Las Vegas is part of the entire sunshine-drenched state of Nevada.

**Converse:** If it is sunny in Las Vegas, then it is sunny in Nevada.

**False.** It could be raining in Reno, Nevada while it is sunny in Las Vegas.

4. **False.** 300 is greater than -500 but not greater than 500.

**Converse:** If a number is greater than 500, then it is greater than -500.

**True.** All numbers greater than 500 are positive, and by definition all positive numbers must be greater than negative numbers like -500.

5. **True.** This is the definition of a teenager.

**Converse:** If a person is between 13 and 19 years old, then that person is a teenager.

**True:** This is the definition of a teenager.

6. **False.** For example, Segovia was a famous male classical guitarist.

**Converse:** If a musician is a girl, then the musician plays classical guitar.

### Texas Assessment of Knowledge and Skills:

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Bayou City  
Geometry



**False:** The Dixie Chicks do not play classical guitar, but they are girl musicians. (This is, of course, open to debate. Like many non-mathematical statements, we can't always be perfectly sure if they are true or false.)

7. **True.** Classical guitar players are all musicians.

**Converse:** If a girl is a musician, then she plays classical guitar.

**False.** For example, Gloria Estafan, who is a percussionist.

8. **False.** Students can argue that there are many ways to have intelligence which have nothing to do with reading.

**Converse:** If a person is smart, then they read a lot.

**False:** For the same reason as above. (This is another example of a non-mathematical statement whose truth is open to debate and interpretation.)

9. **True.** There are no even numbers that are not divisible by 2.

**Converse:** If a number is divisible by 2, then it is even.

**True.** There are no numbers that are even that are not divisible by 2. Or, True by definition of an even number.

10. **True.** Since all cars have 4 wheels and an SUV is a type of car.

**Converse:** If a car has 4 wheels, then it is an SUV.

**False.** A Volkswagen Beetle is not an SUV and it is a car with 4 wheels.

11. **False.** -2.

**Converse.** If a number is greater than 1, then the number squared is greater than 1.

**True.** Squaring a number greater than 1 makes the number even larger.

12. **True.** If B is the midpoint of  $\overline{AC}$ , then by definition,  $AB = BC$ .

**Converse:** If the distance from point A to point B is the same as the distance from point B to point C, then B is the midpoint of  $\overline{AC}$ .

**False.**  $\overline{AB}$  and  $\overline{BC}$  can be the legs of an isosceles triangle.

13. **True.** This is the definition of an octagon.

**Converse:** If a polygon is an octagon, then it has eight sides.

**True.** By definition, an eight-sided polygon is an octagon.

14. **False.** It could be a rectangle.

**Converse:** If a parallelogram is a square, then it has a right angle.



**True:** All squares have right angles by definition.

15. **True.** This is the definition of a trapezoid.

**Converse:** If a quadrilateral is a trapezoid, then it has exactly one pair of parallel sides.

**True.** Definition of a trapezoid.

## Problem Set 2

Answers will vary. Check Student Work.

1. Question 5, Problem Set 1 is a true conditional with a true converse.
2. Question 1, Problem Set 1 is a true conditional with a false converse.
3. Question 14, Problem Set 1 is a false conditional with a true converse.
4. Question 2, Problem Set 2 is a false conditional with a false converse.

## Problem Set 3

1. and 2. Answers will vary. Students should be able to articulate that, in general, it's easier to decide the truth/falseness of mathematical statements. The terms in mathematical statements are precisely defined, and their truth depends on logic. In contrast, the truth/falseness of non-mathematical statements often depends on people's opinions and personal understanding of what words mean.
3. The statements that make good definitions are the conditional statements that have true converses.

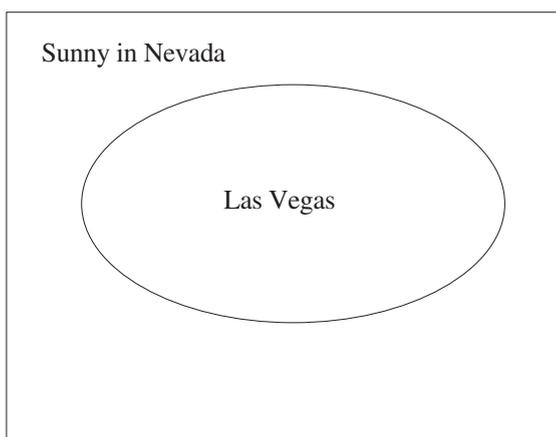


### Extension Questions:

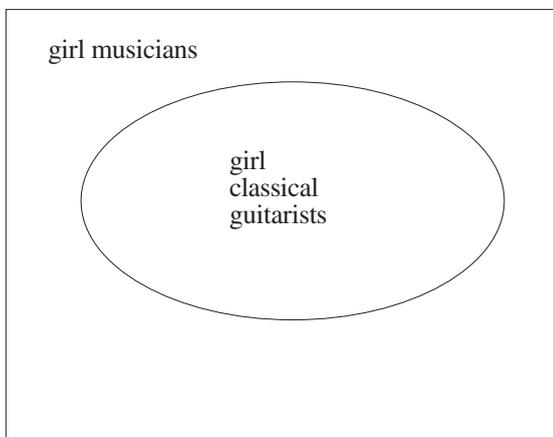
- Refer to questions 3, 7, and 10 in Problem Set 1. For each of these questions, represent the conditional statement in an Euler Diagram.

Note: Euler (pronounced “oiler”) diagrams are often called Venn Diagrams.

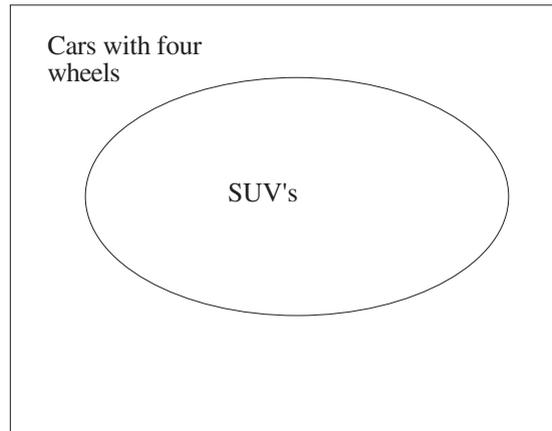
#### Problem 3



#### Problem 7



Problem 10



- Write definitions of the following terms as true conditional statements with true converses. Then, rewrite the definitions as biconditional statements.
  - a) complementary angles
  - b) isosceles triangle
  - c) polygon
    - a) *If two angles are complementary, then the sum of their measures is  $90^\circ$ . Two angles are complementary if and only if the sum of their measures is  $90^\circ$ .*
    - b) *If a triangle is isosceles, then it has at least 2 sides that are congruent. A triangle is isosceles if and only if it has at least two sides that are congruent.*
    - c) *If a shape is a polygon, then it is a plane figure formed from 3 or more line segments, such that each segment intersects exactly 2 other segments, one at each endpoint, and no 2 segments with a common endpoint are collinear.*

*A shape is a polygon if and only if it is a plane figure formed from 3 or more line segments, such that each segment intersects exactly 2 other segments, one at each endpoint, and no two segments with a common endpoint are collinear.*
- Create and name your own object. It does not have to be mathematical, but it should be something you can draw. Then, write the definition of your object as a biconditional statement.

*Answers will vary. Check student work.*





## Going the Distance in Taxicab Land Assessment

Write a paragraph comparing and contrasting the characteristics of the geometric objects you studied in the Going the Distance in Taxicab Land lesson as they are represented on the traditional coordinate grid and as they are represented on a taxicab coordinate grid.

1. Be sure to include all of the following objects in your analysis:

line segments

circles

perpendicular bisectors

2. Discuss whether you think the definitions or characteristics of these objects are valid and useful for both geometric systems.

3. Can you think of situations that might be better represented in Taxicab geometry than in traditional geometry?



**Going the Distance in Taxicab Land Lesson:  
Describing and Analyzing Objects in Two Different Geometric Systems.**

**Activity 1:**

Draw accurate sketches to represent the following definitions, theorems, or postulates, and answer the questions that follow.

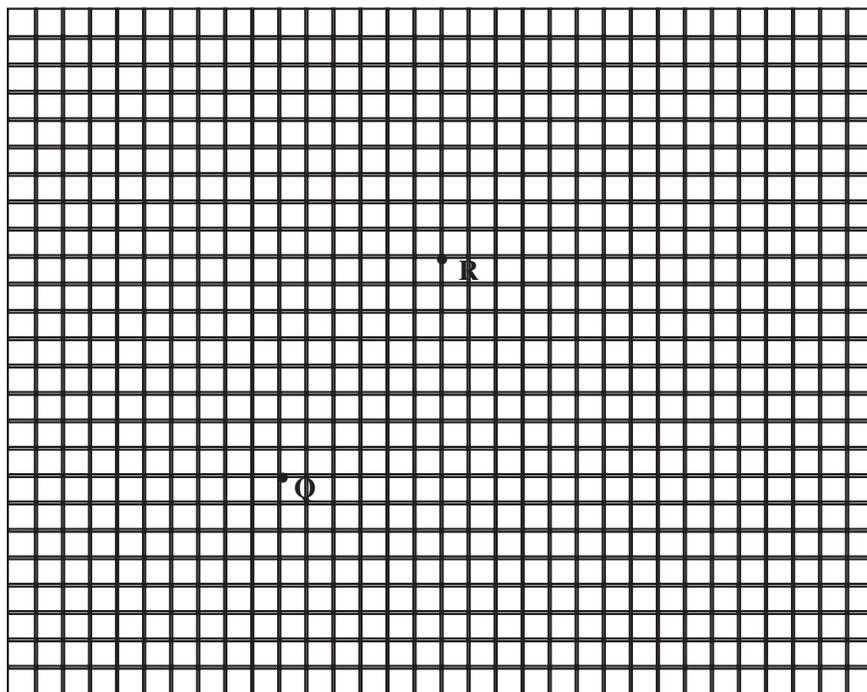
**1. Two points determine a line segment.**

• R

• Q

- a) How many line segments can be drawn between the two given points?
- b) The line segment drawn between the two points represents the \_\_\_\_\_ distance between the two points.

**2. Make the same sketch on a coordinate grid.**



- a) How many line segments can be drawn between the two points?
- b) The line segment drawn between the two points represents the \_\_\_\_\_ distance between the two points.
- c) Does placing the geometric object on the coordinate grid change its characteristics? Explain your answer.

**3. A circle is the set of points in a plane that are the same distance from a given point in the plane.**

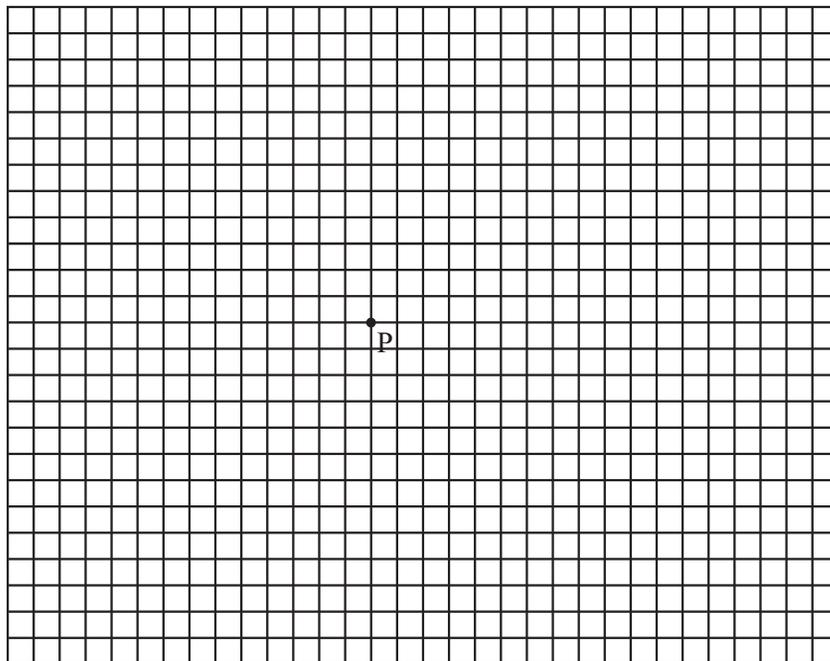
Sketch a circle with center at point P and a radius of 6 cm.



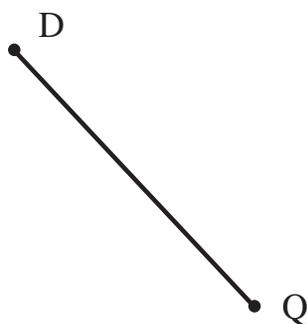
- a) How many different circles can you draw with this center and radius?



4. Make the same sketch on a coordinate grid. (For convenience, make the radius of the circle 6 grid units.)



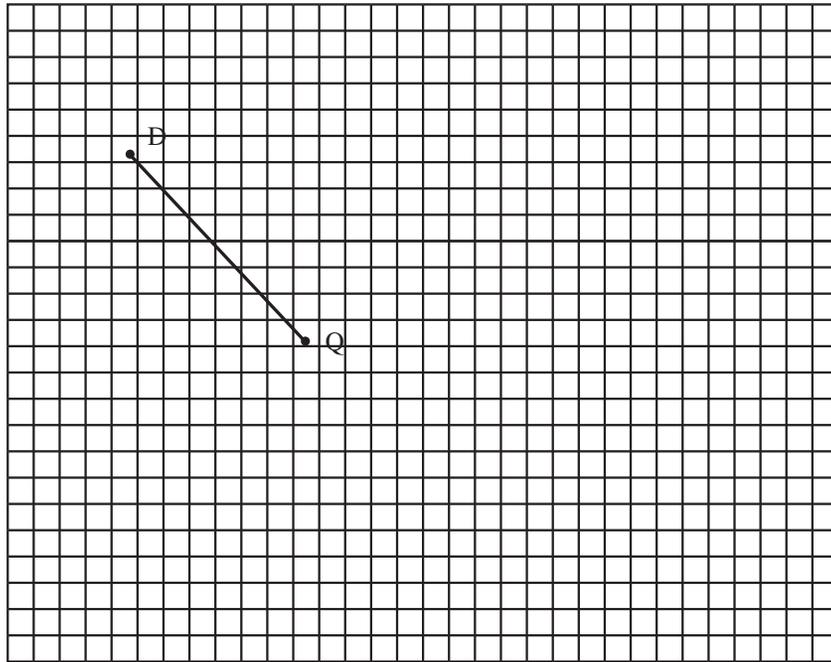
- a) How many different circles can you draw with this center and radius?
- b) Does placing the geometric object on the coordinate grid change its characteristics? Why?
- c) Does changing the location of the geometric object on the coordinate grid change its characteristics?
5. **A point is on a segment's perpendicular bisector if and only if it is the same distance from each of the segment's endpoints.** (Represent all the points that satisfy this requirement.)



- a) How many different perpendicular bisectors is it possible to draw for the given line segment?



6. Make the same sketch on a coordinate grid.

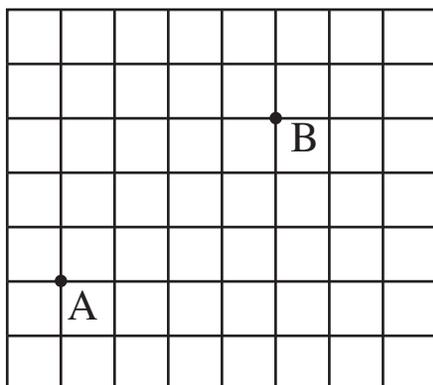


- a) How many different perpendicular bisectors is it possible to draw for the given line segment?
- b) Does placing the geometric object on the coordinate grid change its characteristics? Why or why not?
- c) Describe how changing the position of the geometric object on the coordinate grid changes its characteristics.



## Activity 2:

1. Determine the distance between points A and B on the coordinate grid below.



2. Imagine you are a taxicab driver and the coordinate grid above represents the grid of city streets you can travel on. Determine the **taxidistance** from point A to point B in Taxicab Land.
3. In what ways do you think the coordinate grid in Taxicab Land is different from the traditional coordinate grid you worked with in Activity 1?

## Activity 3:

You have seen that representing a geometric object on a traditional coordinate grid does not change any of its characteristics. Neither does repositioning the object on the grid.

Your task is to analyze what happens when geometric objects are placed on a taxicab grid which has the following characteristics:

Points on a taxicab grid can only be located at the intersections of horizontal and vertical lines.

One unit will be one grid unit.

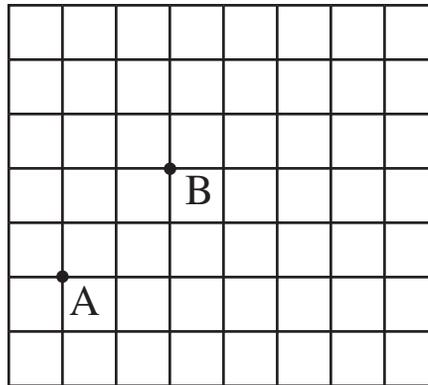
The numerical coordinates of points in **taxicab geometry** must therefore always be integers.

The **taxidistance** between 2 points is the smallest number of grid units that an imaginary taxi must travel to get from one point to the other. In Activity 2, the **taxidistance** between point A and point B is 7.



Draw accurate sketches on a **taxicab geometry** coordinate grid to represent the following definitions, theorems, or postulates, and answer the questions that follow.

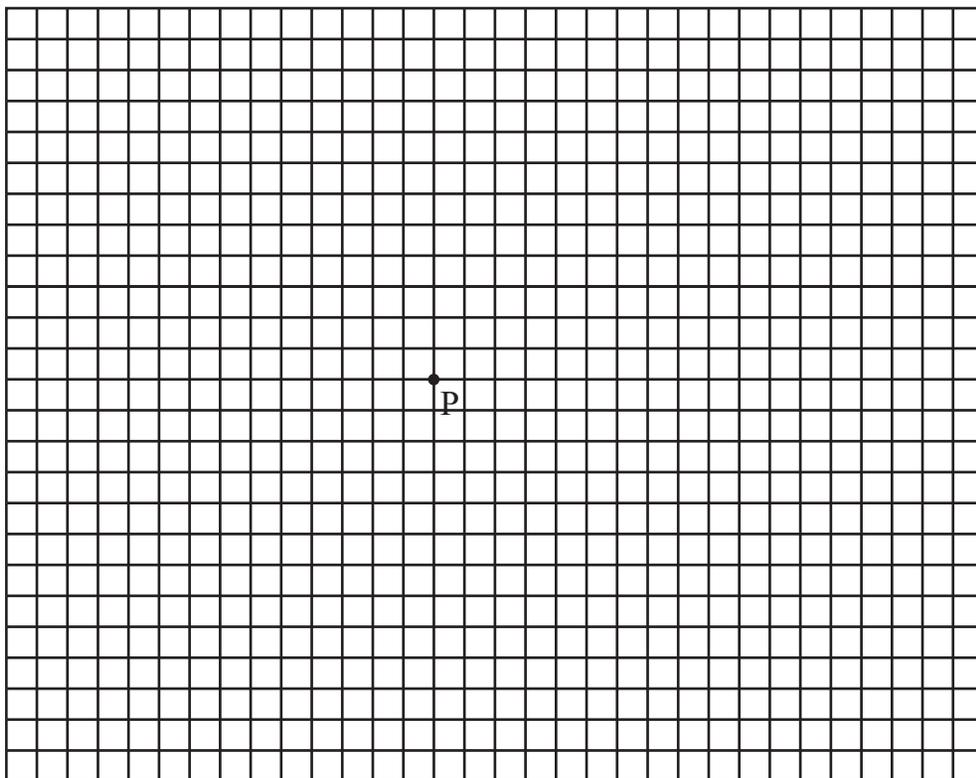
1. **Two points determine a line segment.**



- a) Is this the only line segment that can be drawn between the two given points? Explain.
- b) The line segment(s) drawn between the two points represents the \_\_\_\_\_ taxidistance between the two points.
- c) Does changing the position of the geometric object on the taxicab grid change its characteristics? Explain.



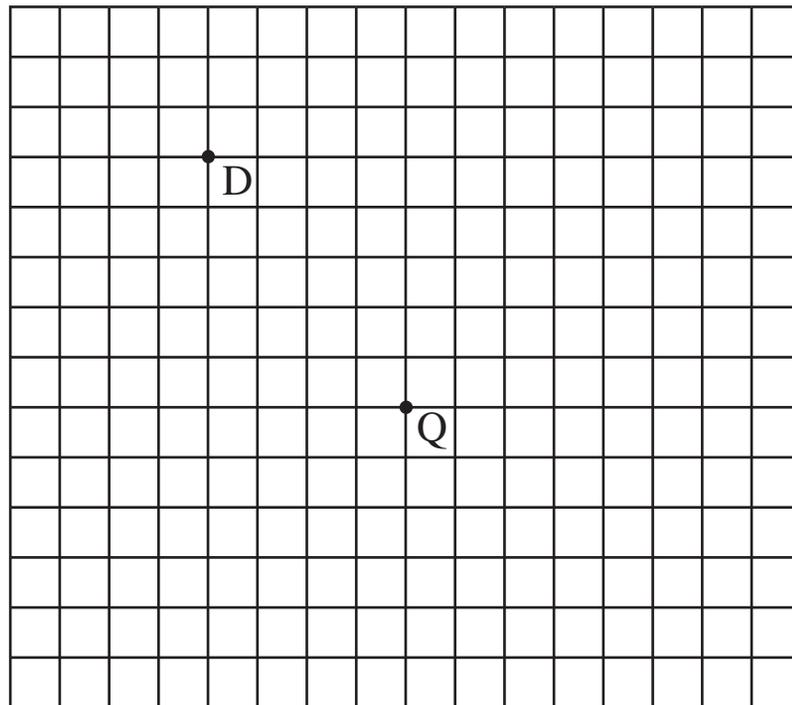
2. **A circle is the set of points in a plane that are the same distance from a given point in the plane.** (Sketch a circle with a radius of 6.)



- a) How many different circles can you draw with this center and radius?  
Explain your answer.
- b) Explain how changing the position of the geometric object on the taxicab grid changes its characteristics.



3. **A point is on a segment's perpendicular bisector if and only if it is the same distance from each of the segment's endpoints.** (Represent all the points that satisfy this requirement.)



- a) Is the set of points you represented in your drawing the only perpendicular bisector for the two given endpoints? Explain.
- b) Does changing the position of the geometric object on the coordinate grid change its characteristics? Explain.



## Teacher Notes

### Materials:

One compass and ruler per student.

### Connections to Geometry TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(C) compares and contrasts the structures and implications of Euclidean and non-Euclidean geometries.

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(C) develops and uses formulas including distance and midpoint.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures

The lesson is intended to allow students to develop “an awareness of the structure of a mathematical system,” (See TEK b.1 (A)). If students are already familiar with taxicab geometry, the assessment may be used to evaluate their understanding of this mathematical system.

### Scaffolding Questions:

#### Assessment

Encourage students to review all the information in the lessons. They may want to organize the information in a chart form before they write their summary paragraph.

#### Activity 1:

- What does the word “determine” mean in the sentence: “Two points determine a line segment”?
- Is the coordinate grid line segment drawn between point Q and R unique? Does it represent the shortest distance between those two points?
4. Students may notice that point P does not lie at a grid intersection.
  - Is it possible to draw a circle with a radius of 6 with point P positioned this way?
  - After students draw the initial circle, they may position point P at a grid point and redraw the circle.
6. Students may notice that the segment does not have endpoints that lie on grid intersections.
  - Is it possible to draw the perpendicular bisector with the points in their present position?
  - Students may reposition the segment so that its endpoints lie on grid intersections and then complete the sketch.



## Activity 2:

1.
  - Is it possible to find the distance between A and B by counting along the grid lines?
  - What formula could we use to determine the distance between A and B?
2.
  - In Taxicab Land is it possible to find the distance between A and B by counting along the grid lines?
  - Is there more than one way to count out the distance?
  - Which way (or ways) of counting out the distance do you think is correct?
3.
  - Where can points be located on a traditional coordinate grid?
  - Where can points be located on a taxicab coordinate grid?
  - What does this tell us about the numerical coordinates of the points on each type of coordinate grid?
  - How is the distance between two points on a taxicab coordinate grid different from the distance on a traditional coordinate grid?

## Activity 3:

1. Students may notice that there are many ways to draw a minimum distance pathway between the two points.
  - Do all these segments represent the shortest taxidistance between the two points?
2.
  - Do the points you drew satisfy the definition of a circle?
  - Should you connect the points by drawing along the grid lines so that it looks more like an enclosed shape?

about the properties of parallel and perpendicular lines;

(C) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them.

### Texas Assessment of Knowledge and Skills:

Objective 7. The student will demonstrate an understanding of two and three dimensional representations of geometric relationships and shapes.

### Connection to High School Geometry: Supporting the TEKS and TAKS Institute:

- I. Structure: Bayou City Geometry

### Student's Response to Assessment Question 2:

"In normal Geometry, a line is the shortest distance between two points. In Taxicab it is not. In normal Geometry circles have points in which the shortest distance from the center to each point is equal. In taxicab geometry, the circles are different. They are not curved. In normal geometry, all segments have a perpendicular bisector. In taxicab geometry only segments with an even taxicab distance have perpendicular bisectors. The definitions for geometric shapes are only useful in normal geometry, not taxicab. You would use taxicab geometry if you wanted to find out how far a car travel to get from one street corner to another."



3.

- Do you think there will be any points that are the same distance from both endpoints?

### **Sample Solution:**

### **Assessment**

#### **Traditional Coordinate System**

Line Segment: Uniquely determined by two points.  
Represents shortest distance between two points.

Circle: Uniquely determined by given center and radius.  
Characteristics do not change when repositioned.

Perpendicular Bisector: Uniquely determined by endpoints of segment.  
Characteristics do not change when repositioned.

#### **Taxicab Coordinate System**

Line segment: Not necessarily uniquely determined by two points.  
Represents shortest distance between two points.

Circle: Uniquely determined by center and radius.  
Characteristics do not change when repositioned.

Perpendicular Bisector: May not exist at all. If it does exist, however, then it is uniquely determined by endpoints of segment.  
  
Characteristics change when repositioned.

Students should realize that the definitions and characteristics may be more useful and familiar in the traditional system, but they are equally valid in both systems.

Any sort of problem where one needs to find locations along a grid of city streets might be more successfully represented using taxicab geometry.



### Activity 1:

1. a) One  
b) The line segment drawn between the two points represents the shortest distance between the two points.
2. a) One  
b) The line segment drawn between the two points represents the shortest distance between the two points.  
c) No. There is one and only one line segment between the two points, and it represents the shortest distance between them.

3. Draw circle with a compass.

a) Only one circle may be drawn with a given radius.

4. Circle can be drawn on coordinate grid with a radius of 6 grid units by using a compass.

If students subsequently reposition point P on a grid point, then they can sketch the circle by counting grid units, make a ruler in grid units, etc.

a) One

b) No. There is only one circle with the given radius and center.

c) No. The coordinate name of the point changes but the properties of the circle stay the same.

5. Compass and straightedge construction is the most accurate way to do this.

a) One

6. Compass and straightedge construction is still an appropriate way to do this. If students reposition points D and Q to the nearest grid intersections, then the 10 unit segment's midpoint will also fall on a grid intersection.

a) One

b) No. There is only one set of points that satisfy the requirement of being the same distance from the endpoints of segment  $\overline{DQ}$ .

c) No

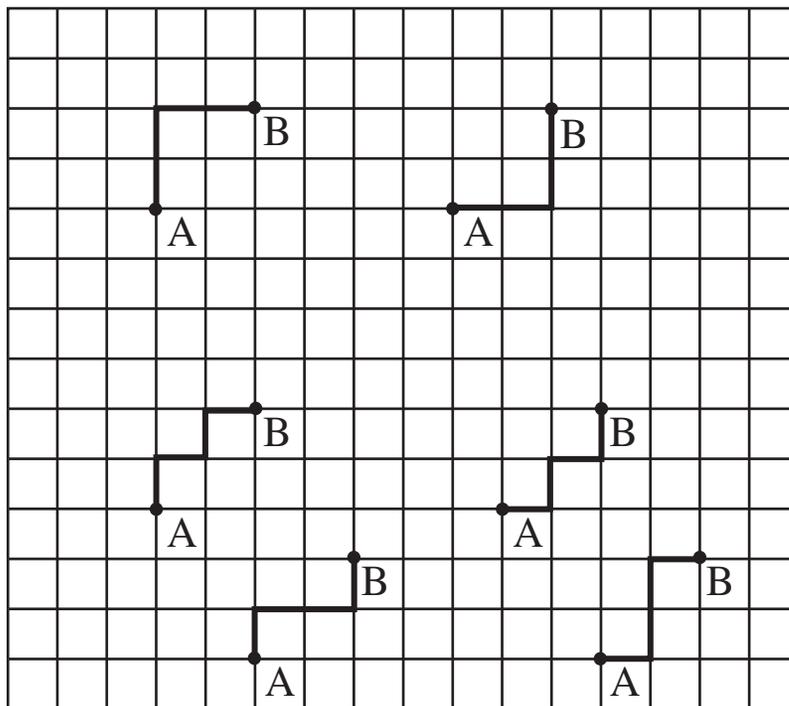


### Activity 2:

1.  $\overline{AB}$  is the hypotenuse of a right triangle with sides of length 3 units and 4 units. By the Pythagorean Theorem the length of  $\overline{AB}$  is 5 units.
2. The taxi driver must travel on the streets so he would travel 3 blocks vertically and 4 blocks horizontally. He would travel a total distance of 7 blocks.
3. Points on a taxicab grid can only be located at grid intersections. This makes their numerical coordinates integers. The taxidistance between two points must always be an integer.

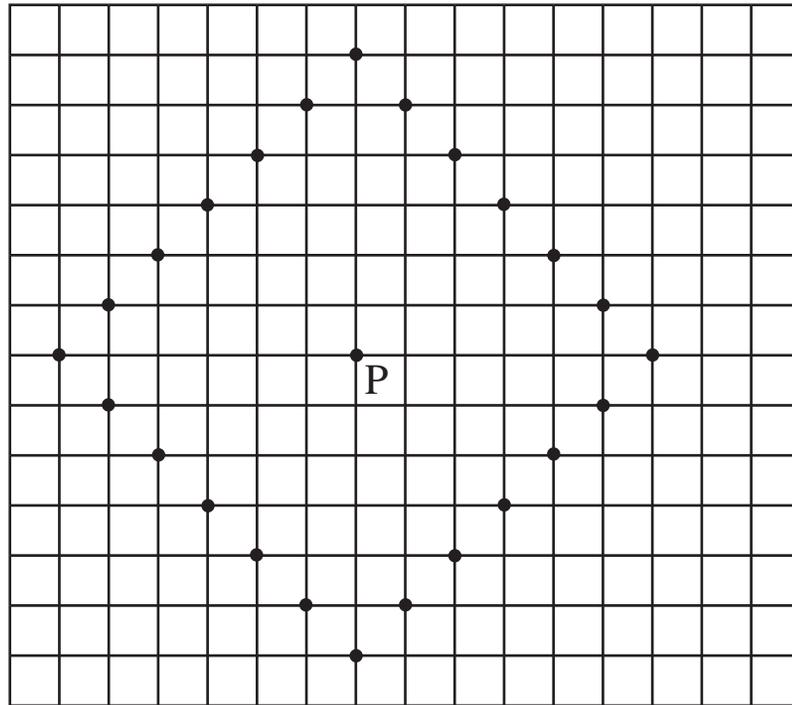
### Activity 3:

1. a) There are six minimum distance pathways that can be drawn between the two points.



- b) The line segment(s) drawn between the two points represent the shortest taxidistance between the two points.
- c) Yes. Students should realize that if they position the two points along either a horizontal or a vertical grid line, then there will only be one minimum distance segment between them.

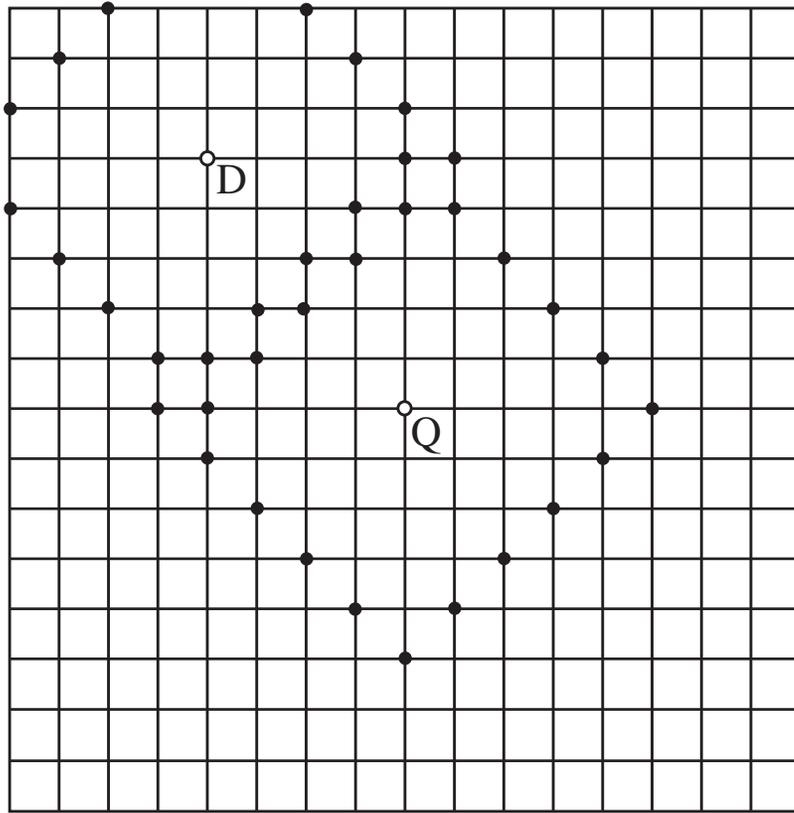




2. a) There is only one set of points that are all 6 units from P. They are points that lie on “street corners,” such that the sum of the horizontal and vertical distances is 6 units.
- b) The coordinates of the points would change, but the shape of the figure would remain the same.
3. a) There are no points that satisfy the requirement of being the same distance from points D and Q. The distance from point D to point Q is 9 units.

The graph shows the set of points that are 5 units from D and the set of shaded points that are 5 units from Q. There are no common points. The two taxicab circles will not intersect.



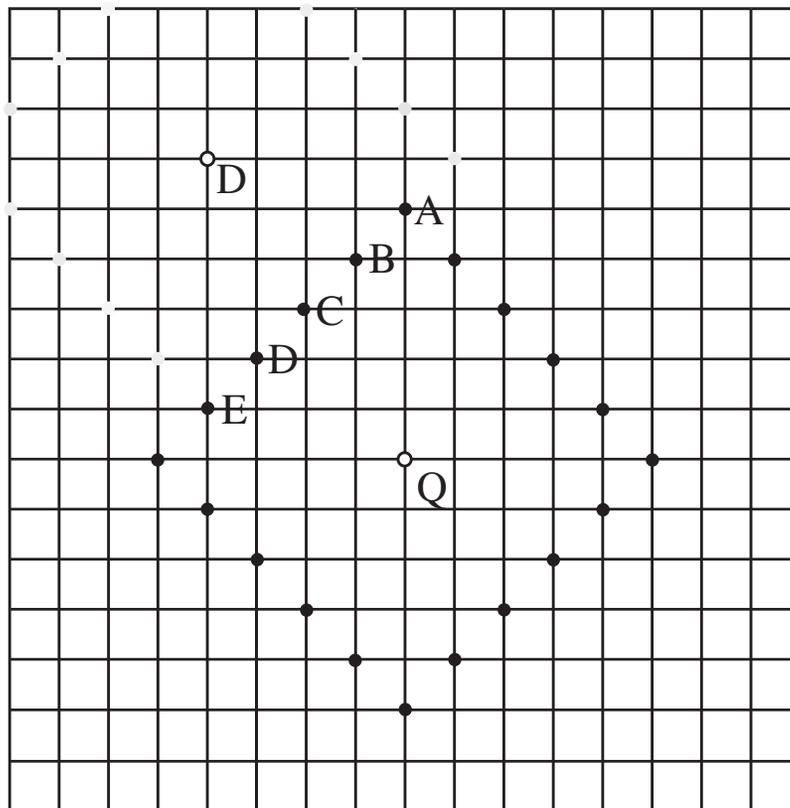


- b) Students may experiment and discover that if the taxidistance between the two points is odd, then there will be no perpendicular bisector.

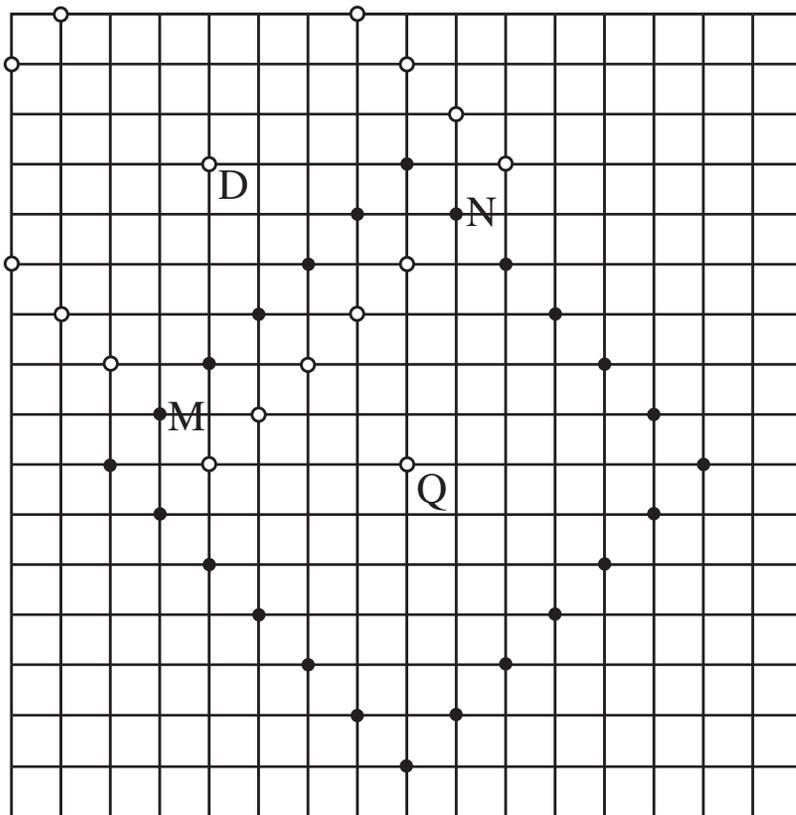
If, however, the taxidistance between the two points is even, there will be a perpendicular bisector, and it can take on a variety of configurations, depending on how the points are positioned.



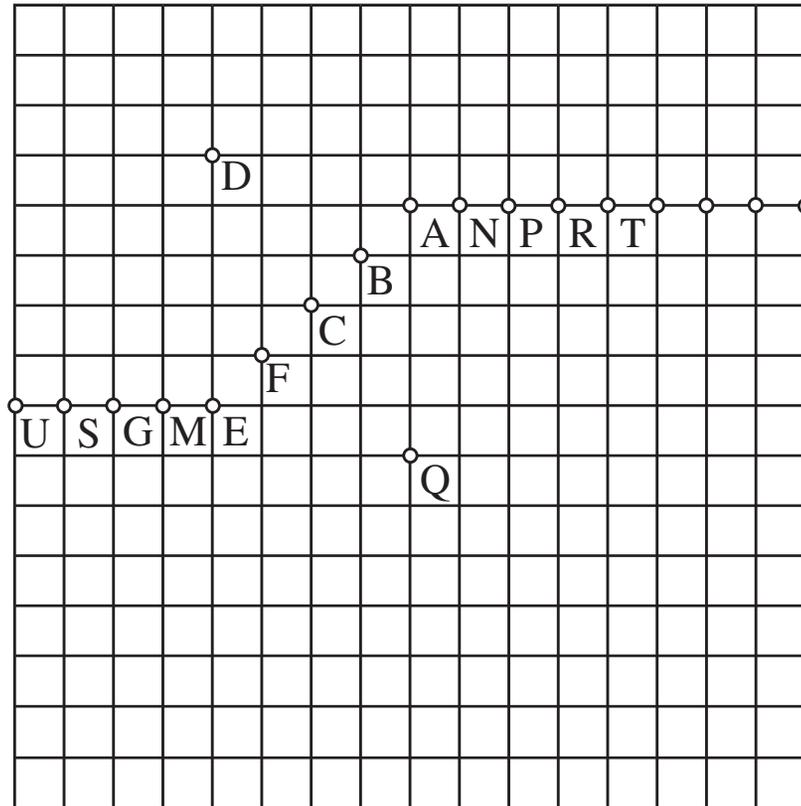
D and Q in this graph are 10 units apart. The next graph shows the two circles that have a radius of 5 units and centers D and Q. The points that the two circles have in common (A, B, C, D, and E) are all 5 units from both D and Q. Then A, B, C, D, and E are points on the perpendicular bisector of D and Q.



The next graph shows the two circles that have a radius of 6 units and centers D and Q. The points that the two circles have in common, M and N, are 6 units from both D and Q.



This next graph shows the points A, B, C, D, and E, 5 units from D and Q, M and N (six units from D and Q), P and Q (seven units from D and Q), R and S (eight units from D and Q), T and U (nine units from D and Q). This collection of points is a part of the set of points equidistant from D and Q.



### Extension Questions:

- For each pair of points given, decide if it is possible to draw a perpendicular bisector in a taxicab coordinate system.

$(1, 3)$  and  $(5, 3)$                       *yes*

$(-2, 0)$  and  $(-6, -4)$                       *yes*

$(5, -2)$  and  $(2, 1)$                       *yes*

$(-3, 3)$  and  $(2, 1)$                       *no*

$(5, -2)$  and  $(-3, 3)$                       *no*

$(5, 3)$  and  $(5, -1)$                       *yes*

- Write a conjecture about what must be true in order for it to be possible to draw a perpendicular bisector in a taxicab coordinate system.

*The taxidistance between the two endpoints must be an even number.*

- Experiment with circles of varying radii on a taxicab coordinate system. Write a conjecture about the value of  $\pi$  in taxicab geometry.

*The value of  $\pi$  in taxicab geometry is 4.*



**Chapter 3:**  
*Properties and  
Relationships of  
Geometric Figures*





# Introduction

Chapter three's set of assessments requires students to analyze the properties and describe relationships in geometric figures, including parallel and perpendicular lines, circles and lines that intersect them, and polygons and their angles. The students will use explorations to formulate and test conjectures.

Many processes underlie all content areas in mathematics. As they do mathematics, students continually use problem-solving, computation in problem solving contexts, language and communication, connections within and outside mathematics, and reasoning, as well as multiple representations, applications and modeling, and justification and proof. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999).





## Angle Bisectors and Parallel Lines

Construct two parallel lines. Draw a transversal. Bisect two interior angles on the same side of the transversal.

Develop and write a conjecture about the intersection of the angle bisectors.



## Teacher Notes

### Materials:

Patty paper and one straightedge per student

Or geometry software

### Connections to the Geometry TEKS:

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines;

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures.

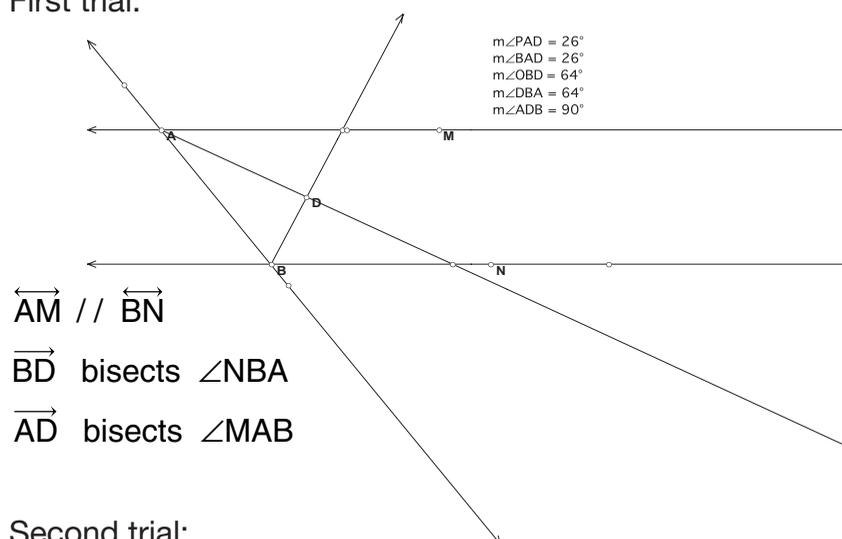
### Scaffolding Questions:

- What does the given information imply about the angles in the figure?
- Which special properties of parallel lines can you use to describe the relationships between or among the angles?

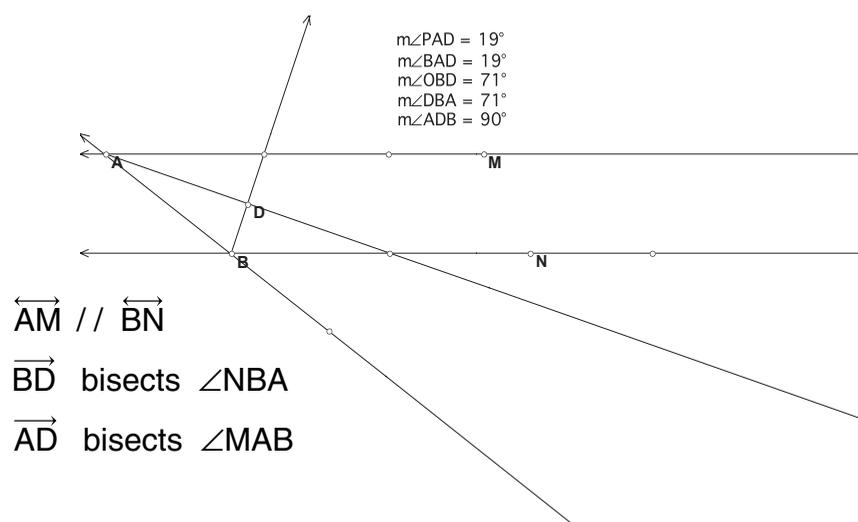
### Sample Solutions:

Construct the figure using geometry software. Measure the resulting angles.

First trial:



Second trial:



Conjecture: The two angle bisectors are perpendicular.



### Extension Questions:

- Prove your conjecture using an axiomatic approach.

$$m\angle BAD = \frac{1}{2}m\angle BAM \text{ because } \overrightarrow{AD} \text{ bisects } \angle BAM.$$

$$m\angle MBA = \frac{1}{2}m\angle ABN \text{ because } \overrightarrow{BD} \text{ bisects } \angle ABN.$$

The two interior angles on the same side of the transversal of two parallel lines are supplementary.

$$\text{Thus, } m\angle ABN + m\angle BAM = 180^\circ$$

$$\frac{1}{2}m\angle ABN + \frac{1}{2}m\angle BAM = 90^\circ$$

By substitution:

$$m\angle MBA + m\angle BAD = 90^\circ \text{ (Equation 1)}$$

The sum of the angles of a triangle is  $180^\circ$ , so in  $\triangle BDA$

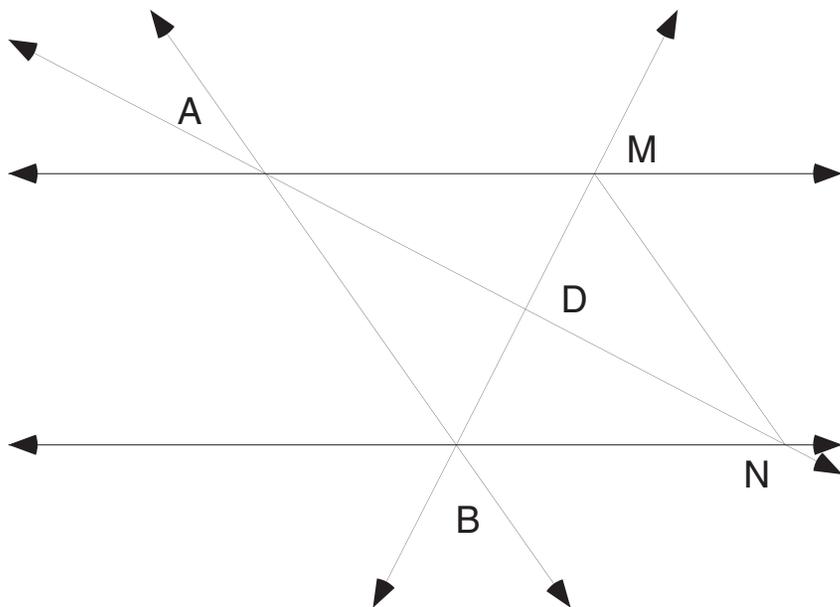
$$m\angle MBA + m\angle BAD + m\angle ADB = 180^\circ. \text{ (Equation 2)}$$

Subtract Equation 1 from Equation 2.

$$m\angle ADB = 90^\circ$$

Therefore  $\angle ADB$  is a right angle and the lines are perpendicular.

- Connect the two points that are the intersection of the angle bisectors and the parallel lines. Describe the resulting triangles and the quadrilateral AMNB.



### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Triangles Tell All

The four triangles are congruent, and  $AMNB$  is a rhombus. The triangles  $AMD$  and  $ABD$  are congruent right triangles with common leg  $\overline{AD}$  and congruent angles ( $\angle DAB \cong \angle DAM$ ).  
 $AM = AB$

Also, the triangles  $NBD$  and  $ABD$  are congruent right triangles with common leg  $\overline{BD}$  and congruent angles ( $\angle DBA \cong \angle DBN$ ).

$BN = AB$

Therefore,  $AM = AB = BN$ .

Because  $AM$  is parallel to  $BN$  and  $AM = BN$ , the figure is a parallelogram.

$AB = MN$  because they are opposite sides of the parallelogram.

$MN = AM = AB = BN$

The figure is a rhombus.

Another way to conclude that it is a rhombus is to realize that it is a parallelogram with perpendicular diagonals.





## Student Work Sample

This student's work shows the use of a compass to construct the parallel lines and bisect the angles.

His work exemplifies many of the criteria on the solution guide, especially the following:

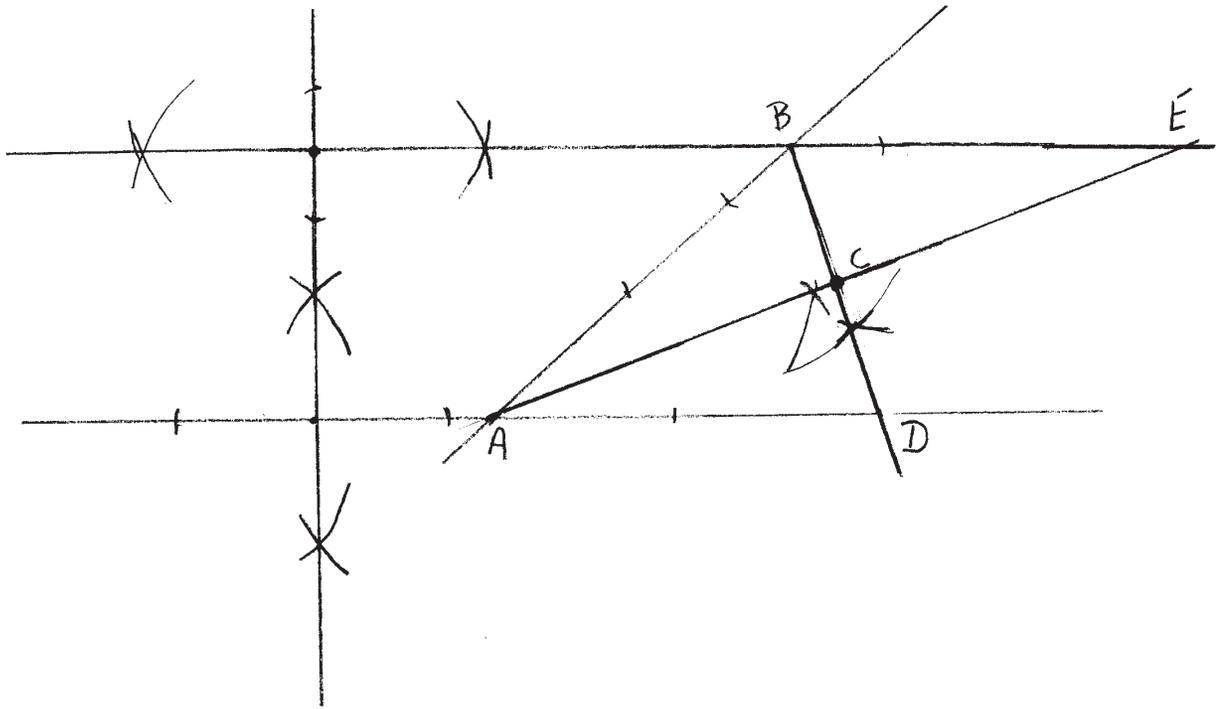
- Identifies the important elements of the problem.

*The student understands what the problem requires. He understands the lines referenced in the problem--the angle bisectors and writes his conjecture about these bisectors.*

- States a clear and accurate solution using correct units.

*This problem requires a construction and a conjecture. The student's construction is accurate, and his conjecture is clearly stated. Note the problem does not require that the student justify his conjecture, but he writes an explanation of how he came to his conjecture. The solution does not require units, but the student used the correct units in his justification.*





$$\begin{aligned} \angle BAD + \angle ABE &= 180^\circ \\ \frac{1}{2} \angle BAD + \frac{1}{2} \angle ABE &= 90^\circ \\ - \quad \frac{1}{2} \angle BAD + \frac{1}{2} \angle ABE + \angle C &= 180^\circ \\ - \angle C &= -90^\circ \\ \angle C &= 90^\circ \end{aligned}$$

When a transversal intersects a pair of parallel lines, and two of the interior angles on the same side are bisected, the bisectors are perpendicular.

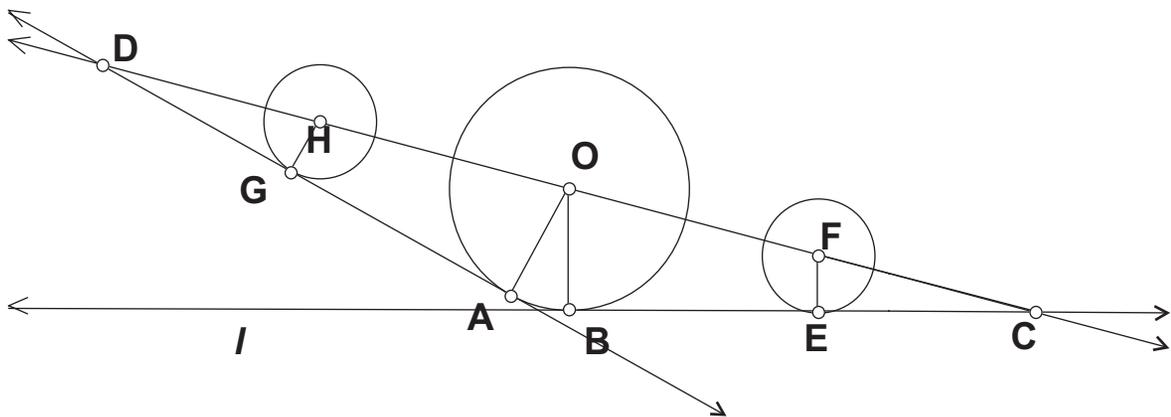




## Circles and Tangents

Maren constructed the figure below by going through the following process:

- Draw a line,  $l$ .
- Construct a circle with center  $O$  tangent to the line at point  $B$ .
- Construct a smaller circle with center  $F$  that is tangent to line  $l$  at point  $E$ . Draw  $\overleftrightarrow{OF}$ .
- Construct another circle on the other side of  $l$ , such that the new circle is congruent to circle  $F$  and has center  $H$  on  $\overleftrightarrow{OF}$ , such that  $HO = FO$ .
- Construct two angles congruent to angle  $COB$ , one with center  $O$  and another with center  $H$  as shown in the diagram.
- The intersection point with circle  $O$  and the angle ray is point  $A$  and the intersection point of the angle ray and circle  $H$  is  $G$ .
- Draw the line  $\overleftrightarrow{GA}$ .



Develop and write a conjecture about the relationships among the angles and the triangles in the figure and between the circles and the line  $\overleftrightarrow{GA}$ .



## Teacher Notes

### Materials:

One straightedge and compass per student

Or geometry software

### Connections to the Geometry TEKS:

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, the student formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them;

(f) **Similarity and the geometry of shape.** The student applies the concepts of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures.

### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

### Scaffolding Questions:

If the student uses constructions to analyze the problem, the following questions may be asked.

- What is the definition of the tangent to a circle?
- Given a line how can you construct a circle tangent to the line?
- What is the relationship between  $\overline{GH}$  and  $\overline{OA}$ ?
- Which segments in the figure must be congruent?
- Describe the relationship between  $\angle DHG$  and  $\angle CFE$ .
- Explain what you know about the relationship between  $\triangle DGH$  and  $\triangle CEF$ .
- What is special about  $\angle DGH$ ?
- What does it tell you about the figure?

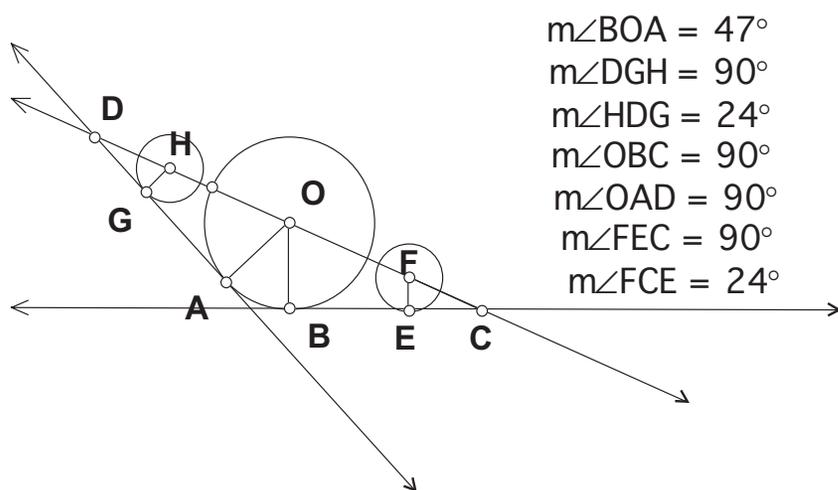
If the student does not have access to geometry software, the problem may be approached analytically.

- What does the given information imply about the triangles in the figure?
- Which special properties of right triangles can you use to describe the relationships between or among the segments in the figure?
- What is the relationship between a central angle and its intercepted arc?



## Sample Solutions:

Construct the figure using geometry software.



$$\begin{aligned} m\angle BOA &= 47^\circ \\ m\angle DGH &= 90^\circ \\ m\angle HDG &= 24^\circ \\ m\angle OBC &= 90^\circ \\ m\angle OAD &= 90^\circ \\ m\angle FEC &= 90^\circ \\ m\angle FCE &= 24^\circ \end{aligned}$$

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

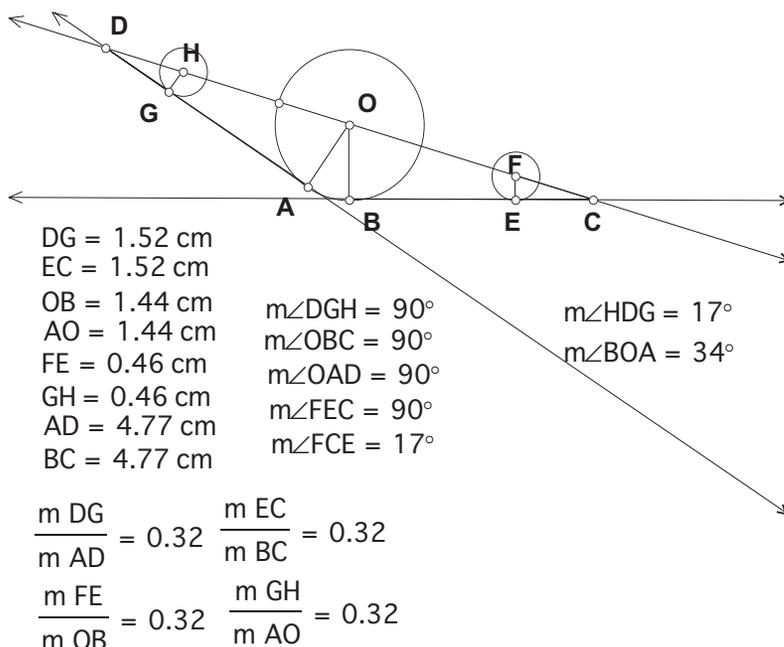
### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

### Teacher's Comment:

"It's hard for some teachers to let students struggle before answering questions for the students. We as teachers need to allow them to struggle and ask their teammates before answering questions. This allows them to work on teamwork skills needed for the work place in their future."

Try a second example. Measure the angles and the ratios of the corresponding sides of triangles.



$$\begin{aligned} DG &= 1.52 \text{ cm} \\ EC &= 1.52 \text{ cm} \\ OB &= 1.44 \text{ cm} \\ AO &= 1.44 \text{ cm} \\ FE &= 0.46 \text{ cm} \\ GH &= 0.46 \text{ cm} \\ AD &= 4.77 \text{ cm} \\ BC &= 4.77 \text{ cm} \end{aligned}$$

$$\begin{aligned} m\angle DGH &= 90^\circ \\ m\angle OBC &= 90^\circ \\ m\angle OAD &= 90^\circ \\ m\angle FEC &= 90^\circ \\ m\angle FCE &= 17^\circ \end{aligned}$$

$$\begin{aligned} m\angle HDG &= 17^\circ \\ m\angle BOA &= 34^\circ \end{aligned}$$

$$\begin{aligned} \frac{m DG}{m AD} &= 0.32 & \frac{m EC}{m BC} &= 0.32 \\ \frac{m FE}{m OB} &= 0.32 & \frac{m GH}{m AO} &= 0.32 \end{aligned}$$





$$m\angle POA \cong m\angle BOT$$

$$m\angle TOP = 180^\circ$$

$$m\angle POA + m\angle AOB + m\angle BOT = m\angle TOP$$

$$m\angle BOT + m\angle AOB + m\angle BOT = 180^\circ$$

$\overleftrightarrow{BC}$  is tangent to circle O

$$\overline{OB} \perp \overleftrightarrow{BC}$$

$$m\angle OBC = 90^\circ$$

$$m\angle BOT + m\angle ECF = 90^\circ$$

$$2(m\angle BOT + m\angle ECF) = 180^\circ$$

$$m\angle BOT + m\angle AOB + m\angle BOT = 2(m\angle BOT) + 2(m\angle ECF) \quad \text{Substitution}$$

$$m\angle AOB = 2(m\angle ECF)$$

Given (constructed to be congruent.)

PT is a diameter of the circle,

An angle is the sum of its parts.

Substitution

Given

Definition of tangent

Definition of perpendicular

Sum of the acute angles of a triangle is 90degrees.

Multiplication

Subtraction

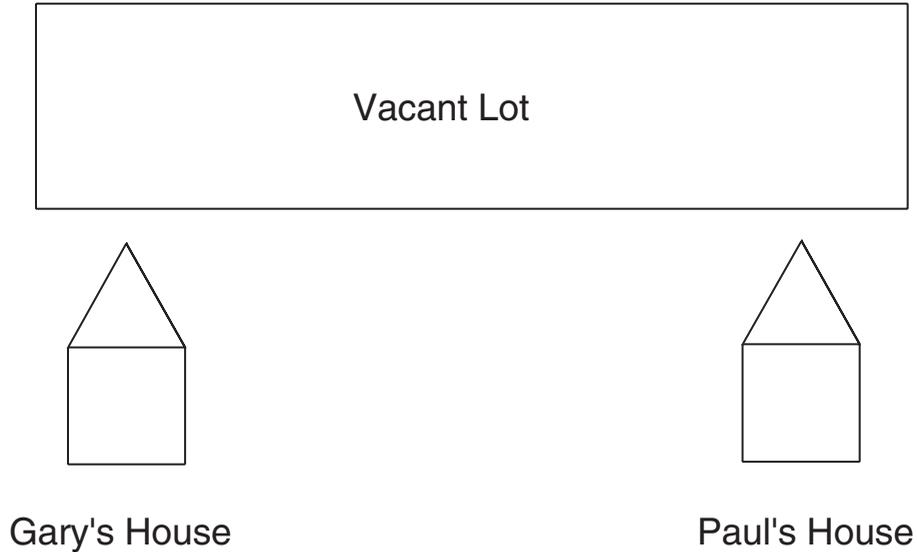




## The Clubhouse

Gary and Paul want to build a clubhouse on the lot next door to their houses. To be fair they decide it has to be the same distance from both of their houses. Describe all the possible locations of the clubhouse so that it is the same distance from both of their houses and in the vacant lot. Justify your answer.

Include a discussion of the properties of the lines and figures you have created. Can you generalize your conjecture?



## Teacher Notes

### Materials:

Tracing paper and one straightedge per student

### Connections to Geometry TEKS:

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(A) based on explorations and using concrete models, formulates and tests conjectures about the properties of parallel and perpendicular lines;

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures.

### Scaffolding Questions:

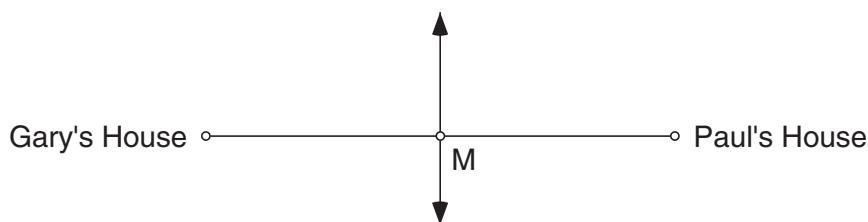
- What are some strategies that you could use to solve this problem?
- Is there more than one location in the vacant lot that they could place the clubhouse?

### Sample Solution:

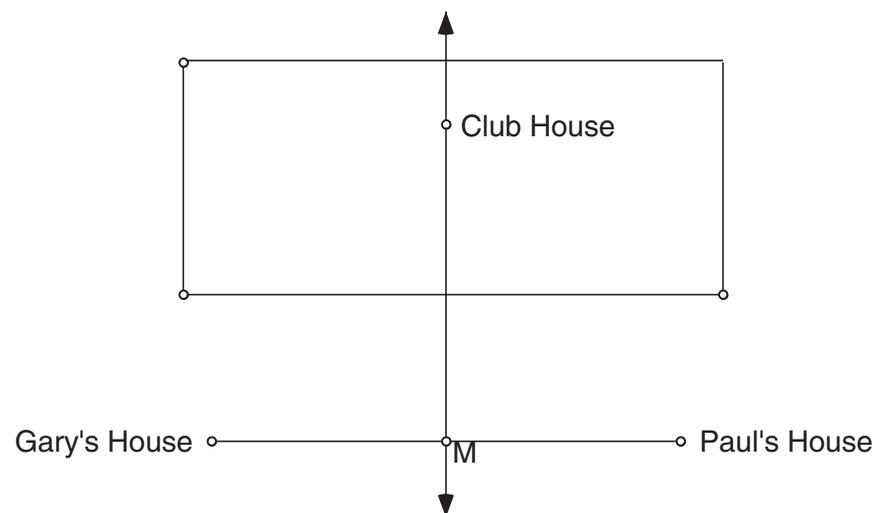
Draw a segment connecting the two houses.

Gary's House ○————○ Paul's House

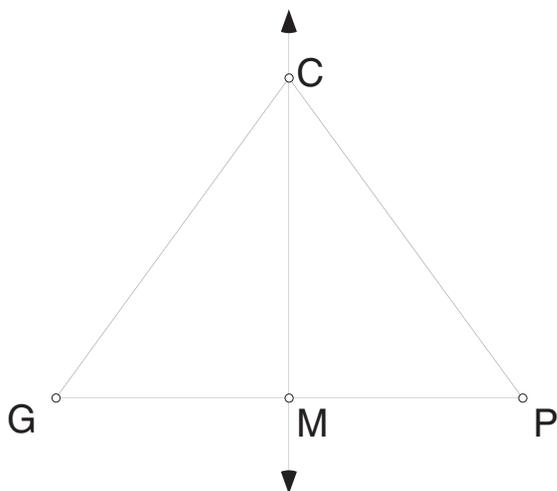
Construct the perpendicular bisector of the connecting segment by reflecting it onto itself.



Place the clubhouse anywhere on this perpendicular bisector and within the lot.



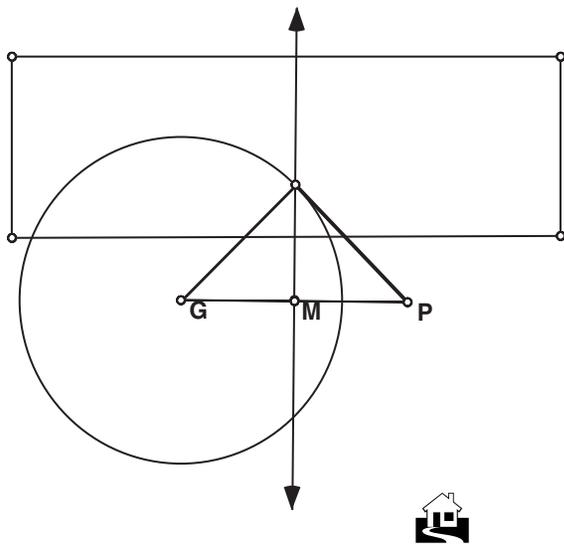
Draw segments connecting their houses to the clubhouse.



Because  $\overleftrightarrow{CM}$  is the perpendicular bisector of  $\overline{GP}$ ,  $\overline{GM} \cong \overline{MP}$  and  $\triangle GMC$  and  $\triangle PMC$  are right triangles with common leg  $\overleftrightarrow{CM}$ . The two right triangles are congruent. Therefore,  $\overline{GC} \cong \overline{PC}$ . This could also be demonstrated by reflecting the segment connecting Gary's house to the clubhouse onto the segment connecting Paul's house to the clubhouse by folding along the perpendicular bisector. The segments are congruent so any point on the perpendicular bisector is equidistant from the endpoints of the segment it bisects.

**Extension Questions:**

- Suppose you have selected the position for the clubhouse. Are there other points in the vacant lot that is the same distance away from Gary's house as the distance between Gary's house and the clubhouse?



**Texas Assessment of Knowledge and Skills:**

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

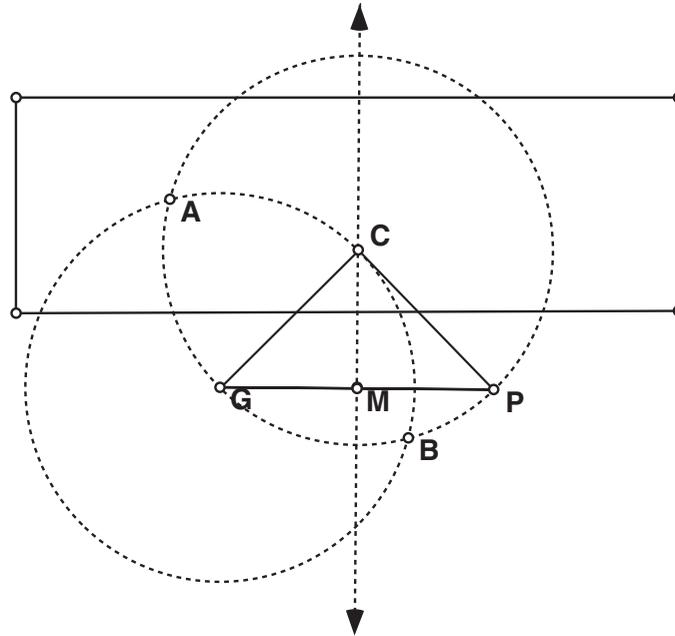
**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

- I. Structure: Triangles Tell All

Any point on the circle with center  $G$  and radius  $\overline{GC}$  that is in the vacant lot will be the same distance from Gary's house as the clubhouse is from Gary's house.

- Describe the location of the points such that the distance from the points to the clubhouse is the same as the distance from Gary's house to the clubhouse.

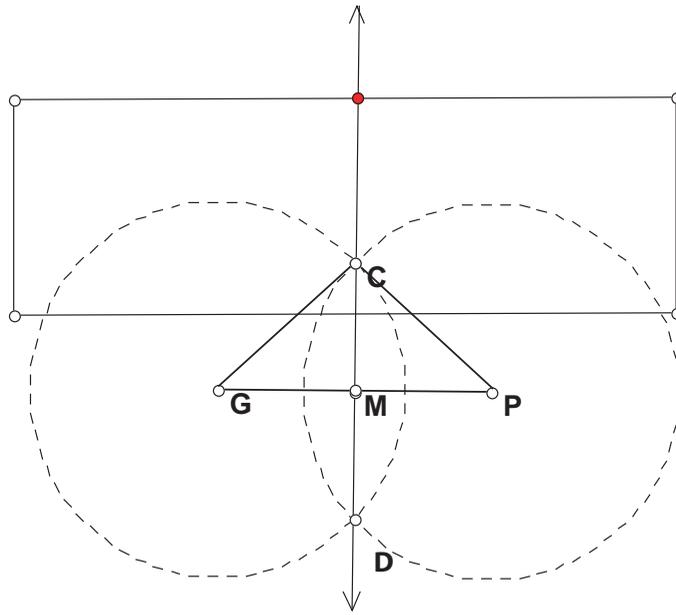
Using  $C$  as a center point and  $GC$  as the radius construct a circle. The intersections of this circle and circle  $G$  are the points,  $A$  and  $B$ , that are the same distance from  $G$  and from  $C$ . One of these points could be in the vacant lot.



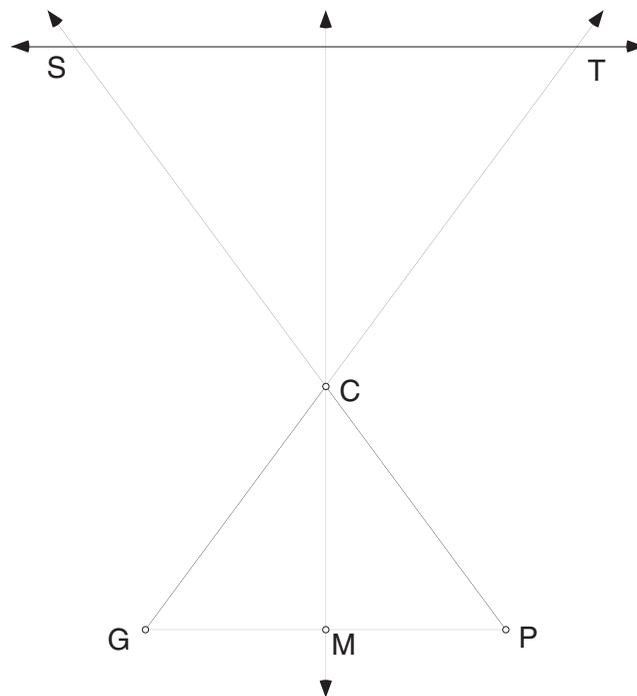
- Is there a point that is the same distance away from  $G$  and  $P$  as the clubhouse?

There is another point,  $D$ , that is on the intersection of the two circles with centers at  $G$  and  $P$ , but it would be in the opposite direction as  $C$  and would not be in the vacant lot.





- Suppose that a street runs parallel to the segment connecting Gary and Paul's houses, and it is on the other side of the vacant lot. Gary and Paul both walk from their own house through the clubhouse to the street as shown in the diagram. Tell me anything you can about the resulting figures.



Possible answers:

If  $\overleftrightarrow{ST} \parallel \overleftrightarrow{GP}$ , then the alternate interior angles are congruent.

$$\angle S \cong \angle P \text{ and } \angle T \cong \angle G$$

Vertical angles are congruent,  $\angle SCT \cong \angle GCP$ .

Two similar triangles are formed because three angles of one triangle are congruent to three angles of the other triangle.

$$\triangle SCT \approx \triangle PCG$$

Because  $CG = CP$ , then triangle  $GCP$  is isosceles.

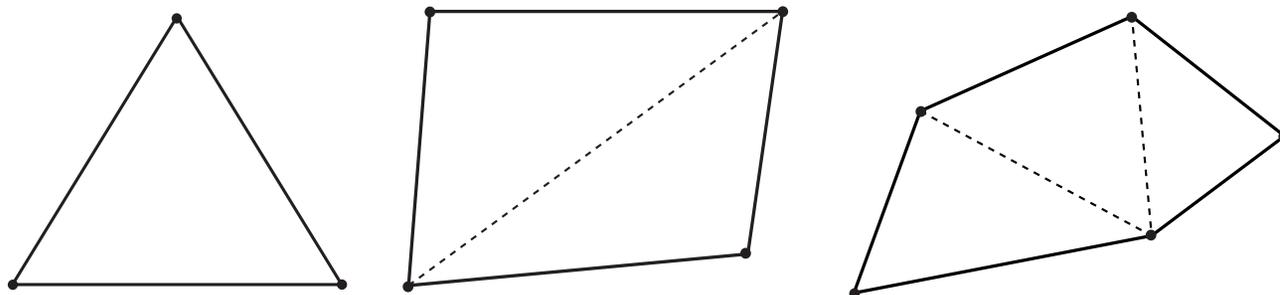
Since  $\triangle SCT \approx \triangle PCG$ , triangle  $SCT$  is also isosceles and  $CS = CT$ .

The distance Gary would walk would be equal to the distance Paul would walk.



## Diagonals and Polygons

Write a conjecture about the relationship between the number of non-intersecting diagonals and the sum of the interior angles in a convex polygon. Justify your conjecture. Represent this function using symbols and a graph.



Polygon Name	Number of Sides	Number of Diagonals From One Vertex	Process to Find the Sum of the Interior Angles	Sum of the Interior Angles
		0		
		1		
		2		
		3		
		4		
		10		
		$n$		



## Teacher Notes

### Materials:

One straightedge, protractor, and graphing calculator per student

### Connections to Geometry

#### TEKS:

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

### Scaffolding Questions:

- How many triangles do the diagonals form?
- What do you know about the sum of the interior angles of a triangle?
- What do you already know about the sum of the interior angles of a polygon?
- How do the number of sides, the number of triangles and the number of diagonals from one vertex relate to the sum of the interior angles?

### Sample Solutions:

Draw the convex polygons; draw the diagonals. The only diagonals that satisfy the requirement to be non-intersecting are the ones drawn from one vertex. If another diagonal is drawn from a different vertex, it will intersect one of the diagonals from the first vertex. Count the number of triangles. There seems to be one more triangle than there are diagonals. To find the sum of their interior angles multiply the number of triangles by 180 degrees.

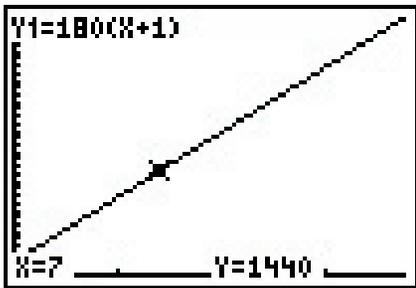
Complete the table.

Polygon Name	Number of Sides	Number of Diagonals From One Vertex	Process to Find the Sum of the Interior Angles	Sum Interior Angles
Triangle	3	0	180 or $180(0+1)$	180
Quadrilateral	4	1	$180(2)$ or $180(1+1)$	360
Pentagon	5	2	$180(3)$ or $180(2+1)$	540
Hexagon	6	3	$180(4)$ or $180(3+1)$	720
Heptagon	7	4	$180(5)$ or $180(4+1)$	900
		10	$180(10+1)$ or $180(11)$	1980
n-gon		$n$	$180(n+1)$	



The number of triangles is 1 more than the number of diagonals. The sum of the interior angles of a convex polygon is 180 times 1 more than the number of non-intersecting diagonals. If  $s$  represents the sum of the angles of the polygon, and  $n$  represents the number of non-intersecting diagonals, then

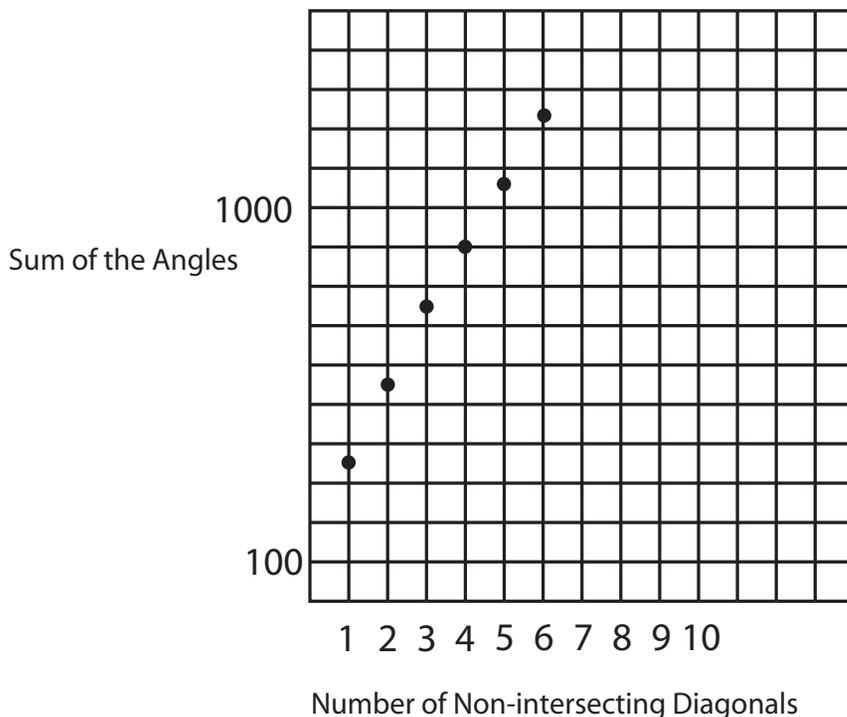
$$s = 180(n + 1).$$



### Extension Questions:

- What are the restrictions on the domain of the function?

*The domain values must be the counting numbers. The graph should be a set of points.*



### Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

Additional objectives are addressed by the Extension Questions that follow the sample solution:

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Objective 10: The student will demonstrate an understanding of the mathematical processes and tools used in problem solving.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

- How does the number of diagonals from one vertex relate to the number of sides?

*There are 3 fewer diagonals than sides. If  $n$  is the number of diagonals, the number of sides is expressed as  $n + 3$ .*

- Will there be a polygon such that the sum of the angles of the polygon is 2000 degrees? Explain why or why not.

*The table shows that the polygon with 10 diagonals has a sum of angles of 1980 degrees and the polygon with 11 diagonals has a sum of angles of 2160 degrees. Since the number of diagonals must be a whole number there is no polygon with an angle sum of 2000 degrees.*

X	Y <sub>1</sub>	
8	1620	
9	1800	
10	1980	
11	2160	
12	2340	
13	2520	
14	2700	
X=10		

- Use your table or graph to determine the sum of the angles of a polygon with 20 diagonals.

*The sum of the angles of a polygon with 20 diagonals is 3780 degrees.*

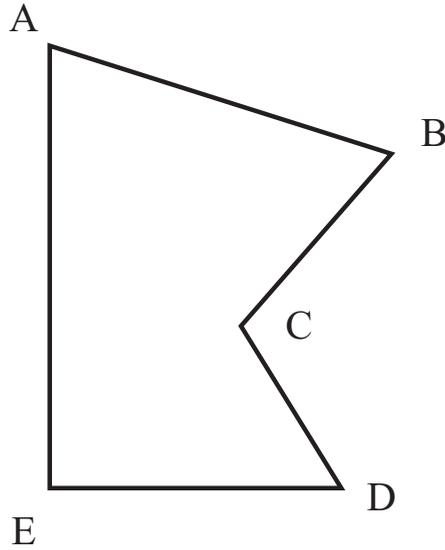
X	Y <sub>1</sub>	
15	2880	
16	3060	
17	3240	
18	3420	
19	3600	
20	3780	
21	3960	
X=20		

- Does the relationship hold true if the polygons are concave?

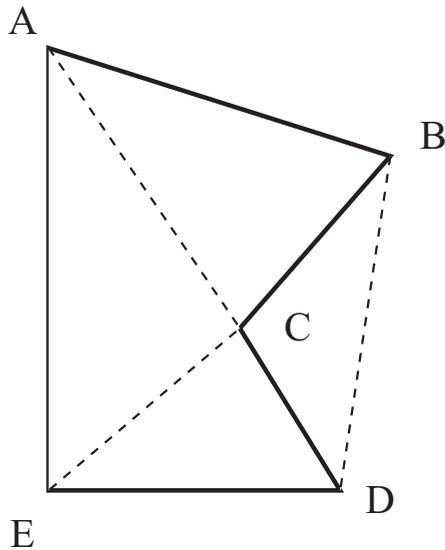


The relationship does not hold true for concave polygons. See the example below.

This is a concave polygon. A diagonal is a segment that has endpoints that are non-adjacent vertices of the polygon.



Draw non-intersecting diagonals.



If the formula is applied using the number 3, the sum of the interior angles is  $(3+1)180 = 4(180)$  or 720 degrees. The sum of the interior angles of this polygon is not 720 degrees. To determine the sum, note that the quadrilateral is composed of three triangles,  $\triangle ACE$ ,  $\triangle ACB$ , and  $\triangle ECD$ . The sum of the angles of the 3 triangles is  $3(180)$  or 540 degrees. The formula does not work for this concave polygon.



## Student Work Sample

This student completed the given table and then wrote the explanation and constructed the graph on the next page.

- Evaluates reasonableness or significance of the solution in the context of the problem.

*The graph is an accurate representation of the situation. The student plots points on the graph and shows a dashed line to indicate that the whole line does not represent the situation. (The number of diagonals must be a whole number.)*

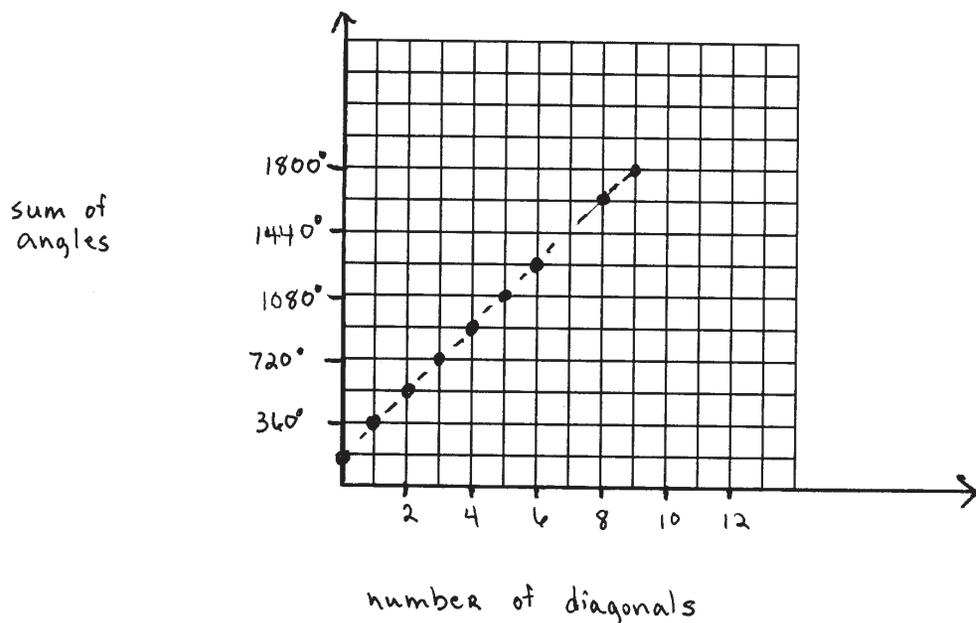
- Uses appropriate terminology and notation.

*In his conjecture, the student identifies the variables, describes the relationships using an accurate verbal description of the relationship between the variables.*



## Diagonals and Polygons

A polygon with  $n$  diagonals from each vertex can be split into  $n + 1$  triangles. Since triangles have an interior angle sum of  $180^\circ$ , and the triangles interior  $\angle$ s make up the larger polygon's interior  $\angle$ s, the sum of the interior angles is  $180^\circ$  times the number of triangles formed by the diagonals of one vertex, or (the number of diagonals from 1 vertex + 1) times  $180^\circ$ .





## The Most Juice

You are in charge of buying containers for a new juice product called Super-Size Juice. Your company wants a container that provides the greatest volume for the given parameters.

- A six-inch straw must touch every point on the base with at least one inch remaining outside of the container.
- The base of the container must either be a square, 4 inches by 4 inches, or a circle that would be inscribed in that square.
- The hole that the straw is inserted into in the top of the container must be exactly one inch from the side of the cylinder or from both sides of the prism.

You have two containers to choose from—a rectangular prism or a right cylinder. Which one would you choose? Justify your answer.



## Teacher Notes

### Materials:

One straightedge and one graphing calculator per student

### Connections to Geometry TEKS:

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

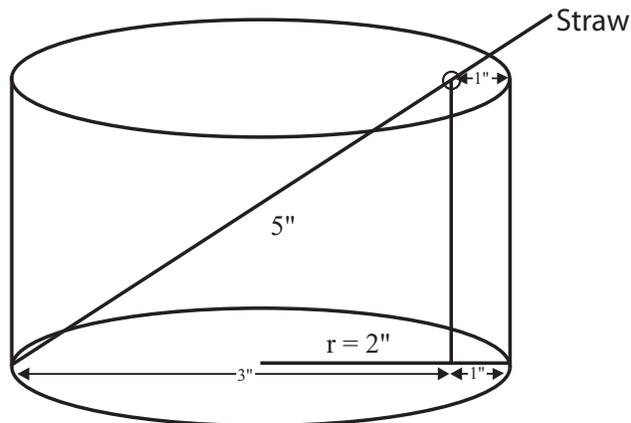
### Scaffolding Questions:

- Can you place the hole for the straw in more than one place?
- Does one inch from the side mean one inch from every side?
- What would pictures of the various containers and hole positions look like?

### Sample Solutions:

For the cylinder:

If the base of the figure is a circle inscribed in a square of sides 4 inches, then the diameter of the inscribed circle is 4 inches. The cylinder has a radius of 2 inches.



If the hole is 1 inch from the edge, a segment from the hole through the center to the other side of the cylinder is  $4 - 1$  or 3 inches. One inch of the straw is outside of the can, so 5 inches are inside the can. When meeting the parameters for the straw, a right triangle with a leg of 3 inches and a hypotenuse of 5 inches is formed. This is a Pythagorean triple, so the other leg must be 4 inches. This makes the minimum height for the cylinder 4 inches.



The volume is:

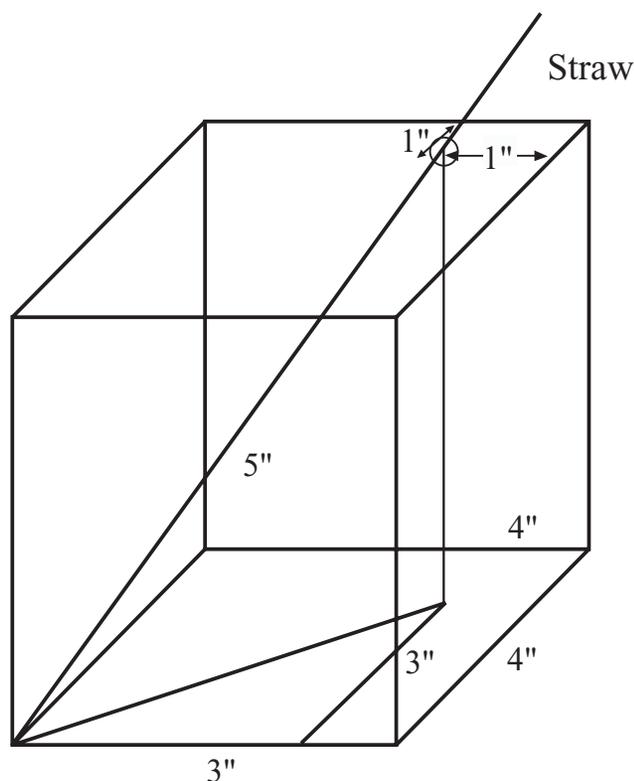
$$V = Bh$$

$$V = \pi r^2 h$$

$$V = 3.14 \times 2^2 \times 4$$

$$V = 50.24 \text{ in}^3$$

For the rectangular prism with a square base:



The triangle is a right triangle with legs of measure 3 inches. The hypotenuse is  $3\sqrt{2}$ . Use the Pythagorean Theorem to determine the height of the box.

$$(3\sqrt{2})^2 + h^2 = 5^2$$

$$18 + h^2 = 25$$

$$h^2 = 7$$

$$h \approx 2.65 \text{ in.}$$

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(C) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of circles and the lines that intersect them; and

(D) analyzes the characteristics of three-dimensional figures and their component parts.

#### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of concepts and uses of measurement and similarity.

Additional objectives are addressed by the Extension Questions that follow the sample solution:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.



**Texas Assessment of Knowledge and Skills: (cont.)**

Objective 3: The student will demonstrate an understanding of linear functions.

Objective 4: The student will formulate and use linear equations and inequalities.

Now find the volume of this box.

$$V = Bh$$

$$V = s^2h$$

$$V \approx 4^2 \times 2.65$$

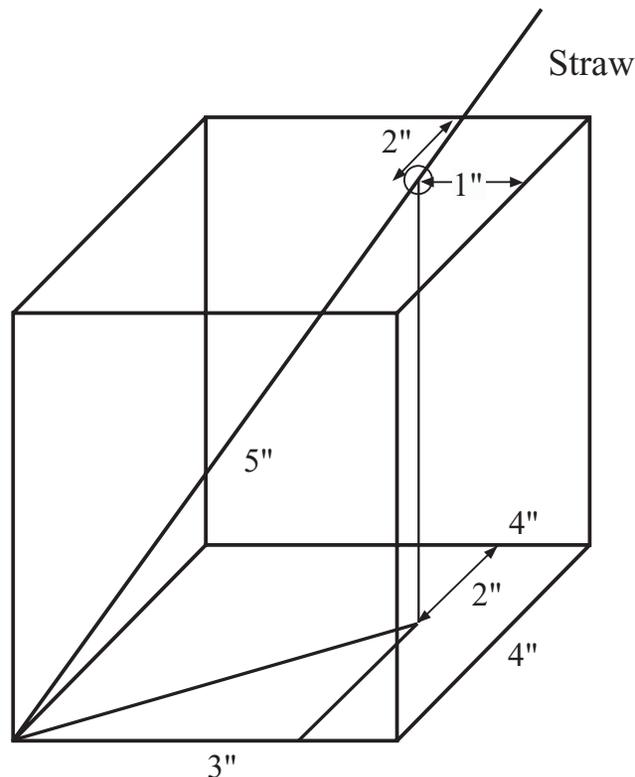
$$V \approx 42.4 \text{ in}^3$$

The volume of the cylinder is approximately  $50.24 \text{ in}^3$ ; the cylinder has the larger volume and is a better choice than the rectangular prism.

**Extension Questions:**

- Suppose that the straw must be 1 inch from one edge and 2 inches from the other edge. How does the volume of this box compare to the other two choices?

*The second yields the largest volume and places the hole 1" from one side in the center of the box.*



*We must use the Pythagorean Theorem to find the base leg of our right triangle.*



$$a^2 + b^2 = c^2$$

$$\sqrt{a^2 + b^2} = c$$

$$\sqrt{2^2 + 3^2} = c$$

$$\sqrt{13} = c$$

$$c = \sqrt{13} \text{ in}$$

Now use the Pythagorean Theorem again to find the height of the box.

$$a^2 + b^2 = c^2$$

$$(\sqrt{13})^2 + b^2 = 5^2$$

$$13 + b^2 = 25$$

$$b^2 = 12$$

$$b \approx 3.46 \text{ in}$$

Now find the volume of this box.

$$V = Bh$$

$$V = s^2 h$$

$$V \approx 4^2 \times 3.46$$

$$V \approx 55.36 \text{ in}^3$$

This box is the best choice because it has the greatest volume.

- Suppose that the length of the straw and the diameter of the base of the cylinder are doubled. How will the volume be affected?

Usually if the dimensions of a solid are doubled, the volume is affected by a factor of 2 cubed. However, there are other factors to consider. The diameter of the base is 8 inches. The cylinder has a radius of 4 inches.

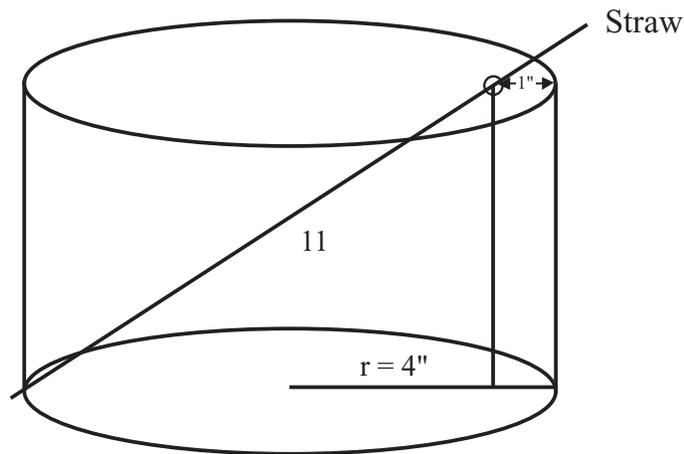
**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

- III. Triangles: Distance in Space
- V. Solid Figure: Dorothy You are Not in Bayou City Anymore

**Teacher's Comment:**

"The students needed clarification on the straw placement. I asked them about drinking out of a cup with a straw and how the straw reaches inside the cup to the corner."





If the hole is 1 inch from the edge, a segment from the hole through the center to the other side of the cylinder is  $8 - 1$  or 7 inches. The part of the straw that is inside the can is  $12 - 1$  or 11 inches.

The height is found using the Pythagorean Theorem.

$$11^2 = 7^2 + h^2$$

$$h^2 = 121 - 49 = 72$$

$$h = \sqrt{72} \approx 8.48$$

The volume is:

$$V = Bh$$

$$V = \pi r^2 h$$

$$V \approx 3.14 \times 4^2 \times 8.48$$

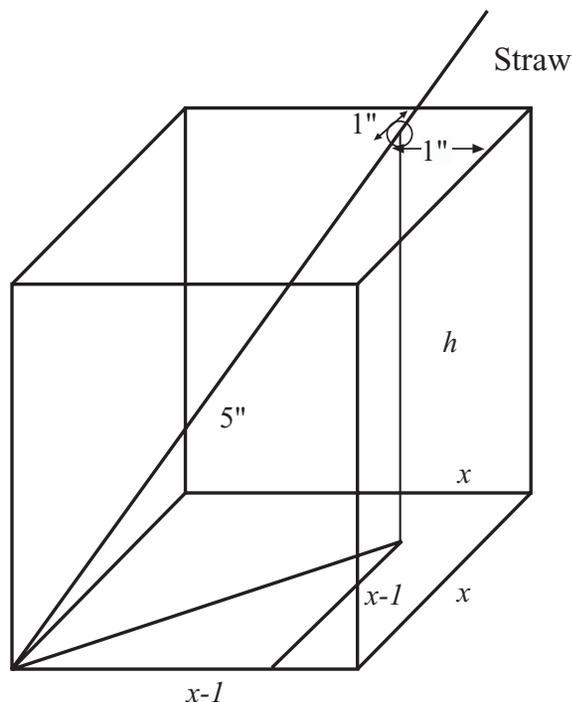
$$V \approx 426.04 \text{ in}^3$$

This number is more than 8 times 50.24 cubic inches. Not all of the dimensions were changed by a multiple of 2, so the theory of changing the volume by a factor of 2 cubed does not apply. Since the height of the new cylinder is 2.12 times the height of the original cylinder, the volume of the new cylinder is  $2(2)(2.12)$  times the volume of the original cylinder.



- Suppose that the base of the rectangular prism could be any square that had dimension 4 inches or less. If the straw hole is one inch from each side of the base, what can you tell about the variables and the volume?

Let the side of the square base be  $x$  inches. The length of one side of the right triangle shown on the base is represented by  $(x-1)$  inches.



The hypotenuse of the right triangle is  $(x-1)\sqrt{2}$ .

The relationship between the side,  $x$ , and the height,  $h$ , is  $5^2 = [(x-1)\sqrt{2}]^2 + h^2$ .

The equation may be solved for  $h$ .

$$h^2 = 5^2 - [(x-1)\sqrt{2}]^2$$

$$h = \pm \sqrt{5^2 - [(x-1)\sqrt{2}]^2}$$

$$h = \pm \sqrt{25 - 2(x-1)^2}$$

Because the height cannot be negative,  $h = +\sqrt{25 - 2(x-1)^2}$ .

The volume of the box is

$$V = x^2 h$$

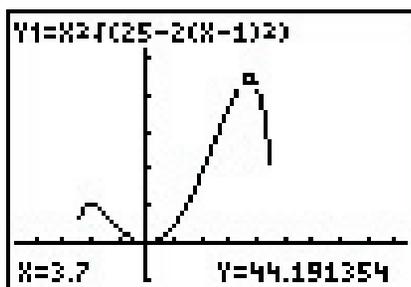
$$V = x^2 \sqrt{25 - 2(x-1)^2}$$



It is possible to graph this function to realize that the maximum volume occurs when  $x$  is about 3.7 inches. Under the given restrictions that  $x$  be less than or equal to 4, a base of about 3.7 inches gives the greatest volume.

```

WINDOW
Xmin=-4.7
Xmax=9.4
Xscl=1
Ymin=-10
Ymax=60
Yscl=10
Xres=1
    
```



X	Y1
3.5	43.31
3.6	43.911
3.7	44.191
3.8	44.083
3.9	43.502
4	42.332
4.1	40.414

X=3.7

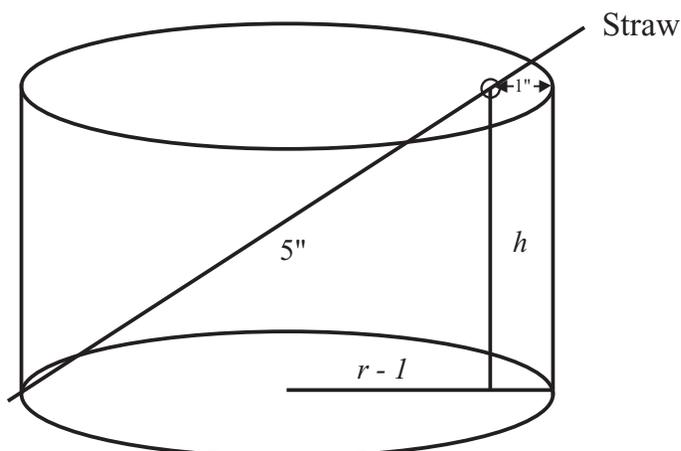
Making the increments on the table smaller would show that the value is between 3.72 and 3.73 inches.

X	Y1
3.7	44.191
3.71	44.199
3.72	44.203
3.73	44.203
3.74	44.199
3.75	44.191
3.76	44.178

X=3.72



- If the base may be a circle with a diameter that is less than or equal to 4, what can you conclude about the volume?



Let the radius of the base be represented by  $r$  and let the height of the cylinder be represented by  $h$ .

The right triangle with hypotenuse 5 inches has legs represented by  $2r-1$  and  $h$ . Apply the Pythagorean Theorem.

$$h^2 + (2r - 1)^2 = 5^2$$

$$h^2 = 5^2 - (2r - 1)^2$$

$$h = \pm \sqrt{5^2 - (2r - 1)^2}$$

$$h = \pm \sqrt{25 - (2r - 1)^2}$$

The height must be positive.  $h = \sqrt{25 - (2r - 1)^2}$

The volume of the cylinder is

$$V = \pi r^2 h$$

$$V = \pi r^2 \sqrt{25 - (2r - 1)^2}$$

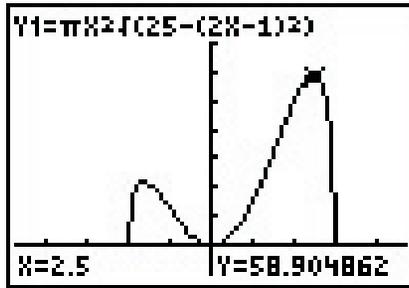
The graph or table of this function illustrates that the maximum volume occurs at a radius of 2.5 inches. However, because of the given restriction that the diameter is less than or equal to 4, the radius must be less than or equal to 2 inches. The maximum volume occurs when the radius is 2 inches.



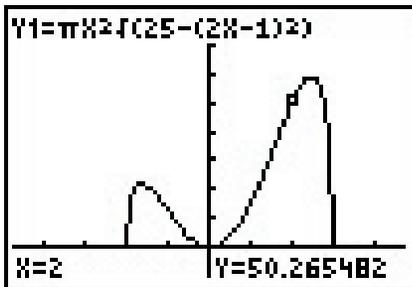
```

WINDOW
Xmin=-4.7
Xmax=4.7
Xscl=1
Ymin=-10
Ymax=80
Yscl=10
Xres=1

```



The maximum volume is approximately  $50.27 \text{ in}^3$ .



## Student Work Sample

The work on the next two pages was created by a group of students.

The solution illustrates many of the solution guide criteria. For example,

- Shows an understanding of the relationships among elements.

*The diagrams and explanations demonstrate the understanding of the relationships between the length and position of the straw and the radius length. These measurements are used to determine the height of the cylinder. Their work shows that they know what measurements they must determine to find the volume of the cylinders.*

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

*The Pythagorean Theorem was used to compute the heights of the cylinder and the prism.*

- Communicates clear, detailed, and organized solution strategy.

*The students described what they were doing to solve the problem. The mathematical processes are detailed. They presented the equations in organized and detailed manner. There is a clear, step-by-step process showing the formulas that are used. They wrote detailed statements to explain their work.*



## The Most Juice

1. First, we are given that we are looking for a juice container that holds the most juice. We know the container has to have a six-in. straw that can touch every point on the base with one inch remaining outside. Also, the hole that contains the straw must be 1 inch from the side of the container. Also, the base must be a square with dimensions of  $4 \times 4$  or a circle that would be inscribed in that square. Finally, the containers could either be a rt. cylinder or a rectangular prism.

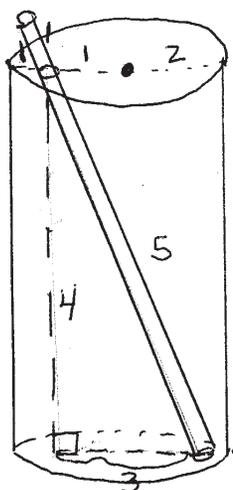
So we came up with these diagrams.

$$3^2 + x^2 = 5^2$$

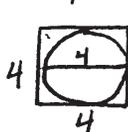
$$9 + x^2 = 25$$

$$x^2 = 16$$

$$x = 4$$



2. For a circle to be inscribed in a square its diameter must be the side of a square



the radius is also 2

$$3^2 + 3^2 = x^2$$

$$x = \sqrt{18}$$

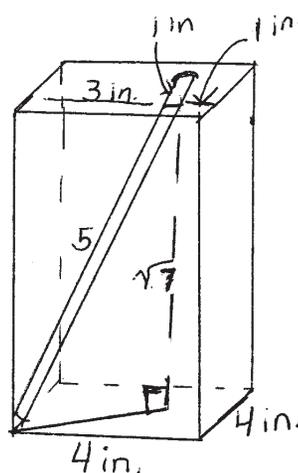
We use the Pythagorean Theorem to find the height.

$$\sqrt{18}^2 + x^2 = 5^2$$

$$18 + x^2 = 25$$

$$x^2 = 7$$

$$x = \sqrt{7}$$



3. Now, since the straw has to have at least 1 inch of it above the surface we know that the farthest reach the straw has in the cylinder must be 5 inches. Therefore we get a triangle with 3, 4, 5 in which four is the height.

3. cont... Now, since the straw has to have at least 1 inch of it above the surface we know that the farthest reach the straw has in the cylinder must be 5 inches. Therefore we get a right triangle with a height of  $\sqrt{7}$

4. Then we find the volume of each container.

Volume of cylinder

$$V = bh$$

$$V = \pi r^2 h$$

$$V = \pi 2^2 (4)$$

$$V = 16\pi$$

$$V = 50.2 \text{ in}^3$$

Volume of Rectangular Prism

$$V = bh$$

$$V = 4^2(\sqrt{7})$$

$$V = 16(\sqrt{7})$$

$$V = 42.3 \text{ in}^3$$

5. In conclusion, we have concluded that the rt. cylinder would hold the most juice for the given specifications because it has more volume than the rectangular prism.





## **Chapter 4:**

*Area,  
Perimeter,  
and Volume*





# Introduction

The problems in this chapter focus on applying the properties of triangles and polygons to compute area, perimeter, and volume.

As presented in Grades K-8, the basic understandings of number, operation, and quantitative reasoning; patterns, relationships, and algebraic thinking; geometry; measurement; and probability and statistics are essential foundations for all work in high school mathematics. Students continue to build on this foundation as they expand their understanding through other mathematical experiences. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999).





## Boxing Basketballs

1. A basketball (sphere) with a circumference of approximately 30 inches is packed in a box (cube) so that it touches each side of the interior of the box. Answer the following question, ignoring the thickness of the surface of the ball and the surface of the box.

What is the volume of the wasted space in the box?

2. A box (cube) that has a side length of 9.5 inches is packed inside a ball (sphere) so that the corners of the box touch the interior of the sphere. Answer the following question, ignoring the thickness of the surface of the ball and the surface of the box.

What is the volume of the wasted space in the ball?



## Teacher Notes

### Materials:

One pencil and graphing calculator per student

### Connections to Geometry

#### TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

(e.1) **Congruence and the Geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

### Scaffolding Questions:

- For the first situation how do the cube and the sphere come in contact with each other (sides or vertices)?
- Do the cube and the sphere have any common measurements?
- In the second problem how do the cube and the sphere come in contact with each other?
- What triangle is formed by the diagonal of the base of the cube and the edges of the base of the cube?
- What triangle is formed by the diagonal of the cube, the diagonal of the base, and one vertical edge of the cube?

### Sample Solutions:

1. First, find the volume of the ball and the box. The diameter of the ball is the side length of the box. Use the circumference,  $C$ , to find the diameter,  $d$ .

$$C = \pi d$$

$$30 = \pi d$$

$$d = 9.55 \text{ in}$$

The volume of the box,  $V$ , in terms of the length of the side,  $s$ , is found by using the formula

$$V = s^3.$$

$$V = 9.55^3$$

$$V = 870.98 \text{ in}^3$$

Find the volume of the ball.

$$V = \frac{4}{3} \pi r^3$$

$$V = \frac{4}{3} \pi \left( \frac{9.55}{2} \right)^3$$

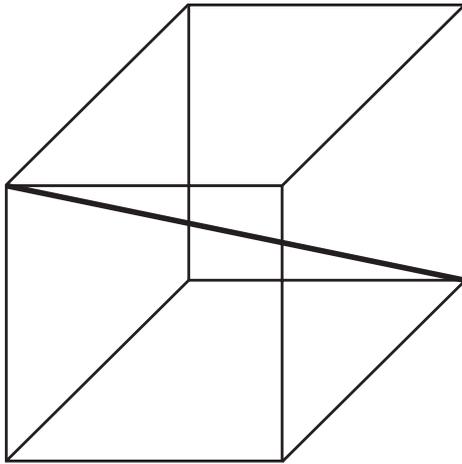
$$V = 456.05 \text{ in}^3$$



Subtract the volume of the ball from the volume of the box to find the volume of wasted space.

$$870.98 \text{ in}^3 - 456.05 \text{ in}^3 = 414.93 \text{ in}^3$$

2. If the cube is surrounded by the ball, the cube will touch the interior of the sphere at eight vertex points. To determine the diameter of the sphere we must find the diagonal of the cube. Since the diagonal of the cube passes through the center of the sphere and its endpoints touch the sphere, its length is the diameter of the sphere.



This can be found by using the Pythagorean Theorem or 45-45-90 special right triangles.

**Texas Assessment of Knowledge and Skills:**

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

- III. Triangles
- IV. Planar Figures: Stained Glass Circles
- V. Solid Figures: Dwellings, Bayou City Dome

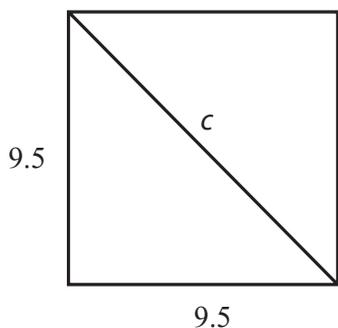
**Teacher's Comments:**

"I handed out a copy of the problem and we approached it first in a whole group discussion. I asked students to draw a picture and prompted them to determine what values were given for the critical attributes and how the parts were related."

"We reviewed formulas for the volume of spheres and rectangular prisms. We discussed the given scaffolding questions.... We need to work with 3-D models more often."



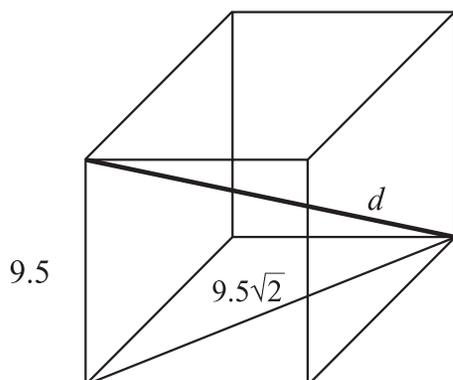
First find a diagonal of a side of the cube. The right triangle has sides measuring 9.5 inches.



$$9.5^2 + 9.5^2 = c^2$$

$$c = \sqrt{2(9.5^2)} = 9.5\sqrt{2}$$

Now use the Pythagorean Theorem again to find the diagonal of the cube. This diagonal is the hypotenuse of a right triangle. One leg of the triangle is the side of the square, which measures 9.5 inches, and the other leg is the diagonal of the side face,  $9.5\sqrt{2}$  inches.



$$9.5^2 + (9.5\sqrt{2})^2 = d^2$$

$$d \approx 16.45 \text{ inches}$$

Now find the volume of the sphere and the cube.

Cube

$$V = s^3$$

$$V = 9.5^3$$

$$V = 857.375 \text{ in}^3 \approx 857.38 \text{ in}^3$$

Sphere

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{4}{3}\pi \left(\frac{16.45}{2}\right)^3$$

$$V = 2330.75 \text{ in}^3$$

The volume of the wasted space is the volume of the sphere minus the volume of the cube.

$$2330.75 - 857.38 = 1473.37$$

The volume of the wasted space is approximately  $1473.37 \text{ in}^3$ .



### Extension Questions:

- If the circumference of the ball were doubled in problem 1, how would the wasted space be affected?

*If the circumference is doubled, the diameter and the radius would also be doubled. The volume should be multiplied by two cubed or eight. This idea may be demonstrated by looking at the general formula.*

*The volume of the wasted space, in terms of the diameter for the original situation, is*

$$(d)^3 - \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = (d)^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right]$$

*If the diameter is doubled, the equation becomes*

$$(2d)^3 - \frac{4}{3}\pi\left(\frac{2d}{2}\right)^3 = (2d)^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right] = 8(d)^3\left[1 - \frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right]$$

*The wasted space is multiplied by 8.*

- If the length of the side of the box in problem 2 is multiplied by one-third, how will the diameter of the sphere be affected?

*The diameter will also be multiplied by one-third. The diameter of the sphere is the diagonal of the cube. The diagonal of the cube is the hypotenuse of a right triangle. The right triangle is similar to the right triangle in the original cube. The corresponding sides of the two right triangles are proportional. Since the side of the new cube (also the leg of the right triangle) is one-third times the side of the original cube, the diagonal of the new cube (also the hypotenuse of the similar triangle) is one-third times the diagonal of the original cube.*



## Student Work Sample

The work on the next page was as an individual assessment following a unit on volumes. The student's work exemplifies many of the criteria from the solution guide. Note the following criteria:

- Shows an understanding of the relationships among elements.

*The student uses arrows in the first problem to show the circumference is used to determine the diameter which is equal to the length of the box's edges. In the second problem he shows how he used the edge of the box and the right triangles to find the radius of the sphere. He shows how he determines the answers in both problems by subtracting the two volumes, the volume of the box, and the volume of the sphere.*

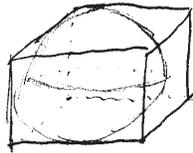
- States a clear and accurate solution using correct units.

*The student not only uses the correct units "in<sup>3</sup>" but indicates that it is the volume of the wasted space.*



# Boxing Basketballs

1



Circumference of a Basketball = 30 in

So the diameter of it is =  $\frac{30}{\pi}$  - the length of box's edges

the radius of it is =  $\frac{30}{2\pi} = \frac{15}{\pi}$  in

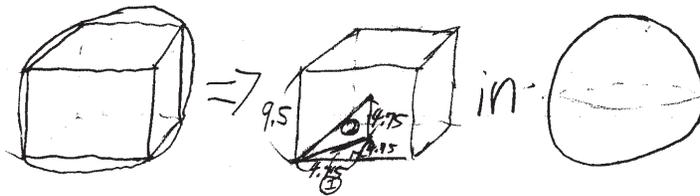
$$V \text{ of the box} = \left(\frac{30}{\pi}\right)^3$$

$$V \text{ of the ball} = \left(\frac{15}{\pi}\right)^3 \pi \times \frac{4}{3}$$

$$V \text{ of the wasted space} = \left(\frac{30}{\pi}\right)^3 - \left(\frac{15}{\pi}\right)^3 \pi \times \frac{4}{3}$$

$$= \boxed{414.85 \text{ in}^3}$$

2



To find the radius of the sphere (look at the diagram):

① Use special triangle:  $4.75\sqrt{2}$

then ② use Pythagorean theorem:  $\sqrt{4.75^2 + (4.75\sqrt{2})^2}$

↓  
This is the radius of the sphere

$$V \text{ of the box} = 9.5^3$$

$$V \text{ of the sphere} = \left(\sqrt{4.75^2 + (4.75\sqrt{2})^2}\right)^3 \times 3.14 \times \frac{4}{3}$$

$$V \text{ of the wasted space} = \left(\sqrt{4.75^2 + (4.75\sqrt{2})^2}\right)^3 \times 3.14 \times \frac{4}{3} - 9.5^3$$

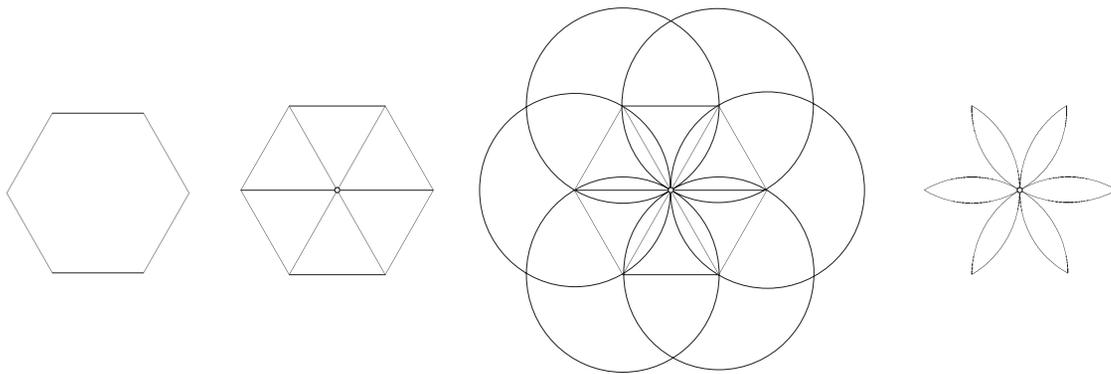
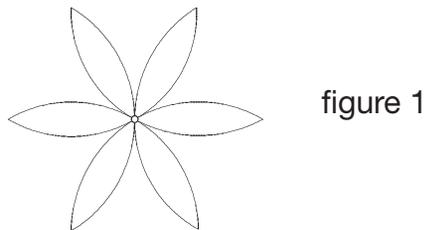
$$= \boxed{1474.10 \text{ in}^3}$$





## Flower

A landscape company has produced the garden design below (see figure 1). The petals are constructed by drawing circles from the vertex to the center of a regular hexagon with a radius of 20 feet (see figure 2).



The company must know the area of the petals in order to determine how many flowers to purchase for planting. Find the area of the petals. Give answers correct to the nearest hundredth.



**Materials:**

One graphing calculator per student

**Connections to Geometry****TEKS:**

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(B) finds areas of sectors and arc lengths of circles using proportional reasoning; and

(C) develops, extends, and uses the Pythagorean Theorem.

## Teacher Notes

**Scaffolding Questions:**

- Are all of the petals the same size?
- What is the combination of figures that form a petal?
- How can the measure of the arc be determined?

**Sample Solutions:**

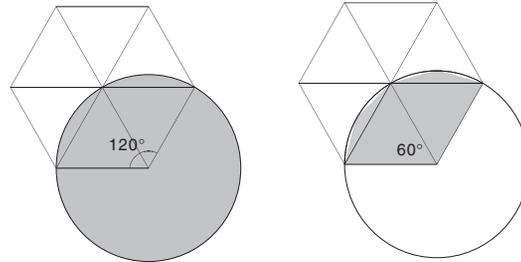
The hexagon eases the solution in that it can be subdivided into 6 equilateral triangles.

First, find the area of one of the circles.

$$A = \pi r^2$$

$$A = \pi(20)^2 = 1256.64 \text{ ft}^2$$

Each angle of the hexagon is  $120^\circ$ . Find the area of the sector of the circle.

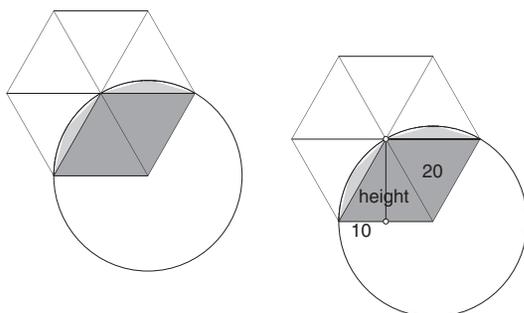


The hexagon is composed of six equilateral triangles. The area of half the sector shown is  $\frac{60}{360}$  or  $\frac{1}{6}$  of the area of the circle. Thus, to determine the area of the sector multiply the area of the circle by one-sixth.

$$A = \frac{1}{6}(1256.64) = 209.44 \text{ ft}^2$$



In order to find the area of the segments of the circle that form a petal, the area of the triangle must be subtracted from the area of the one-sixth sector of the circle.



Determine the height of the triangle.

$$h^2 + 10^2 = 20^2$$

$$h^2 = 300$$

$$h = 10\sqrt{3} \approx 17.32 \text{ ft}$$

The area of the triangle is

$$A = \frac{1}{2}bh = \frac{1}{2}(20)(17.32) = 173.2 \text{ ft.}$$

The area of one-half of a petal is the area of the sector minus the area of the triangle.

$$209.44 - 173.2 = 36.24 \text{ ft}^2.$$

The area of one petal is  $2(36.24) \text{ ft}^2 = 72.48 \text{ ft}^2$

Multiplying by 6 yields the total area of the petals.

$$\text{Total area} = 6(72.48) = 434.88 \text{ ft}^2$$

### Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

IV. Planar Figures: Stained Glass Circles

### Teacher's Comment:

“Stress to students that they must write explanations of steps in words and that they must explain how they found anything that was not given.”



### Extension Questions:

- If the length of the hexagon side is doubled, how is the area affected?

*If the length is doubled, then the area is multiplied by a factor of two squared.*

- Is it possible to use other regular polygons to create flowers using the same process?

*What makes this activity possible is the fact that the side of the hexagon becomes the radius of the circle. The hexagon is the only one for which this relationship is true.*





## Student Work Sample

The next page shows the work of an individual student. The teacher required her students to give a verbal description of the process used to solve the problem. The second page shows the students computations. The work exemplifies the following criteria:

- Communicates clear, detailed, and organized solution strategy.

*The student provides a step by step description of the steps. For each step he gives a reason for each step. “you know that the hexagon is regular, which means that all of the sides are equal.... If you draw a circle around a regular hexagon...”*

- Demonstrated geometric concepts, process, and skills.

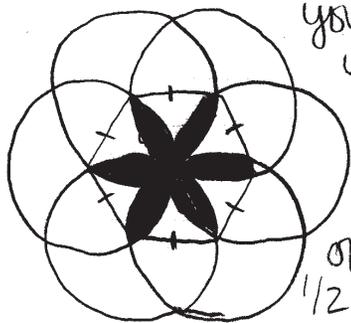
*He used the Pythagorean Theorem. He showed how to get the area of the hexagon and the area of a circle. He showed how to find the area of the petal region by subtracting the area of the hexagon from the area of the circle.*



## Flower

A landscaping company has a flower design. The petals are constructed by drawing circles from the vertices to the center of a regular hexagon, with a radius of 20 feet. The company must know the area of the petals so they will know how many flowers to buy. Find the area of the petals.

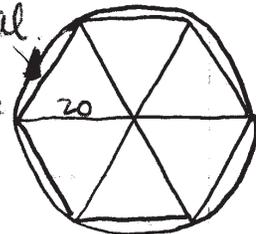
To start off the problem, you know that the hexagon is regular, which means that all of the sides are equal.



You also know that if you draw a circle around a regular hexagon, so where it touches each of the 6 corners, the space outside of the hexagon is equal to  $\frac{1}{2}$  of a petal.

$\frac{1}{2}$  of a single petal.

You find the area of the hexagon, and then subtract that from the area of the circle that you draw.

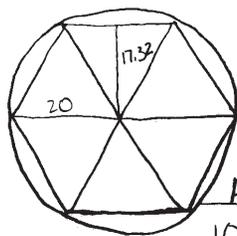


$(10\sqrt{300})/2 \cdot 12 = 1039.23$ . Then the circle  $\pi 20^2$  so subtract,  $\pi 20^2 - (10\sqrt{300})/2 \cdot 12$  and you get 217.41. You know that's  $\frac{1}{2}$  of all of the petals, so you multiply that by 2 to get 434.81  $\text{ft}^2$ .

→



This problem used the Pythagorean theorem, so that you could figure out the area of the hexagon. Also, you used  $\pi r^2$  to find the area of the circle. You then used subtraction and addition to find the total area of the petals.



Work

Height of 1 hexagon

$$10^2 + x^2 = 20^2$$

$$x^2 = 300$$

$$x = \sqrt{300} \text{ or } 17.32$$

Area of all Hexagons

$$10(\sqrt{300}) = 173.21 / 2 = 86.60(12) = 1039.23$$

Area of circle

$$\pi 20^2 = 1256.64$$

$$1256.64$$

$$- 1039.23$$

$$\hline 217.41$$

$$\times \quad 2$$

Area of flower  $\rightarrow 434.81$

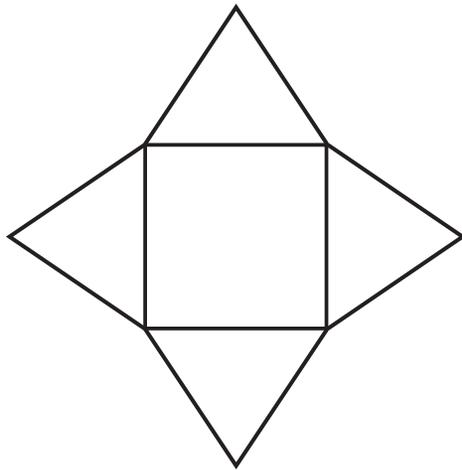
Area of flower petals:  $434.81 \text{ ft}^2$



## Great Pyramid

About 2,550 B.C., King Khufu, the second pharaoh of the fourth dynasty, commissioned the building of his tomb at Giza. This monument was built of solid stone and was completed in just under 30 years. It presides over the plateau of Giza on the outskirts of Cairo, and is the last survivor of the Seven Wonders of the World. At 481 feet high, the Great Pyramid stood as the tallest structure in the world for more than 4,000 years. The base of the Great Pyramid was a square with each side measuring 756 feet.

1. If the average stone used in the construction was a cube measuring 3.4 feet on a side, approximately how many stones were used to build this solid monument? Justify your answer.
2. If you could unfold the pyramid like the figure below, how many acres of land would it cover? (1 acre = 43,560 square feet)



**Materials:**

One graphing calculator per student

**Connections to Geometry TEKS:**

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(B) uses nets to represent and construct three-dimensional objects

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(D) analyzes the characteristics of three-dimensional figures and their component parts.

## Teacher Notes

**Scaffolding Questions:**

- What shape is the solid figure? How do you know?
- What formula do you need in order to find the number of stones?
- Explain what measurements you will use in order to determine the volume of the pyramid.
- What are the component parts of the net of the pyramid?
- Describe how you could find the surface area of the pyramid.

**Sample Solutions:**

1. Find the volume of a stone and the pyramid.

**Stone**

$$V = s^3$$

$$V = 3.4^3$$

$$V = 39.304 \text{ ft}^3$$

**Pyramid**

$$V = \frac{1}{3}Bh$$

$$V = \frac{1}{3}(756)^2(481)$$

$$V = 91,636,272 \text{ ft}^3$$

$$\text{Number of stones} = \frac{V_{\text{pyramid}}}{V_{\text{stone}}} = \frac{91636272}{39.304} \approx 2,331,474.456$$

If whole number of stones are required, the number of stones would be 2,332,475.

2. Find the area of component parts.

**Square**

$$A = s^2$$

$$A = 756^2$$

$$A = 571536 \text{ ft}^2$$

**Triangle**

Find the slant height of the pyramid; this is the hypotenuse of the cross section of the pyramid.

$$a^2 + b^2 = c^2$$

$$481^2 + \left(\frac{1}{2}756\right)^2 = (\text{slant height})^2$$

$$\sqrt{374245} = \text{slant height}$$

$$611.76 \text{ ft} \approx \text{slant height}$$



Now find the area of a side.

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2} \cdot 756 \cdot 611.76$$

$$A \approx 231245.28 \text{ ft}^2$$

The four sides of the pyramid are congruent. The total surface area of the pyramid is the area of the square base plus the area of the four sides.

$$\begin{aligned} \text{Total area} &\approx 571536 + 4(231245.28) \\ &\approx 1496517.12 \text{ ft}^2 \end{aligned}$$

$$\text{Convert to acres: } 1496517.12 \text{ ft}^2 \left( \frac{1 \text{ acre}}{43560 \text{ ft}^2} \right) \approx 34 \text{ acres}$$

### Extension Questions:

- If all the dimensions of the pyramid were multiplied by 4, how would the volume of the pyramid be changed?

*If a side is multiplied by a factor of 4, the volume is multiplied by a factor of  $4^3$ .*

- If you want to have the net cover twice the surface area, how would the sides of the pyramid need to be changed?

*If a side is multiplied by a factor of  $k$ , the area is multiplied by a factor of  $k^2$ . If the area is multiplied by a factor of  $h$ , the side is multiplied by a factor of  $\sqrt{h}$ .*

### Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connections to High School Geometry: Supporting TEKS and TAKS Institute:

III. Pythagorean Theorem: A Patty Paper Proof

V. Solid Figures: Dwellings

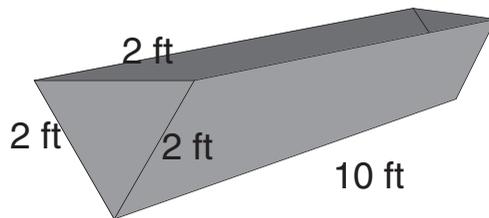




## Walter and Juanita's Water Troughs

You have been hired as chief mathematician by a company named Walter and Juanita's Water Troughs. This company builds water troughs for various agricultural uses. The company has one design (see figure 1). Your job is to perform mathematical analysis for the owners.

figure 1



1. A customer would like to know what the depth of the water is (in inches) if the trough only has 32 gallons in it.
2. The interior of the troughs must be coated with a sealant in order to hold water. One container of sealant covers 400 square feet. Will one container of sealant be enough to seal ten troughs? Why or why not?
3. Walter and Juanita would like to explore some minor modifications of their original design. They would like to know which change will produce a water trough that would hold more water—adding one foot to the length of the trough, making it 11 feet long, or adding three inches to each side of the triangular bases, making them 2 feet 3 inches on each side (see figures 2 and 3). Justify your answer.

figure 2

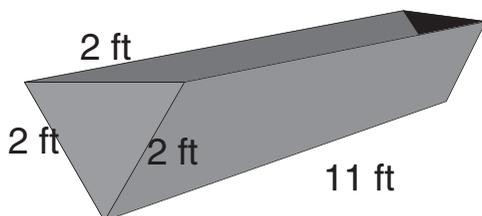
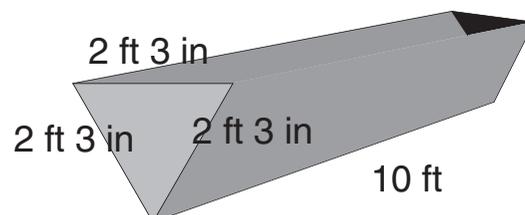


figure 3



## Teacher Notes

### Materials:

One calculator and straightedge per student

### Connections to Geometry

#### TEKS:

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem.

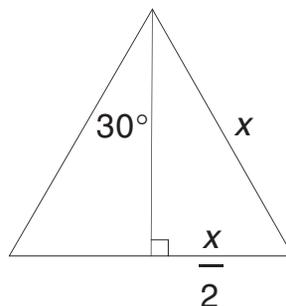
(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

### Scaffolding Questions:

- What important components of the trough are needed to solve the problem?
- What is your prediction for which trough will hold the most water in number 3? Justify your answer.

### Sample Solutions:

1. To solve this problem the area of the triangular base must be found in terms of an unknown side. If water is poured into the trough, the height of the water is a function of the side of the triangle. Consider the end of the trough that is an equilateral triangle.



Using 30-60-90 special right triangle properties or the Pythagorean Theorem the altitude is  $\frac{x\sqrt{3}}{2}$  ft.

Area of the base can be found by:

$$\text{Area} = \frac{1}{2}(x)\left(\frac{x\sqrt{3}}{2}\right)$$

$$\text{Area} = \frac{x^2\sqrt{3}}{4} \text{ ft}^2$$

The desired volume is given in gallons and must be converted to cubic feet because the measurements are given in feet.

$$32 \text{ gal.} \cdot \frac{1 \text{ ft}^3}{7.48 \text{ gal}} \approx 4.28 \text{ ft}^3$$

The volume of the tank is the area of the triangular base times the length of the tank.



$$\left(\frac{x^2\sqrt{3}}{4}\right)10 = 4.28 \text{ ft}^3$$

$$(x^2\sqrt{3})10 \approx 17.12$$

$$x \approx \pm\sqrt{0.988} \approx 0.994 \text{ ft}$$

The side length is 0.994 ft.

The depth of the water is the altitude of the base.

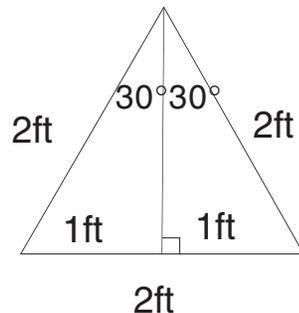
$$\text{altitude} = \frac{0.994\sqrt{3}}{2}$$

$$\text{altitude} \approx 0.861 \text{ ft}$$

$$\text{depth of water} \approx 10.3 \text{ in}$$

2. Find the interior surface area of one trough and multiply by 10.

Figure 2: First, find the area of the base (the ends of the trough). In order to do this, find the altitude of the triangle. The base is an equilateral triangle, so the altitude bisects the side and the intercepted angle. The angles of an equilateral triangle are all  $60^\circ$ .



Use the Pythagorean Theorem to find the length of the altitude.

$$a^2 + b^2 = c^2$$

$$a^2 + 1^2 = 2^2$$

$$a^2 + 1 = 4$$

$$a^2 = 3$$

$$a = \pm\sqrt{3}$$

### Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connections to High School Geometry: Supporting TEKS and TAKS Institute:

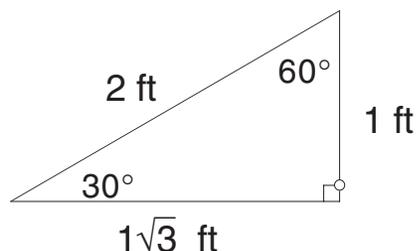
- III. Pythagorean Theorem: A Patty Paper Proof
- V. Solid Figures: Dwellings



The altitude must be positive.

$$a = \sqrt{3}$$

Another approach to determining the altitude is to use 30-60-90 special right triangle properties. The hypotenuse is twice the shortest side. The side opposite the 60-degree angle is the shorter leg times  $\sqrt{3}$ .



The altitude is  $1\sqrt{3} \approx 1.73$  ft.

The area of the base may be found using the altitude and the base of the triangle.

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3} \approx 1.73 \text{ ft}^2$$

The surface area is the sum of the areas of the 2 ends and the 2 sides.

$$2\sqrt{3} + 2(20) \approx 43.64 \text{ ft}^2$$

The surface area of ten troughs is 10 times the surface area of one trough.

$$(10)43.46 = 434.6 \text{ ft}^2$$

Since the gallon of paint covers  $400 \text{ ft}^2$ , there will not be enough sealant to seal ten troughs.

3. The volume of a right prism is the area of the base times the height.

The area of the base for the first case was found in problem 2.

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(2)(\sqrt{3}) = \sqrt{3} \approx 1.73 \text{ ft}^2$$

$$V = Bh = \sqrt{3}(11) \approx 1.73(11) \approx 19.05 \text{ ft}^3$$



For the second case, find the altitude of the right triangle base with side of 2.25 feet and the shorter leg, one-half of 2.25 or 1.125. Use the Pythagorean Theorem.

$$a^2 + b^2 = c^2$$

$$a^2 + (1.125)^2 = (2.25)^2$$

$$a^2 \approx 5.0625 - 1.266$$

$$a \approx \sqrt{3.8}$$

$$a \approx 1.95 \text{ ft}$$

Find the area of the base.

$$A = \frac{1}{2}bh$$

$$A \approx \frac{1}{2}(2.25)(1.95)$$

$$A \approx 2.19 \text{ ft}^2$$

Find the volume.

$$V = Bh$$

$$V \approx (2.19)(10)$$

$$V \approx 21.9 \text{ ft}^3$$

Adding 3 inches to each side of the base will produce a greater increase in volume than adding a foot to the distance between the bases.



### Extension Questions:

- A new tank is designed in the shape of a hemisphere (half of a sphere) to hold the same volume of water as the tank in figure one. What is the depth of the water if the tank is full?

The area of the base (the ends of the trough) is the same as the area for figure 2. The area is  $\sqrt{3} \approx 1.73 \text{ ft}^2$ . The volume of a right prism is the area of the base times the height.

$$V = Bh = \sqrt{3}(10) \approx 17.3 \text{ ft}^3.$$

Volume of a sphere =  $\frac{4}{3}\pi r^3$ , but we only want  $\frac{1}{2}$  of this amount.

$$V = \frac{1}{2} \left( \frac{4}{3} \pi r^3 \right)$$

$$17.3 = \frac{1}{2} \left( \frac{4}{3} \pi r^3 \right)$$

$$17.3 = \frac{2}{3} \pi r^3$$

$$\frac{3}{2} \cdot 17.3 = \frac{3}{2} \cdot \frac{2}{3} \pi r^3$$

$$\frac{25.95}{\pi} = \frac{\pi}{\pi} r^3$$

$$\sqrt[3]{8.26} = \sqrt[3]{r^3}$$

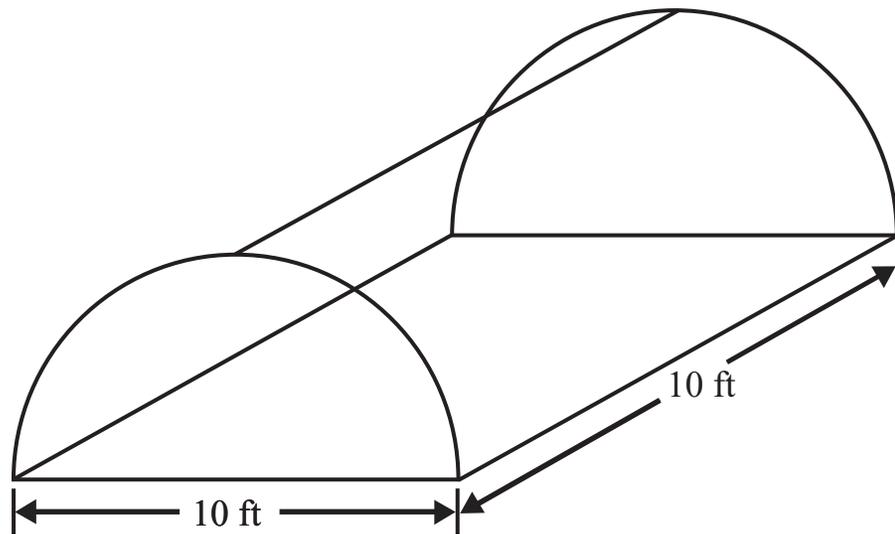
$$2.02 \approx r$$

The depth is approximately 2.02 feet.



## Greenhouse

The backyard greenhouse in the figure below uses plastic tubing for framing and plastic sheeting for wall covering. The end walls are semicircles, and the greenhouse is built to the dimensions in the figure below. All walls and the floor are covered. The door is formed by cutting a slit in one of the end walls.



1. How many square feet of plastic sheeting will it take to cover top, sides, and floor of the greenhouse?
2. What is the volume of the greenhouse?



**Materials:**

One graphing calculator per student

**Connections to Geometry****TEKS:**

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds area of regular polygons and composite figures

(B) finds areas of sectors and arc lengths of circles using proportional reasoning; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

## Teacher Notes

**Scaffolding Questions:**

- Describe the shape of the greenhouse.
- What dimensions do you need to determine the surface area?

**Sample Solutions:**

1. The ends of the greenhouse are semicircles that have a diameter of ten. Area of each end:

$$A = \pi r^2$$

$$A = \pi(5)^2 \approx 78.54$$

The area of one-half of the circle is  $A \approx 78.54 \cdot \frac{1}{2} \approx 39.27 \text{ ft}^2$

Area of side:

Since the greenhouse is half of a cylinder, we can unwrap the sides, find the area of the rectangle, and divide by 2. One dimension of the rectangle is the circumference, and the other is 10.

$$C = \pi d$$

$$C = \pi 10$$

$$C \approx 31.4 \text{ ft}$$

Find the area.

$$\text{Surface area of the cylinder} \approx 31.4 \cdot 10 \approx 314 \text{ ft}^2$$

$$\begin{aligned} \text{Surface area of half of the cylinder} &\approx 314 \text{ ft}^2 \cdot \frac{1}{2} \\ &\approx 157 \text{ ft}^2 \end{aligned}$$

The floor is the area of the square.

$$A = 10 \cdot 10$$

$$A = 100 \text{ ft}^2$$



Total square feet of plastic needed is

$$39.27 + 39.27 + 157 + 100 = 335.54 \text{ ft}^2.$$

2. The volume of the greenhouse is the area of the base multiplied by the distance between the bases. The base is a semicircle with a diameter of 10.

$$A = \frac{1}{2} \pi r^2 \text{ because we only need half of the circle.}$$

$$A = \frac{1}{2} \pi (5)^2$$

$$A \approx 39.27 \text{ ft}^2$$

The volume of the greenhouse may be computed using the formula.

$$V = Bh$$

$$V = 39.27 \cdot 10$$

$$V \approx 392.7 \text{ ft}^3$$

### Extension Question:

- What would the dimensions of a new square floor greenhouse have to be in order to double the volume of the greenhouse in this problem?

*Let the side length be  $x$ . The radius of the semicircular ends would be expressed as  $\frac{x}{2}$ .*

*The area of the base is one-half of the area of a circle with radius  $\frac{x}{2}$ .*

$$A = \frac{1}{2} \pi \left( \frac{1}{2} x \right)^2$$

$$A = \frac{1}{8} \pi x^2$$

*The volume of the greenhouse is the area of the base (the area of the semicircle) times the length,  $x$ .*

### Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Dwellings



The rule for the volume as a function of the side is:

$$V = \frac{1}{8}\pi x^2 \cdot x$$

The volume must be double the original volume, or two times  $392.7 \text{ ft}^3$ . The volume must be  $785.4 \text{ ft}^3$ .

$$785.4 = \frac{1}{8}\pi x^3$$

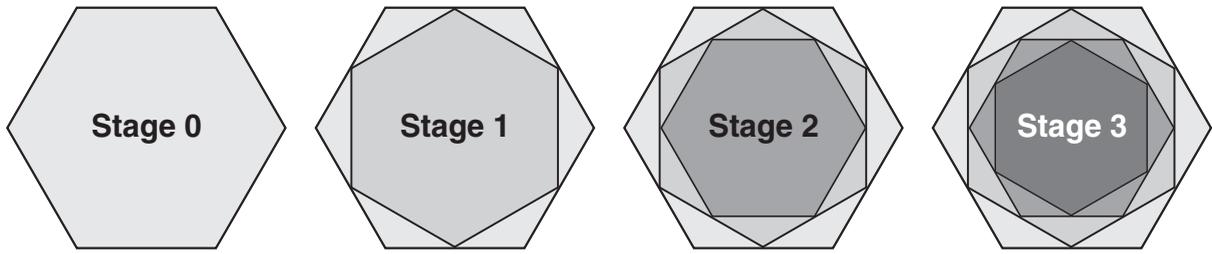
$$x \approx 12.6 \text{ ft}$$

The length of the side of the base must be about 12.6 feet.



## Nesting Hexagons

The first four stages of nested hexagons are shown below. Connecting the midpoints of the sides of the previous stage creates each successive stage. The side of the hexagon in Stage 0 measures 3 units.



1. Create a function rule to find the area of the innermost hexagon of any stage.
2. Find the area of the innermost hexagon in stage 10.
3. What is the domain of your function rule?
4. What is the range of your function rule?



**Materials:**

One graphing calculator and straightedge per student

**Connections to Geometry****TEKS:**

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem.

## Teacher Notes

**Note:** It may prove beneficial to create a geometry software sketch of “Nesting Hexagons” once students have performed the calculations for Stages 1 and 2. Students can explore whether the size of the hexagon has any impact on the relationship between the stages. They can also use the measurements to help determine a function rule for this situation.

It may also prove beneficial to model the organization of data in a table such as the one in the possible solution strategies. Encourage students to round area values to the nearest hundredth.

**Scaffolding Questions:**

If students have created a table, the following questions may be asked:

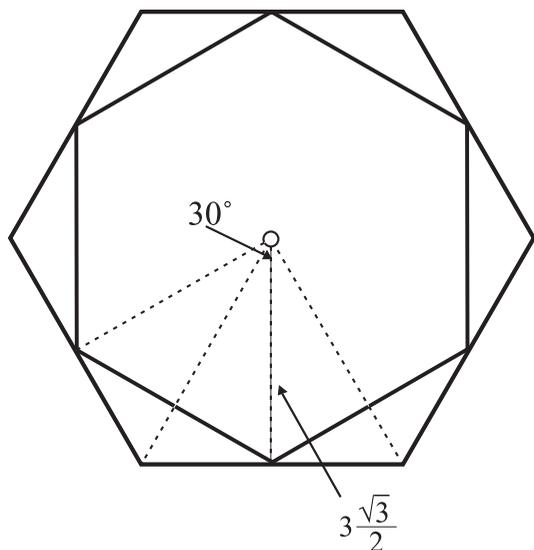
- Is there a common difference between the terms in the area column of the table?
- Is there a common second difference between the terms in the table?
- Is there a common ratio between the terms in the table?
- How is each stage related to the previous stage?

**Sample Solutions:**

1. The perimeter is 18. Use  $A = \frac{1}{2}ap$  to find the area of the regular hexagon. Each interior angle of a regular hexagon is  $120^\circ$ . The segments from each vertex to the center bisect the interior angles forming 6 equilateral triangles. Therefore the radius and the sides of the hexagon have the same measure.

Using 30-60-90 right triangle properties, the length of the apothem is found by dividing the hypotenuse, or in this case the radius, by 2 and multiplying by  $\sqrt{3}$ . The length of the apothem for stage 0 is  $\frac{3\sqrt{3}}{2}$ , and the length of the radius is 3. The perimeter,  $p$ , is 6 times 3 units or 18 units.





$$A = \frac{1}{2}ap$$

$$A = \frac{1}{2} \left( 3 \frac{\sqrt{3}}{2} \right) (18)$$

$$A \approx \frac{1}{2} \cdot 3 \cdot 18 \cdot \frac{\sqrt{3}}{2}$$

$$A = 27 \frac{\sqrt{3}}{2} \text{ or}$$

$$A \approx 23.38 \text{ square units}$$

Stage	Apothem	Radius	Side length	Perimeter	Area	Area Rounded to nearest hundredth
0	$3 \frac{\sqrt{3}}{2}$	3	3	18	$27 \frac{\sqrt{3}}{2}$	23.38

The apothem of the stage 0 hexagon is now the radius of the stage 1 hexagon. The length of the side of the stage 1 hexagon is  $3 \frac{\sqrt{3}}{2}$ .

Using 30-60-90 right triangle properties, the length of the side is  $3 \frac{\sqrt{3}}{2}$ , and the length of the apothem is  $3 \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$ .

### Texas Assessment of Knowledge and Skills:

Objective 1: The student will describe functional relationships in a variety of ways.

Objective 2: The student will demonstrate an understanding of the properties and attributes of functions.

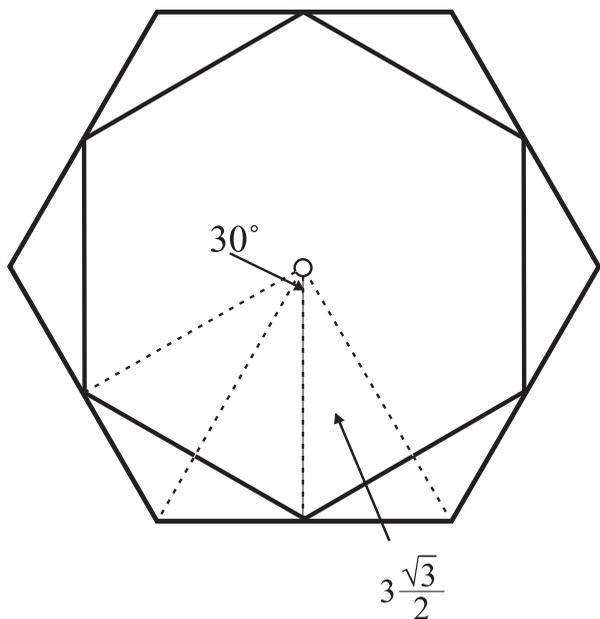
Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connections to High School Geometry: Supporting TEKS and TAKS:

- II. Transformations: Fractals
- IV. Planar Figures: Stained Glass Circles





$$A = \frac{1}{2} ap$$

$$A = \frac{1}{2} \left( 3 \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \right) \left( 18 \frac{\sqrt{3}}{2} \right)$$

$$A = \frac{1}{2} \cdot 3 \cdot 18 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$A = 27 \frac{\sqrt{3}}{2} \cdot \frac{3}{4} \approx 17.54 \text{ square units}$$

Stage	Apothem	Radius	Side length	Perimeter	Area	Area Rounded to nearest hundredth
0	$3 \frac{\sqrt{3}}{2}$	3	3	18	$27 \frac{\sqrt{3}}{2}$	23.38
1	$3 \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3 \frac{\sqrt{3}}{2}$	$3 \frac{\sqrt{3}}{2}$	$18 \frac{\sqrt{3}}{2}$	$27 \frac{\sqrt{3}}{2} * \frac{3}{4}$	17.54

A pattern emerges when the rest of the table is completed. In each new stage the side length, radius, apothem, and perimeter change by a factor of  $\frac{\sqrt{3}}{2}$  times the amount in the previous stage. The area in the new stage changes by a factor of  $\frac{3}{4}$  times the amount in the previous stage.



Stage	Apothem	Radius	Side Length	Perimeter	Area	Area Rounded to nearest hundredth
0	$3\frac{\sqrt{3}}{2}$	3	3	18	$27\frac{\sqrt{3}}{2}$	23.38
1	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} * \frac{3}{4}$	17.54
2	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4}$	13.15
3	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$3\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$18\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}$	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4} * \frac{3}{4}$	9.86

A recursive process of repeated multiplication can be used to generate the function rule for the area.

Stage	Area
0	$27\frac{\sqrt{3}}{2}$
1	$27\frac{\sqrt{3}}{2} * \frac{3}{4}$
2	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4}$
3	$27\frac{\sqrt{3}}{2} * \frac{3}{4} * \frac{3}{4} * \frac{3}{4}$
4	$27\frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^4$
$x$	$27\frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^x$

Another approach is to look for patterns in the area values themselves. It will facilitate the development of a pattern if the values are rounded to the nearest hundredth. If the difference between each term is found, no pattern emerges. If the difference between each of those differences is found, no pattern emerges. However, if the ratios of the terms are found, a common ratio emerges. This means that each successive term can be found by multiplying the previous term by this common ratio. This leads to the generation of the exponential rule for this situation.



Stage	Process	Area	1 <sup>st</sup> differences	2 <sup>nd</sup> differences	common ratio
0	23.38 or $23.38 = 23.38 (0.75)^0$	23.38	5.84	1.45	$\frac{17.54}{23.38} \approx 0.75$
1	23.38 (0.75) or $23.38 (0.75) = 23.38 (0.75)^1$	17.54			
2	17.54 (0.75) or $23.38 (0.75)(0.75) = 23.38 (0.75)^2$	13.15	4.39	1.1	
3	13.15 (0.75) or $23.38(0.75)(0.75)(0.75) = 23.38 (0.75)^3$	9.86	3.29		$\frac{9.86}{13.15} \approx 0.75$
Description	The area of the current stage hexagon is the area of the stage 0 hexagon multiplied by 0.75 raised to the current stage number.				
$x$	$23.38 (0.75)^x$	$A$			

$$A = 27 \frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^x$$

$$2. A = 27 \frac{\sqrt{3}}{2} \left(\frac{3}{4}\right)^{10} \approx 1.32$$

3. Possible notation for the domain of the function might include one of the following: all real numbers  $-\infty < x < +\infty$  or  $(-\infty, +\infty)$ .

The domain for the problem situation would be non-negative integers.

4. Possible notation for the range of the function might include one of the following: real numbers greater than  $0 < y < +\infty$  or  $(0, +\infty)$ .

The range for the problem situation is more complicated. The range is the specific set of numbers corresponding to the domain values for the problem situation. For example,

$$\left\{ \frac{27\sqrt{3}}{2}, \frac{81\sqrt{3}}{8}, \frac{243\sqrt{3}}{32}, \dots \right\}$$



### Extension Questions:

- What is the relationship between the ratio of side lengths  $\frac{\text{stage}(n)}{\text{stage}(n-1)}$  and the ratio of perimeters? How is this number related to the ratio of areas?

*The ratio of side lengths and the ratios of perimeters are both  $\frac{\sqrt{3}}{2}$ . This ratio squared is the ratio of the areas.*

- What is the largest possible area in this problem? Justify your answer.

*The largest possible area is  $27\frac{\sqrt{3}}{2}$  or 23.38.*

*The area of each successive stage is  $\frac{3}{4}$  times the area of the hexagon of the previous stage so the area is always smaller than the area of the hexagon of stage 0.*

- What is the smallest possible area in this problem? Justify your answer.

*There will never be a smallest area. Any area thought to be the smallest area can be multiplied by  $\frac{3}{4}$  according to our rule in order to find a new stage with a smaller area.*

*The area will approach 0 as the stage number gets very large but will never actually be 0.*





**Chapter 5:**  
*Solids,  
and Nets*





# Introduction

The assessment problems in this chapter require students to analyze relationships between two- and three-dimensional objects and use these relationships to solve problems.

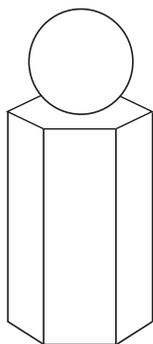
Spatial reasoning plays a critical role in geometry; shapes and figures provide powerful ways to represent mathematical situations and to express generalizations about space and spatial relationships. Students use geometric thinking to understand mathematical concepts, and the relationships among them. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999).



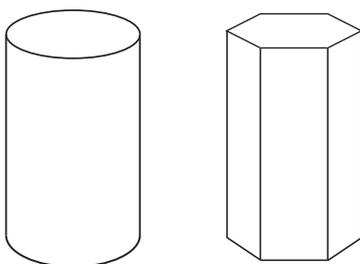


## Perfume Packaging

Ima Smelley would like to package her newest fragrance, *Persuasive*, in an eye-catching, yet cost-efficient box. The *Persuasive* perfume bottle is in the shape of a regular hexagonal prism 10 centimeters high. Each base edge of the prism is 3 centimeters. The cap is a sphere with a radius of 1.5 centimeters.



Ms. Smelley would like to compare the cost of the proposed box designs shown below. It is important that each bottle fit tightly in the box to avoid movement during shipping. The dimensions of the box must be a little bigger to allow for easy movement in getting the bottle in and out of the box. Thus, the box should only be 0.5 cm taller and 0.5 cm wider than the height and widest portion of the base of the bottle.



1. Determine the measurements needed for each dimension on the two boxes. Explain how you determined these dimensions.
2. Sketch the net for each package design, showing how you would make each box from a single sheet of cardboard. (Do not include the flaps needed to glue the box.)
3. Calculate the amount of cardboard used for each design. Which box will be more cost-efficient?



**Materials:**

One graphing calculator per student

**Connections to Geometry****TEKS:**

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

## Teacher Notes

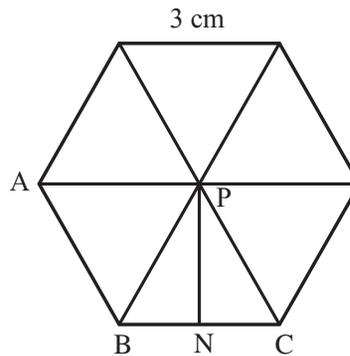
### Scaffolding Questions:

- What is the total height of the perfume bottle and the cap?
- Describe the shape of the sides of each box.
- Describe the bases of each box.
- How will the dimensions of the perfume bottle compare to the dimensions of the box?

### Sample Solutions:

1. The total height of the perfume bottle and cap is 13 centimeters.

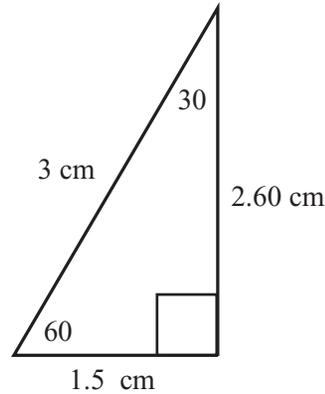
The base of the bottle is a regular hexagon with sides of 3 centimeters. A sketch of the base is shown below. The diagonals of the regular hexagon intersect in the center of the hexagon. The segment connecting the center of the regular hexagon and a vertex is the radius,  $\overline{AP}$  or  $\overline{BP}$ , of the regular hexagon.



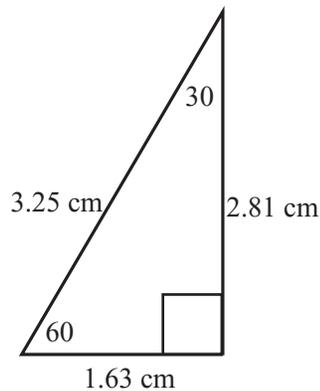
A regular hexagon has a radius equal to the length of the side of the regular hexagon. This can be established by drawing in the apothem of the hexagon and using a 30-60-90 triangle. The regular hexagon has an interior angle,  $\angle ABC$ , measuring 120 degrees. The radius of the hexagon,  $\overline{BP}$ , bisects this angle forming a 60-degree angle,  $\angle CBP$ . The apothem  $\overline{PN}$  forms a 90-degree angle with the side of the hexagon. The apothem also bisects the side of the hexagon; therefore  $\overline{BN}$  measures 1.5 centimeters. The remaining angle of the triangle,  $\angle BNP$ , measures 30 degrees.



Using this information, it can be determined that the hypotenuse of the triangle is 3 centimeters. The hypotenuse of the triangle is also the radius of the hexagon. The following figure shows the measurements in the 30-60-90 triangle. Measurements are rounded to the nearest hundredth.



The required 0.5 cm allowance for packaging space must be considered. This changes the length of the diagonal of the hexagon to 6.5 cm. The diagonal is made up of 2 radii (shown as the hypotenuse in the triangle below). Therefore the actual measurement needed for the packaging must be adjusted on the triangle as follows:



The widest dimension of the base will be 6.5 centimeters. This dimension will be the same for the hexagon and the circular base.

The shorter distance across the hexagon (2 apothems, end to end) will be approximately 5.62 cm. (This amount is rounded.)

The measurement of the distance needed to go around the bottle will be the perimeter of the hexagon and the circumference of the circular base of the cylinder.

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(B) uses nets to represent and construct three-dimensional objects; and

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.



The student:

(A) finds areas of regular polygons and composite figures;

(C) develops, extends, and uses the Pythagorean Theorem; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

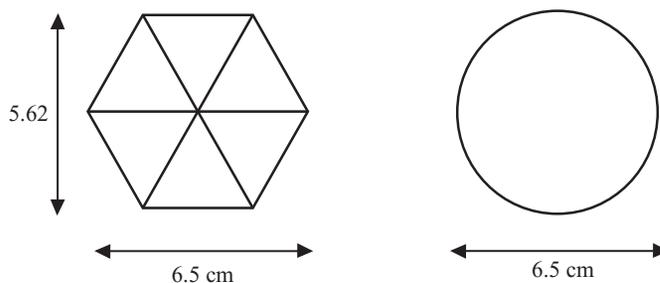
(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(D) analyzes the characteristics of three-dimensional figures and their component parts.

### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.



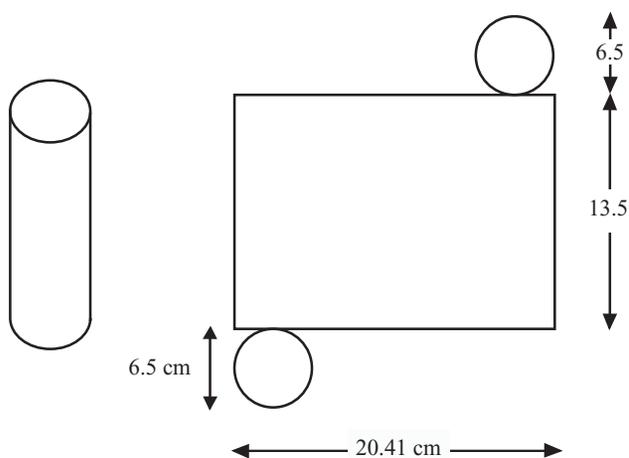
The perimeter of the regular hexagon is calculated by multiplying the length of one side by six.  $P = 3.25 \text{ cm} \cdot 6 = 19.5 \text{ cm}$ .

The circumference of the circular base is calculated using  $C = 2\pi r$ . The radius of the circle is 3.25 cm.  $C = 2\pi(3.25 \text{ cm}) = 6.5\pi \text{ cm}$ . Using 3.14 for  $\pi$ , the length of the circumference is:

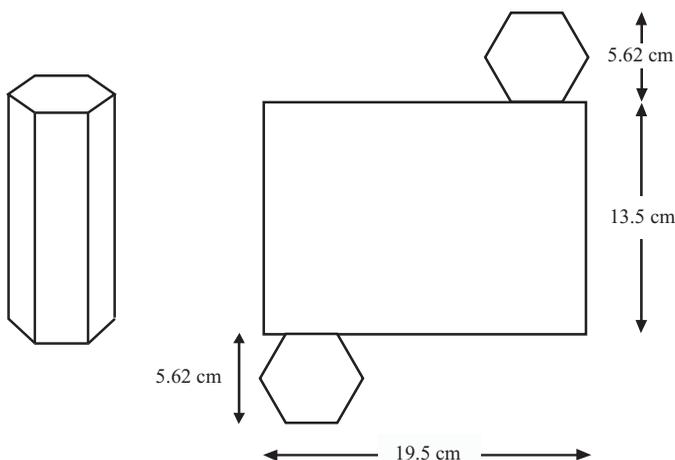
$$C = 6.5(3.14) \text{ cm} = 20.41 \text{ cm}.$$

2. The sketch of the nets for the boxes is shown below along with the surface area of each net. The 0.5 cm allowance for space is included in the dimensions. This space allows for the bottle to slide in and out of the box with ease, yet provides minimum movement during shipping. Values are rounded to the nearest hundredth of a centimeter.

### Cylindrical Box:



## Hexagonal Box:



3. The net of the cylindrical box consists of 2 congruent circles and a rectangle.

The area of each circle is  $A = \pi r^2$ .

$$A = \pi (3.25 \text{ cm})^2$$

$$A = (3.14)(10.56 \text{ cm})$$

$$A = 33.16 \text{ cm}^2$$

The area of the rectangle is  $A = bh$ .

$$A = 20.41 \text{ cm} \cdot 13.5 \text{ cm}$$

$$A = 275.54 \text{ cm}^2$$

The total surface area of the cylinder is found by finding the sum of the areas of the circles and the rectangle. Surface Area =  $33.16 \text{ cm}^2 + 33.16 \text{ cm}^2 + 275.54 \text{ cm}^2 = 341.86 \text{ cm}^2$

The net of the hexagonal box is comprised of 2 regular hexagons and a rectangle.

The area of the rectangle is  $A = bh$ .

$$A = 19.5 \text{ cm} \cdot 13.5 \text{ cm}$$

$$A = 263.25 \text{ cm}^2$$

The area of the regular hexagon is  $A = (ap) / 2$ .

We know that  $a = 2.81 \text{ cm}$  and  $p = 6(3.25 \text{ cm}) = 19.5 \text{ cm}$

$$A = (2.81 \text{ cm} \cdot 19.5 \text{ cm}) / 2$$

$$A = (54.8 \text{ cm}^2) / 2$$

$$A = 27.4 \text{ cm}^2$$

The total surface area of the hexagonal box is

$$A = 263.25 \text{ cm}^2 + 27.4 \text{ cm}^2 + 27.4 \text{ cm}^2$$

$$A = 318.05 \text{ cm}^2$$

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

V. Solid Figures: Dwellings, Bayou City Dome

### Teacher's Comment:

“The students were hesitant to get started. No one wanted to give a wrong answer to the whole class. We share in class in groups, but not often as a class looking at each group's work. I will do more of this to encourage participation and understanding of different approaches. I will give more problems to be worked this way. I need to start this kind of problem at the beginning of the school year.”



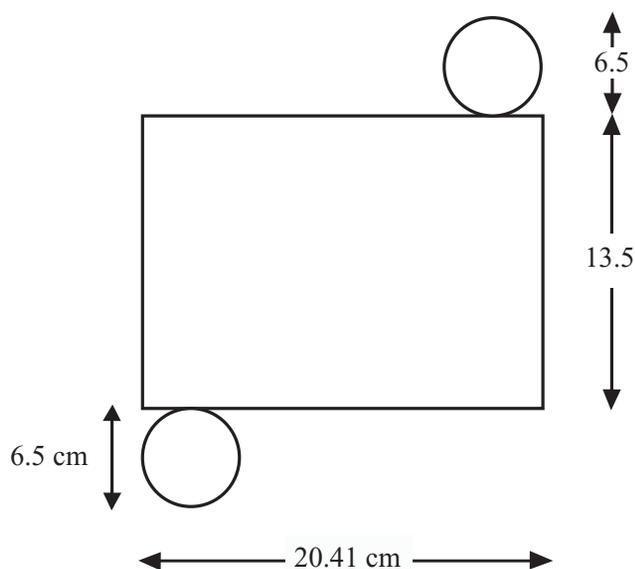
The hexagonal box has a smaller total surface area of  $318.05 \text{ cm}^2$ , while the cylindrical box has a total surface area of  $341.86 \text{ cm}^2$ . Ms. Ima Smelley should select the hexagonal box for the perfume because it is more cost-efficient.

### Extension Questions:

- Suppose the packaging manufacturer used sheets of cardboard 1 meter long and 1 meter wide. How many of each type of packaging could be made from a single sheet? Explain your answer in detail.

*The area of the cardboard sheet is 1 square meter or 10000 square centimeters. You may not, however, divide this total by the total surface area of the box because the shapes, when placed on the sheet, will not use all of the space.*

*The cylindrical shape with tangent circle bases:*



*The total width is 20.41 cm, and the length is  $6.5 + 6.5 + 13.5$  or 26.5 cm. If it is assumed that the shape must be placed with the tangent circles, the question is, "How many rectangles of this size will fit in a rectangle 100 cm by 100 cm?"*

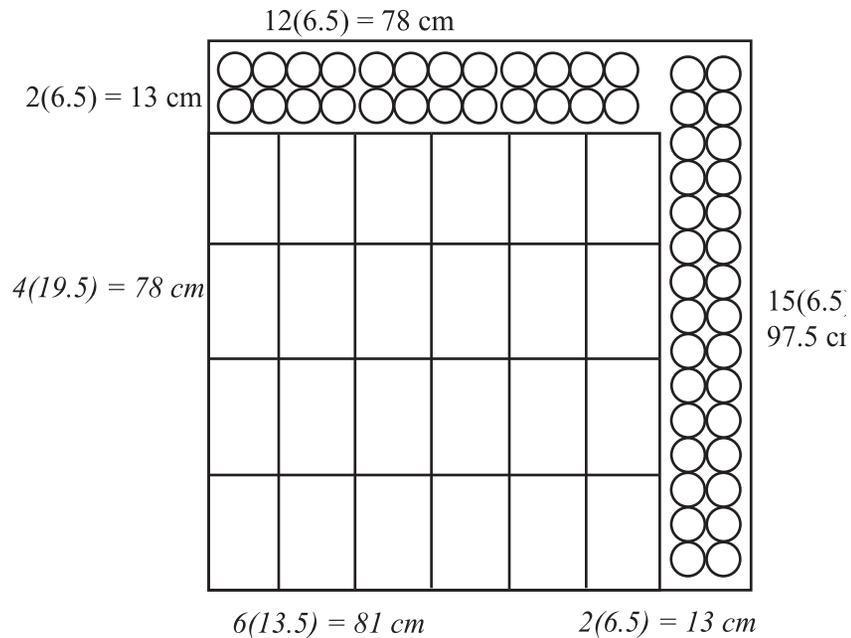
$$100 \div 26.5 \approx 3.77$$

$$100 \div 20.41 \approx 4.9$$

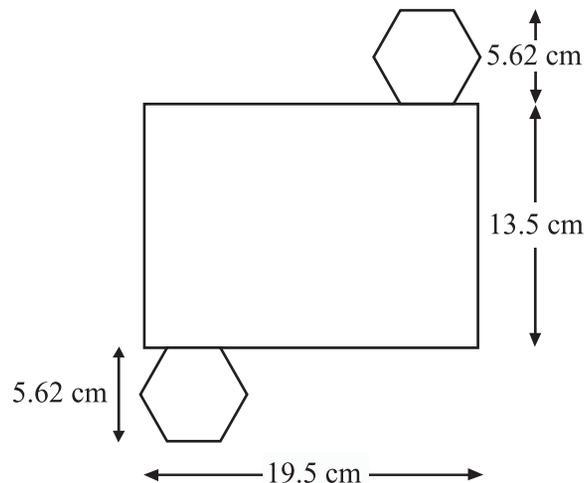
*An array of three rectangles by four rectangles would fit on the sheet. At least 12 shapes would fit on the sheet.*



If the circles may be separated from the rectangle, more of the shapes could be cut out of the cardboard. One possible arrangement is shown below.



24 rectangles and 54 circles could be arranged on the square.



The total width is  $19.5 \text{ cm}$ , and the length is  $5.62 + 5.62 + 13.5$  or  $24.74 \text{ cm}$ . The question is, "How many rectangles of this size will fit in a rectangle  $100 \text{ cm}$  by  $100 \text{ cm}$ ?"

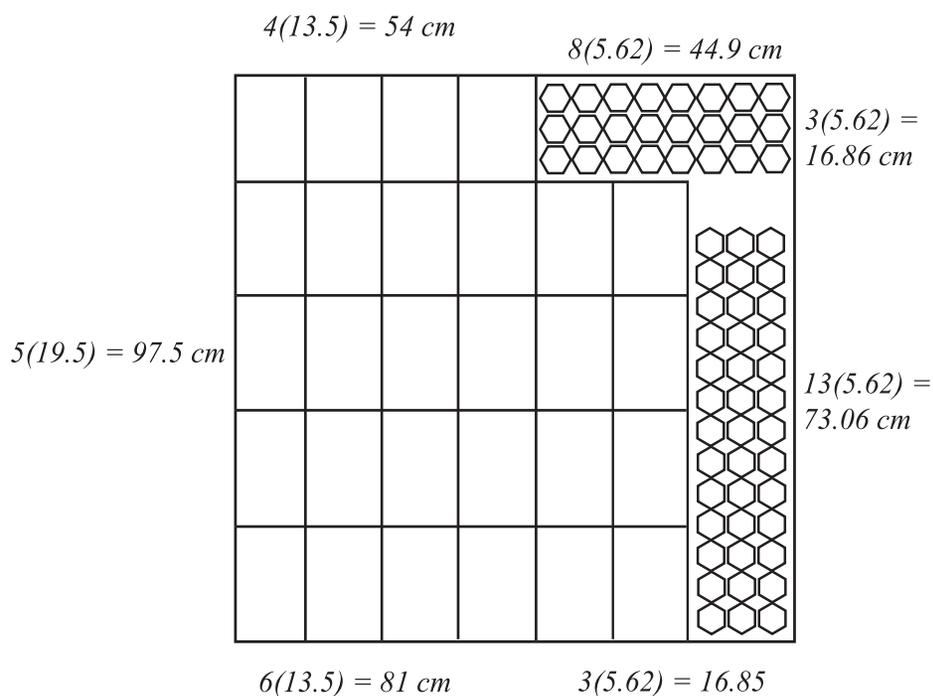
$$100 \div 19.5 \approx 5.128$$

$$100 \div 24.74 \approx 4.04$$

An array of five shapes by four rectangles would fit on the sheet. At least 20 shapes would fit on the sheet.



If the hexagons are not connected to the rectangles, more could be positioned on the cardboard. One possible arrangement is demonstrated in the diagram below.



*Rectangles:*

$$4(5) = 20$$

$$2(4) = 8$$

28 rectangles

*Hexagons:*

$$13(3) = 39$$

$$8(3) = 24$$

63 hexagons

Thus, 28 sides and 56 tops and bottoms will fit in the space.





## Student Work

A group of four students created a poster of their solution to this problem. The work has been copied on the next three pages.

This work exemplifies all of the solution guide criteria. For example:

- Makes an appropriate and accurate representation of the problem using correctly labeled diagrams.

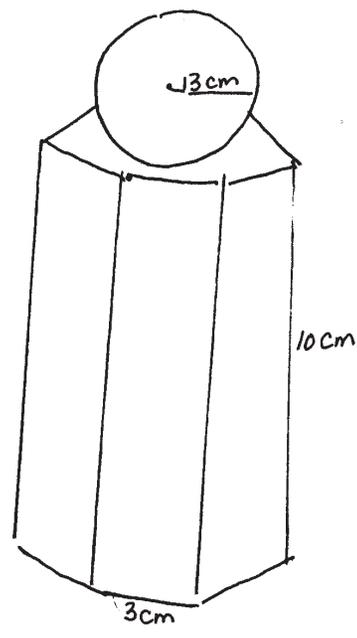
*The students' diagrams of the solids and nets are correct. They have labeled the measurements of the perfume bottle, the containers and the nets.*

- Communicates clear, detailed, and organized solution strategy.

*The students used the words, “first”, “to find these measurements we did the following things”. They described what they did and why. They listed the formulas used, what the variables in the formula represented and what they were for this particular situation. “Formula of the surface area (SA) of a hexagonal prism is  $SA = Ph + 2B$ . So we know the perimeter,  $p$ , of the prism is ....*

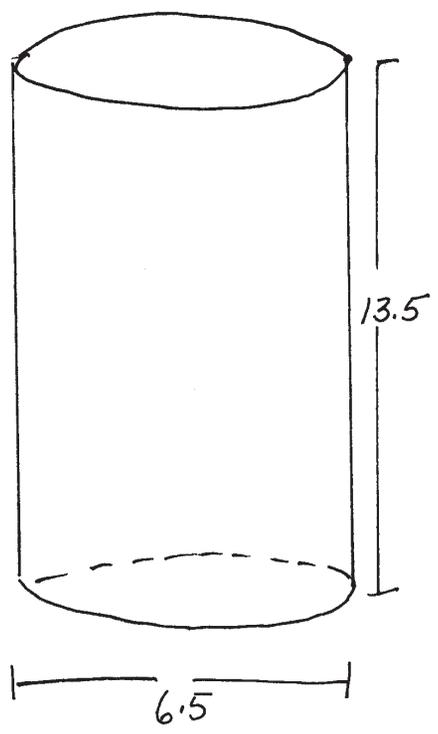


①

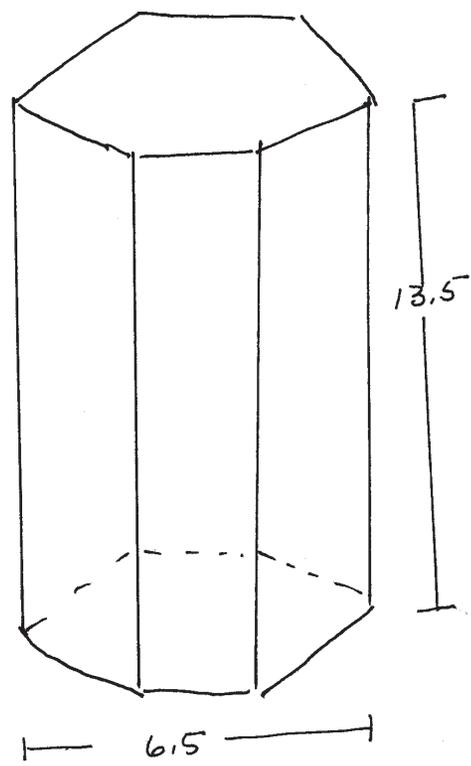


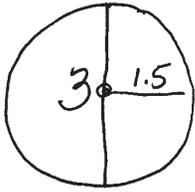
First, we were give the information that this perfume bottle was 10 cm high, each base edge was 3 cm. wide, and the radius of the sphere was 1.5 cm. So we labeled accordingly.

Option #1



Option #2

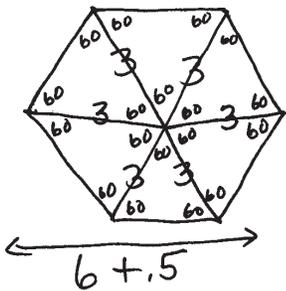




$$\begin{array}{r} +10 \\ \hline 13 \\ +.5 \\ \hline 13.5 \end{array}$$

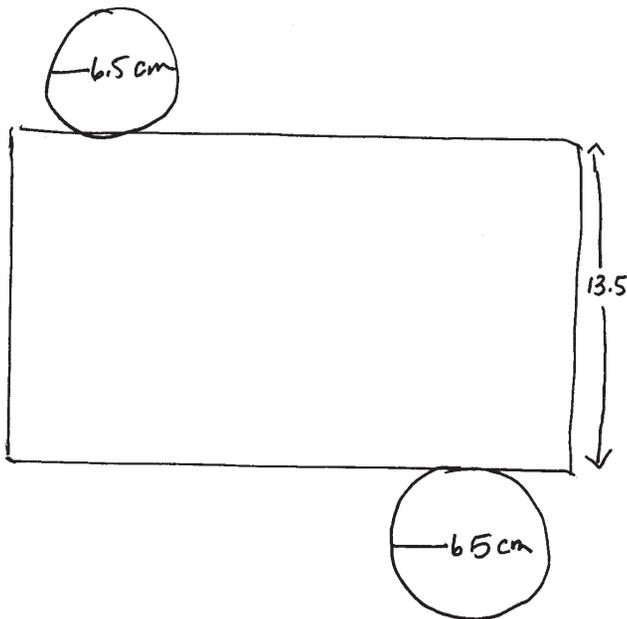
We were given the information that ② the box should be .5 cm bigger at the highest and widest parts. To find these measures we did the following things. The radius of the circle at the top of the bottle is 1.5 cm. The diameter would be 3 cm.

Therefore the height of the bottle is 13 cm. The height of the box must be 13.5 cm (13 + .5). The width is 6 cm on the bottle.



We found that by splitting the hexagon into equilateral triangles the width was 6 cm (3 + 3) to make it as big as needed the width would need to be 6.5 cm on the packing box.

$$\text{Angles } 180(b-2) = \frac{720}{6} = 120 \quad \frac{120}{2} = 60^\circ \text{ each angle.}$$

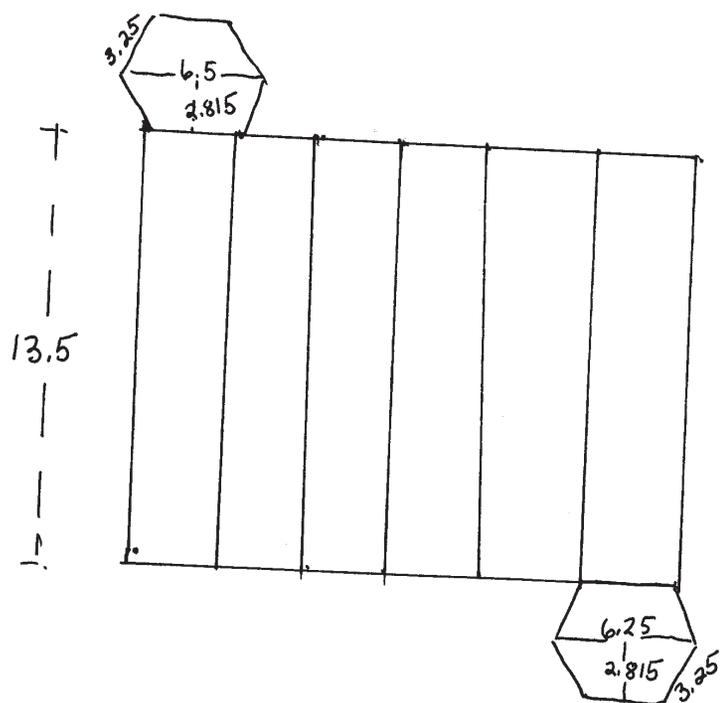


① Formula of Surface Area (SA) of a cylinder is  $2\pi rh + \pi r^2$ . So we know the height is  $h = 13.5$  and the radius is  $r = 3.25$  So we fill in the formula accordingly, and solve.

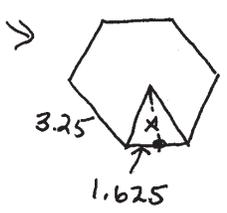
$$\begin{aligned} SA &= 2\pi(3.25)(13.5) + 2\pi(3.25)^2 \\ SA &= 2\pi(43.875) + 2\pi(10.563) \\ SA &= (137.8) + 33.2 \\ SA &= 341.9 \text{ cm}^2 \end{aligned}$$



③



Formula of surface Area (SA) of a hexagonal prism is  $SA = Ph + 2B$ . So we know the perimeter,  $P$ , of the prism is  $6(3.25) = 19.5$  and the height  $h = 13.5$ . In addition, area of a hexagon equals  $SA = \frac{1}{2} Pa$  where  $P$  stands for perimeter and  $a$  stands for apothem. Since  $P = 19.5$  and  $a = 2.815$



then we use Pythagorean Theorem

$$3.25^2 = 1.625^2 + x^2$$

$$x^2 = 7.421875$$

$$x = \sqrt{7.421875}$$

$$x = 2.815$$

So we fill in the formula accordingly...

$$\text{Area of the hexagon} = \frac{1}{2} Pa$$

$$A = \frac{1}{2} (19.5)(2.815)$$

$$A = 27.4$$

Now we know the area of the hexagon or base we fill in the SA of the prism formula.

$$SA = Ph + 2B$$

$$SA = (19.5)(13.5) + 2(27.4)$$

$$SA = 318.1 \text{ cm}^2$$

The hexagon prism would serve as the most cost efficient container because it has a lesser surface area than the cylinder.





## Playing with Pipes

R&B Engineers have been hired to design a piping system for an industrial plant. The design calls for 4 different shaped pipes that will carry water to the industrial plant. The pipes will be constructed out of four rectangular pieces of sheet metal, each with a width of 360 cm and a length of 100 cm. The metal will be folded or rolled to make the 4 pipes, each having cross sections with the same perimeter, 360 cm. One of the pipes has a rectangular cross section with dimensions of 60 cm by 120 cm. One pipe cross section is a square; one is a regular hexagon, and one is circular. The length of each pipe will be 100 cm.

1. Draw a sketch of each pipe, and label the dimensions on each cross section of pipe.
2. The pipe with the greatest flow is considered to be the pipe with the greatest cross section area. Determine which cross section of pipe will allow for the greatest flow of water, and list them in order from greatest to least. Justify your answers.



## Teacher Notes

### Materials:

One graphing calculator per student

### Connections to Geometry

#### TEKS:

**(b.4) Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

**(c) Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

**(d.1) Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

### Scaffolding Questions:

- How can the perimeter of the square cross section be determined? The regular hexagonal cross section? The circular cross section?
- How can you determine the amount of water flow through the pipes?

### Sample Solutions:

1. All of the pipes have a cross section with the same perimeter. The given perimeter is 360 cm. The dimensions of the rectangular cross section are given in the problem. The dimensions of the other cross sections can be calculated as follows:

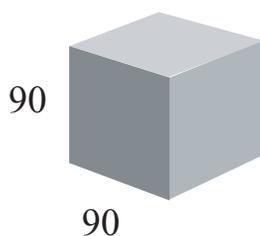
$$\text{Square: } \frac{360 \text{ cm}}{4} = 90 \text{ cm per side}$$

$$\text{Regular Hexagon: } \frac{360 \text{ cm}}{6} = 60 \text{ cm per side}$$

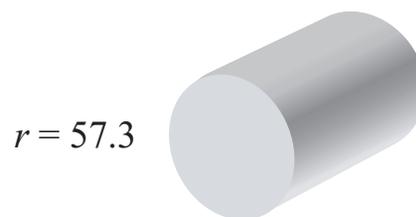
$$\begin{aligned}\text{Circle: } 2\pi r &= 360 \\ \pi r &= 180 \\ \pi r &\approx 57.3 \text{ cm}\end{aligned}$$

The pipes are shown below.

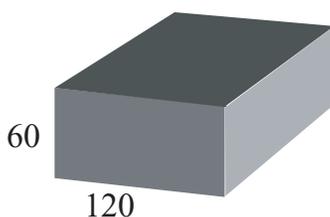
Square



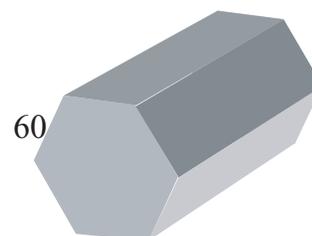
Circular



Rectangle



Regular Hexagon



2. In order to determine the pipe with the greatest water flow, the area of each cross section must be calculated.

Listed in order, from greatest flow to least flow:

The area of the circular region will produce an opening that is:  $A = \pi r^2 = \pi(57.3)^2 = 10,314 \text{ cm}^2$ .

The area of the regular hexagonal region will produce an opening that is:  $A = \frac{1}{2} ap = \frac{1}{2} \cdot 30\sqrt{3} \cdot 360 \approx 9353 \text{ cm}^2$ .

The area of the square region will produce an opening that is:  $A = s^2 = 90 \cdot 90 = 8100 \text{ square cm}$ .

The area of the rectangular region will produce an opening that is:  $A = lw = 60 \cdot 120 = 7200 \text{ square cm}$ .

### Extension Questions:

- A pipe having a circular cross section with an inside diameter of 10 inches is to carry water from a reservoir to a small city in the desert. Neglecting the friction and turbulence of the water against the inside of the pipes, which is the minimum number of 2" inside-diameter pipes of the same length that is needed to carry the same volume of water to the small city in the desert? Explain your answer.

*In order to solve this problem, it will be necessary to find the number of smaller pipes that it will take to provide the same cross sectional area as that of the larger pipe.*

*The cross sectional areas of the pipes are circles having the area  $A = \pi r^2$ .*

*The radius of the 10-inch pipe is 5 inches, and the radius of the 2-inch pipe is 1 inch.*

*The area of the 10-inch pipe is  $A_1 = \pi 5^2 = 25\pi \text{ in}^2$*

*The area of the 2-inch pipe is  $A_2 = \pi 1^2 = 1\pi \text{ in}^2$*

*The ratio of the areas is  $25\pi : 1\pi$  or  $25 : 1$ . This implies that the cross sectional area of the 10-inch pipe is 25 times larger than the cross sectional area of the smaller pipe. It will require 25 of the 2-inch pipes to provide the same cross sectional area as the 10-inch pipe.*

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

**(e.1) Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures;

### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Bayou City Dome



An alternate approach is to realize that the ratio of the diameters is 10 to 2 or 5 to 1. Thus, the ratio of the areas is  $5^2$  to  $1^2$  or 25 to 1. Therefore, the requirement is 25 of the 2-inch pipes.

- Suppose the cross sections of the pipes are squares rather than circles. The diagonal of one square is 10 inches, and the diagonal of the second square is 2 inches. Will the shape of the pipes affect the number of pipes required to produce equal areas? Justify your answer.

It makes no difference what shapes the pipes are, provided that the cross sectional areas are the same. If the pipes had square cross sections, and the diagonals were 10 inches and 2 inches, it will still require 25 smaller pipes to provide the same cross sectional area as the larger pipe. The pipe with the 10-inch diagonal forms two 45-45-90 triangles with leg lengths of  $5\sqrt{2}$  inches. Therefore, the square cross section will have side lengths of  $5\sqrt{2}$  inches. The area of the pipe with the 10-inch diagonal will be  $A = (5\sqrt{2})^2 = 50 \text{ in}^2$ .

The pipe with the 2-inch diagonal forms two 45-45-90 triangles with leg lengths of  $1\sqrt{2}$  inches. Therefore the square cross section will have side lengths of  $1\sqrt{2}$  inches. The area of the pipe with the 2-inch diagonal will be  $A = (1\sqrt{2})^2 = 2 \text{ in}^2$ .

The ratio of the areas of the square cross section is 50 : 2 or 25: 1. Therefore, it will require 25 of the smaller square pipes to provide the same cross sectional area as the larger pipe.



## Circular Security

The Circular Security Company manufactures metal cans for small amounts of medical hazardous waste. They have contracted your marketing firm to create a label for the waste can. The can is a right circular cylinder with a base radius of 9 inches and a height of 15 inches.

The slogan for Circular Security is “*Circular* — For that all-around sense of security!” As an employee of the marketing company, your task is to design a label for the can that has a relationship to the slogan. You have decided that a spiral stripe is to be painted on the label of the can, winding around it exactly once as it reaches from bottom to top. It will reach the top exactly above the spot where it left the bottom.

1. Create a net showing the dimensions of the can and the placement of the stripe. Round dimensions to the nearest tenth of an inch if needed.
2. Determine the length (in inches) of the stripe. Round your answer to the nearest tenth of an inch if needed.
3. Justify your solution.



## Teacher Notes

### Materials:

One graphing calculator per student

### Connections to Geometry

#### TEKS:

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(B) uses nets to represent and construct three-dimensional objects; and

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem.

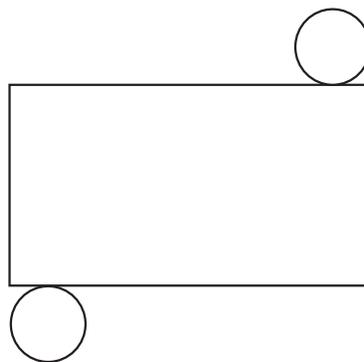
### Scaffolding Questions:

- What 2-dimensional figures comprise a net of a cylinder?
- If the radius of the cylinder is 9 inches, what is the circumference of the cylinder?
- How will knowing the circumference of the cylinder help you determine the dimensions of the net?
- Where would the stripe be on the net of the cylinder?
- How can you calculate the length of the stripe?

### Sample Solution:

This problem begins as a 3-dimensional problem but can be reduced to a 2-dimensional problem to make the solution easier. Some cans, like ones that might be found in a grocery store, have a label all the way around them. If the label is removed and flattened out, the shape of the label is a rectangle. The top and the bottom of the can are circles.

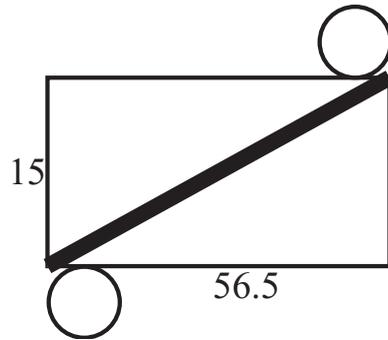
The net for the can is:



The height of the can is 15 inches; this will also be the height of the rectangle representing the label. The radius of the circular base of the can is used to find the circumference of the circle, which will turn out to be the width of the rectangular label.

If  $r = 9$  inches, then the circumference of the circle is  $2\pi(9) \approx 56.5$  inches. Therefore the label's width is also 56.5 inches.





Let the length of the diagonal be represented by  $x$ . The diagonal black line in the net above represents the stripe. The problem states that the stripe goes from the top to the bottom of the can in exactly one circumference; on the rectangle this is the length of the diagonal.

The Pythagorean theorem can be used to find the length of the diagonal as follows:

$$\begin{aligned} 15^2 + 56.5^2 &= x^2 \\ 225 + 3192.25 &= x^2 \\ 58.5 &\approx x \end{aligned}$$

The length of the diagonal stripe, from corner to corner, is approximately 58.5 inches.

#### Extension Questions:

- Suppose a Circular Security customer wants to place a special order for a can that is similar to the original can but will have a height of 10 inches. Determine the dimensions of the new can and the length of the stripe that will encircle the can.

*If a customer wants to order a similar can with a height of 10 inches, the ratio of the corresponding sides is 15 to 10, or 3 to 2. To find the radius of the base, solve the proportion.*

$$\begin{aligned} \frac{3}{2} &= \frac{9}{x} \\ x &= 6 \text{ inches.} \end{aligned}$$

*The radius of the special-order cylinder is 6 inches.*

*The circumference of the base is the same as the width of the rectangle.*

#### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

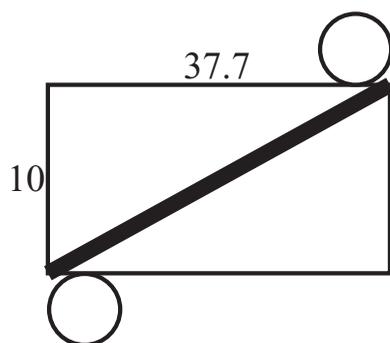
#### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Dwellings



$$C = 2\pi(6)$$

$$C \approx 37.7 \text{ inches}$$



The length of the stripe encircling the special order can is represented by  $x$ :

$$10^2 + 37.7^2 = x^2$$

$$100 + 1421.29 = x^2$$

$$1521.29 = x^2$$

$$x \approx 39.0 \text{ inches}$$

- Compare the stripe of the original can to this new can's stripe-length.

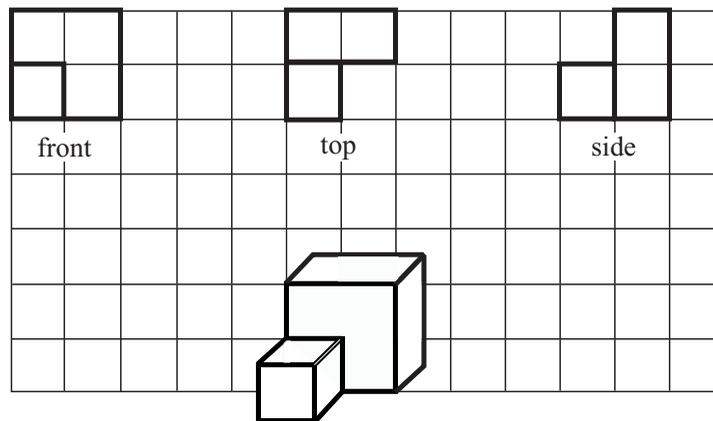
The ratio of the original stripe-length to the new stripe-length is 58.5 in.

to 39.0 in.  $\frac{58.5}{39.0} = \frac{1.5}{1} = \frac{3}{2}$  The ratio of the stripes is the same as the ratio that compares the dimensions of the two cans or 3 to 2.

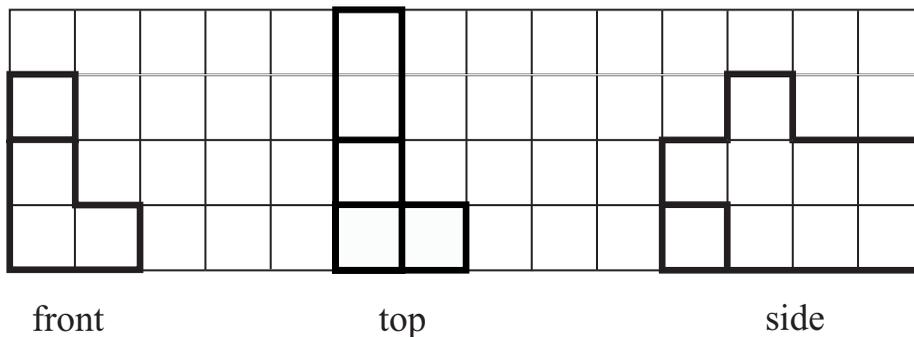


## Different Views

The picture below shows the front, top, and side views of an object as well as the object itself. This object is made of five unit cubes (four in the back row and one in the front), so the volume is 5 units<sup>3</sup>. The surface area is 20 units<sup>2</sup>.



Below are the front, top, and side views of another object. Draw this object, and explain how the given views helped you construct your final object. Calculate the volume of this object, and carefully explain how you got your answer.



**Materials:**

Cubes or blocks for each student (optional)

**Connections to Geometry****TEKS:**

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

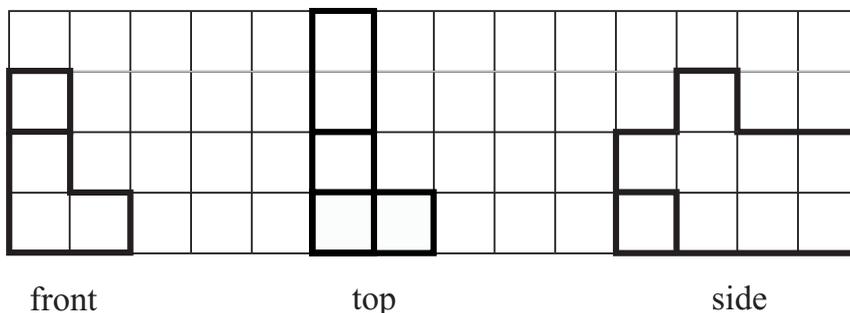
(B) uses nets to represent and construct three-dimensional objects; and

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

## Teacher Notes

**Scaffolding Questions:**

- What does the square on the front view of the very first figure tell you?
- Demonstrate your answer above by using cubes.
- Build the final figure in the first set of diagrams.
- How many cubes make up the final figure?
- What is the volume of one of the cubes?
- How do you know that the volume of the first final figure really is 5 cubic units?
- Determine the surface area of one cube.
- Explain how you know the surface area of the first final figure really is 20 square units.

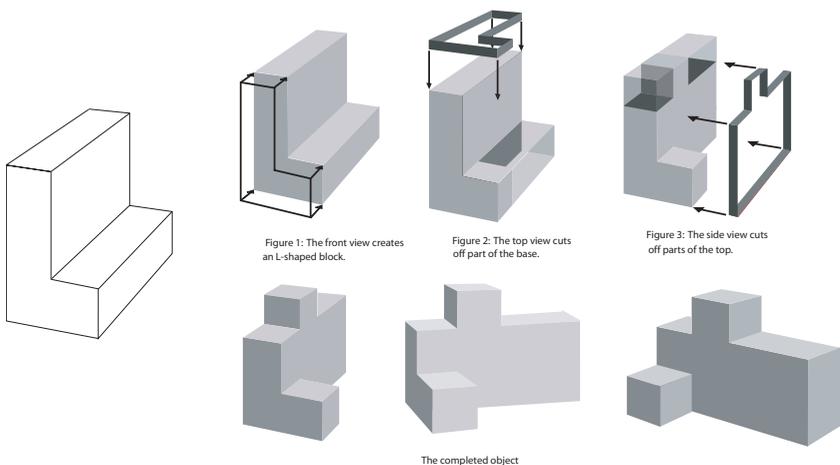
**Sample Solution:**

The front view of the object has 3 blocks in an L shape and another block on top that is either farther in or farther out.

The top view indicates that the front 3 squares have different heights and that there are blocks at the end that are of the same height. The top view also produces an overall L shape.

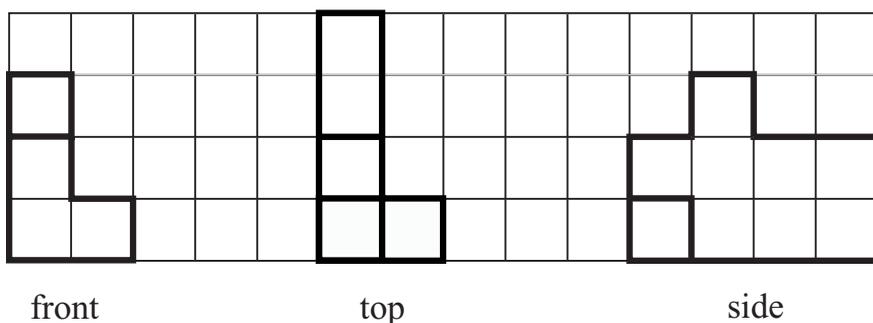
The side view indicates that there is 1 block that sticks out. This is also visible from the front and top views. In the side view there is also a wall 2 blocks high and 4 blocks long. There is a tower on top of the wall consisting of 1 block. From this information, a picture of the figure is drawn.



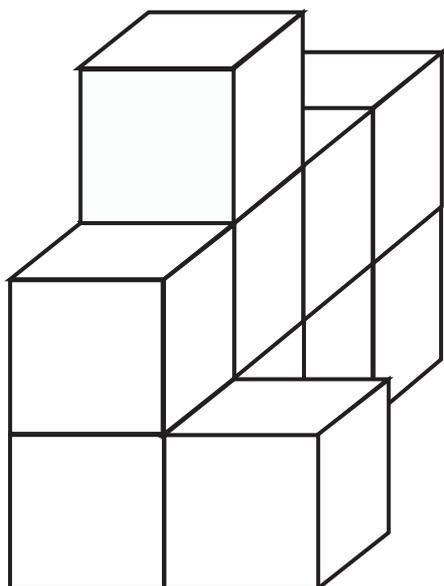


Using the given information in the original diagrams, a sketch is drawn showing the individual blocks.

Given Diagram:



New figure with individual blocks drawn:



**Texas Assessment of Knowledge and Skills:**

Objective 7: The student will demonstrate an understanding of two-and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

V. Solid Figures: What's Your Perspective

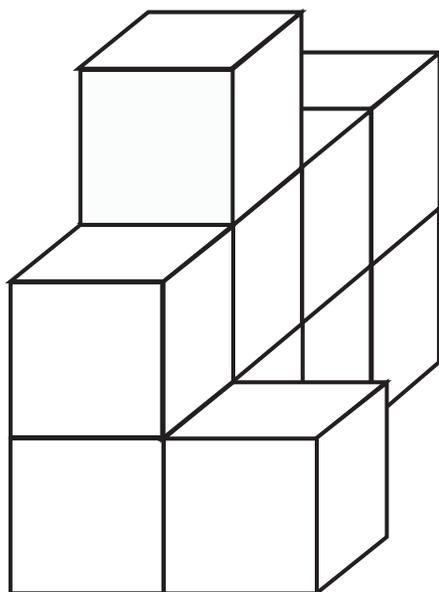


Using the new figure, the total number of blocks is determined as follows. Consider each row to be a tower if the blocks are separated. There are 3 blocks in the front row (the L-shaped front view), 3 blocks are in the second row (a 3-block tower), 2 blocks in the third row, and 2 blocks in the fourth row. The total number of blocks is  $3 + 3 + 2 + 2 = 10$  blocks. Each block is 1 cubic unit; therefore the figure has a volume of 10 cubic units.

### Extension Questions:

- Find the surface area of the second object. Justify your answer.

*To find the surface area, the figure is divided into rows, front to back.*



*Each face on a cube that is showing adds a square unit to the surface area of the figure. Faces that touch each other are not considered part of the surface area. The bottom, back, and left side of the figure is part of the surface area, even though they are not showing.*

*There are 3 blocks in the front row (the L-shaped front view.) This tower has 3 faces showing on the front, 1 face showing on the back, 2 on the left side, 2 on the right side, and 2 on top. There are 2 faces on the bottom of the tower. There are a total of 12 faces on the front tower.*

*The second row is a 3-block tower. This tower has 1 face showing from the front, 1 face on the back, 1 face on the bottom, 1 face on the top, 3 faces on the left side, and 3 faces on the right side. There are a total of 10 faces on the second tower.*



The third row is a tower, 2 blocks high. There are no faces showing from the front or the back. There is 1 face on the top, 1 face on the bottom, 2 faces on the left side, and 2 faces on the right side. There are a total of 6 faces on the third tower.

The fourth row is a tower that is also 2 blocks high. There are no faces showing from the front. There are 2 faces on the back, 1 face on the bottom, 1 face on the top, 2 faces on the left, and 2 faces on the right. There are a total of 8 faces on the fourth row.

The total number of faces that comprises the surface area is found by adding the total from each row:  $12 + 10 + 6 + 8 = 36$  faces. The surface area of this figure is 36 square units.

In an alternate approach, the surface area can be found by subtracting the number of faces that are facing each other from the total number of faces of the 10 cubes.

The total number of faces of 10 cubes is  $10 \cdot 6$  because each cube has 6 faces.

There are 12 cube faces that are touching another face. The cube at the front right faces another cube. When row two is stacked on row one there are 4 horizontal faces touching each other and there are 6 vertical faces touching each other. There is one more face touching another face when row three is stacked. Thus, the total number is  $1 + 4 + 6 + 1$  or 12 faces.

The total number of facing cubes is  $12 \cdot 2$  or 24 faces.

The number of faces in the surface area is  $60 - 24$  or 36 faces. Therefore, the surface area of the figure is 36 square units.





## The Slice Is Right!

A cube has edges that are 4 units in length. At each vertex of the cube, a plane slices off the corner of the cube and hits each of the three edges at a point 1 unit from the vertex of the cube.

1. Draw the cube showing how the corners will be sliced.
2. Draw and label a figure that represents the solid that is cut from each corner of the cube.
3. How many cubic units are in the volume of the solid that remains? Justify your answer.



**Materials:**

One graphing calculator per student

Styrofoam or play dough cubes, fishing line to cut off corners

**Connections to Geometry****TEKS:**

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

The student:

(A) describes and draws cross sections and other slices of three-dimensional objects;

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem; and

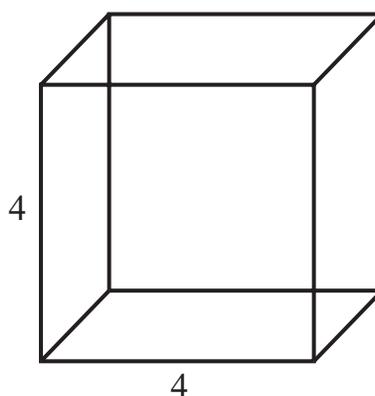
## Teacher Notes:

**Scaffolding Questions:**

- If the plane cuts off the corner one unit from the edge of the cube, what is the shape of the piece that is cut off?
- How many of the corners are cut off?
- What are the dimensions of each piece that is cut off?
- Determine the volume of the original cube.
- Determine the volume of each cut off piece.

**Sample Solutions:**

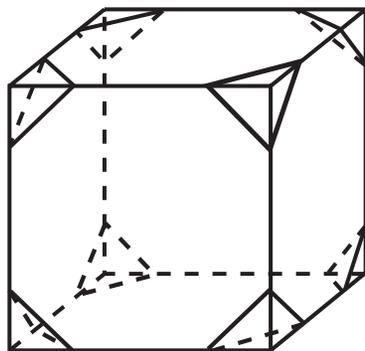
1. Original Cube:



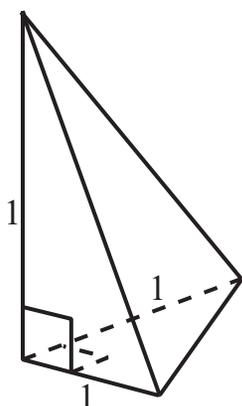
When the corners of the cube are cut off, eight congruent pyramids with four triangular faces are formed. Three of the faces are isosceles right triangles with legs that are 1 unit in length. The fourth face is an equilateral triangle. The side of this triangle is also the hypotenuse of the isosceles right triangle.



A cube with all corners sliced by a plane at a distance of 1 unit from each corner:



2. Each triangle pyramid may be positioned so that its base is one of the isosceles right triangles. The height of the pyramid is also 1 unit. The corner of the cube forms a right angle on each face of the pyramid.



3. Each triangular pyramid that is sliced off has volume:

$$V = \frac{1}{3} Bh = \frac{1}{3} \left( \frac{1}{2} \cdot 1 \cdot 1 \right) \cdot 1 = \frac{1}{6}.$$

The volume of the 8 pyramids that are sliced off the cube is:

$$8 \cdot \frac{1}{6} = \frac{8}{6} = \frac{4}{3}.$$

The volume of the original cube is:  $V = 4^3 = 64$  cubic units.

The volume of the solid after the 8 corners are removed can be found by subtracting the volume of the 8



(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

V. Solid Figures: Bayou City Dome

### Teacher's Comment:

"Some students who don't do as well on traditional assignments were in their element with this problem. It pleased me that several students saw how you would turn the corner pieces around in order to find the volume before I asked any scaffolding questions. It worked well to have them put their solutions on chart paper. They got a chance to be thorough with their thinking as well as their creativity."

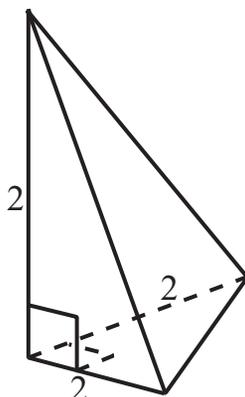
pyramids from the volume of the original cube.

$V = \text{original cube} - 8 \text{ pyramid volumes}$

$$V = 64 - \frac{4}{3} = 62\frac{2}{3} \text{ cubic units.}$$

### Extension Questions:

- Suppose the plane slices off sections that are 2 units from each vertex of the cube. How does the volume of the remaining figure compare to the original slice (1 unit from each vertex)? Calculate the percent of decrease of the volume. Round your answer to the nearest percent. Justify your answer.



*There are 8 corners on the original cube; therefore 8 triangular pyramids are formed. Each triangle pyramid has an isosceles right triangle for the base and 2 faces. The area of the base triangle is  $\frac{1}{2} \cdot 2 \cdot 2$  or 2 square units.*

*The volume of one pyramid is  $\frac{1}{3} \cdot 2 \cdot 2$  or  $\frac{4}{3}$  cubic units.*

*The volume of the 8 pyramids is  $8 \left(\frac{4}{3}\right)$  or  $\frac{32}{3}$  or  $10\frac{2}{3}$  cubic units.*

*The volume of the solid after the eight unit corners have been removed is  $64 - 10\frac{2}{3}$  or  $53\frac{1}{3}$ .*

*The difference of the volume of the solid after the eight 2-unit corners are removed and the cube with the eight 1-unit pyramids removed can be found by subtracting the volumes.*

*$V = \text{volume of cube with 8 1-unit pyramids} - \text{volume of cube with 8 2-unit pyramids}$*

$$V = 62\frac{2}{3} - 53\frac{1}{3} = 9\frac{1}{3} \text{ cubic units}$$



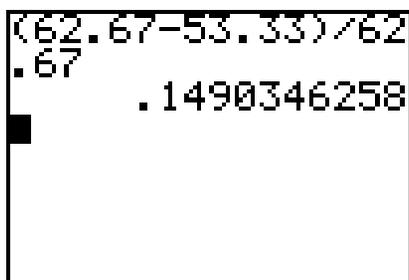
The percent of decrease in the volume of the cubes is found by calculating the following:

$$\frac{(\text{original volume}) - (\text{new volume}) \cdot 100}{(\text{original volume})}$$

Fractions were converted to decimals for ease of use in calculation:

Original volume with 1-inch slices:  $62 \frac{2}{3} = 62.67$  cubic units

New volume with 2-inch slices:  $53 \frac{1}{3} = 53.33$  cubic units



A calculator display showing the calculation  $(62.67 - 53.33) / 62.67 = 0.1490346258$ . The display is divided into two lines: the top line shows  $(62.67 - 53.33) / 62$  and the bottom line shows  $.67$  followed by  $.1490346258$ .

The percent of volume decrease is rounded to 15%.



## Student Work Sample

The student who created the work on the next page

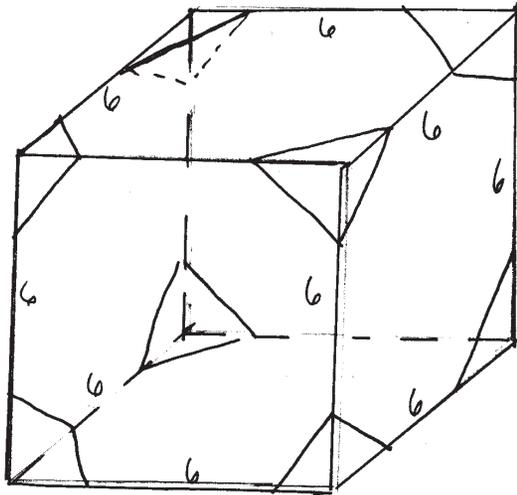
- Shows an understanding of the relationships among elements.

*The student understands what the given information means and how to interpret the given to create the solid that is cut off from the corners of the cube. He correctly draws the resulting triangular pyramid.*

- Demonstrates geometric concepts, processes, and skills.

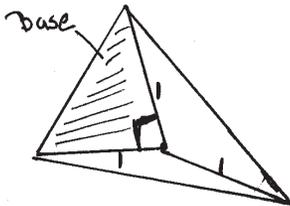
*The student shows the formulas used to determine area and volume. He shows the triangular pyramid, shows the base of the pyramid, and how to determine the volume using the right triangle base.*





A cube has edges of length 6 units at each vertex of the cube a plane slices off the corner of the cube and hits each of three edges at a point 1 unit from the vertex of the cube.

The volume of the original cube is:  $6^3 = 216 \text{ units}^3$



Area of one of the cut off corners.  
+  
Volume

Area of Base:  $\frac{\text{Base} \cdot \text{Height}}{2} = \frac{1 \cdot 1}{2} = 0.5 \text{ units}^2$

Volume:  $0.5 \cdot 1 = \frac{0.5}{3} = \frac{1}{6} \text{ units}^3$

The Volume of one of these pyramids is  $\frac{1}{6} \text{ units}^3$

So the added volume of all these pyramids combined is  $1 \cdot \frac{1}{3} \text{ units}^3$

- Subtract the volume of the corners from the volume of the cube

$$\begin{array}{r} 216 \\ - 1\frac{1}{3} \\ \hline 214\frac{2}{3} \text{ cubic units} \end{array}$$





# Chapter 6:

## *Congruence*





# Introduction

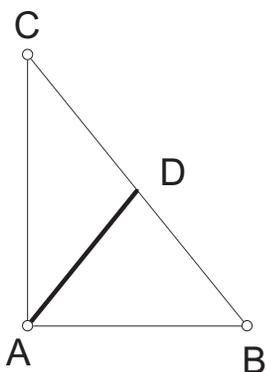
Application problems can provide the context for using geometry and for applying problem-solving techniques. The six problems in this chapter make connections among geometric concepts. The students will use the concept of congruence to solve problems. For example, The Shortest Cable Line involves a reflection of a triangle. Tell Me Everything You Can About... requires constructions. The School Flag assesses area concepts. Median to the Hypotenuse of a Right Triangle requires conjecturing and justification using axiomatic or coordinate methods.





## Median to the Hypotenuse of a Right Triangle

1. On a right triangle how does the length of the median drawn to the hypotenuse compare with the length of the hypotenuse? Use at least two approaches—drawings, constructions, or appropriate geometry technology—to investigate and conjecture a relationship between these lengths.



Triangle ABC is a right triangle with  $\overline{AC} \perp \overline{AB}$ .  $\overline{AD}$  is the median to the hypotenuse.

2. Prove your conjecture using one of these methods:
  - a) congruence transformations (isometries)
  - b) a Euclidean argument
  - c) a coordinate proof
3. Is your conjecture for right triangles valid for other triangles? Why or why not?
4. Is the converse of your conjecture true? Why or why not?



**Materials:**

One compass and ruler per student

Unlined paper, patty paper, and geometry software

**Connections to Geometry****TEKS:**

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

## Teacher Notes

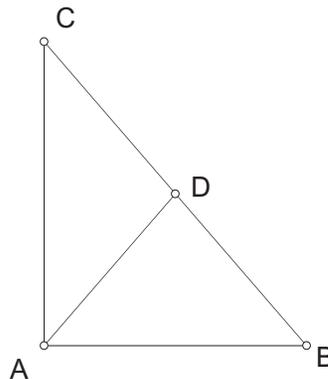
**Scaffolding Questions:**

- What is meant by a median of a triangle?
- Construct a scalene right triangle, an isosceles right triangle, and a 30-60-90 triangle. Construct the median to the hypotenuse. What appears to be true about these medians? How can you use measurements to investigate your observation?
- What is meant by congruence transformations? What are these transformations?
- What is meant by a Euclidean argument?
- A right triangle can always be viewed as “half” of what polygon? How could you extend (add auxiliary segments) the diagram of the right triangle to show this?

**Sample Solutions:**

Use geometry technology or paper and pencil constructions and take measurements of the median to the hypotenuse and the hypotenuse. For example,

AD = 3.54 cm  
 CD = 3.54 cm  
 DB = 3.54 cm  
 CB = 7.08 cm



In  $\triangle BAC$ ,  $\overline{BA} \perp \overline{AC}$  and  $\overline{AD}$  is the median to the hypotenuse.

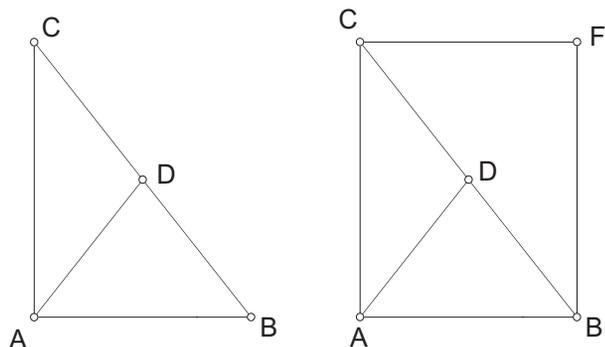
1. Conjecture: The measurements show that the median to the hypotenuse is one-half as long as the hypotenuse, or



the hypotenuse is twice as long as the median drawn to it.

## 2. Congruence Transformations:

Rotate  $\angle ACB$  180 degrees clockwise about point D and  $\angle ABC$  180 degrees counter-clockwise about point D. Label the point of intersection of the angles point F.



These transformations form  $\angle FBC$  and  $\angle FCB$  so that  $\angle ACB \cong \angle FBC$  and  $\angle ABC \cong \angle FCB$

This means that  $\overline{AC}$  is parallel to  $\overline{BF}$  and  $\overline{AB}$  is parallel to  $\overline{CF}$  because these angle pairs are congruent alternate interior angles.

This means that quadrilateral ABFC is a parallelogram. Since  $\angle CAB$  is a right angle, we know that all four angles are right angles.

Now we know that ABFC is a rectangle. Since the diagonals of a parallelogram bisect each other, and the diagonals of a rectangle are congruent, we now know that

$$AD = \frac{1}{2}AF = \frac{1}{2}BC.$$

This proves our conjecture that on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse.

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

(B) constructs and justifies statements about geometric figures and their properties;

(C) demonstrates what it means to prove mathematically that statements are true;



(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

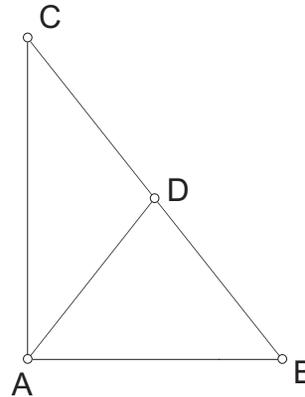
(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

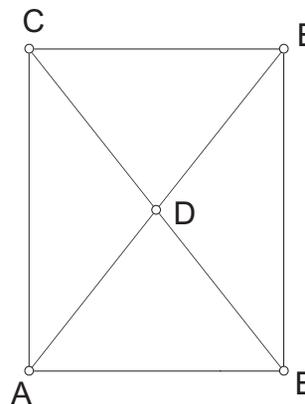
Euclidean Argument:



We are given that  $\triangle BAC$  is a right triangle with a right angle at A and that  $\overline{AD}$  is the median to the hypotenuse.

By the definition of a median, we know that  $CD = DB$ .

Extend  $\overline{AD}$  through point D to point E so that  $AD = DE$ , and draw  $\overline{EC}$  and  $\overline{EB}$  to form quadrilateral ACEB.



By the Vertical Angle Theorem,  $\angle ADC \cong \angle EDB$ .

Now  $\triangle ADC \cong \triangle EDB$  by SAS.

It follows that  $\overline{AC} \cong \overline{EB}$  and  $\angle CAD \cong \angle BED$  because these are corresponding parts of congruent triangles.

These congruent angles are alternate interior angles formed by transversal  $\overleftrightarrow{AE}$  on  $\overleftrightarrow{AC}$  and  $\overleftrightarrow{EB}$ . Therefore,  $\overleftrightarrow{AC}$  and  $\overleftrightarrow{EB}$  are parallel.

Now consider quadrilateral ACEB. Since quadrilateral ACEB

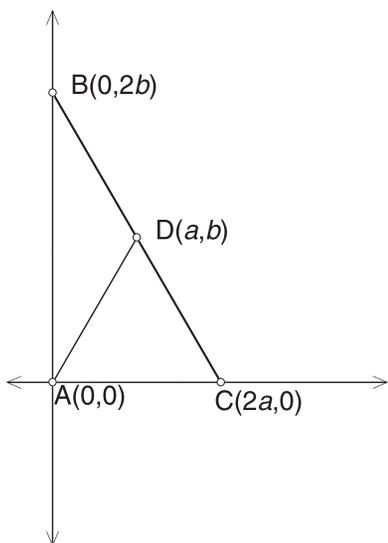


has a pair of opposite sides that are congruent and parallel, the quadrilateral is a parallelogram.

Next, since opposite angles of a parallelogram are congruent, and consecutive angles are supplementary, and since  $\angle BAC$  is a right angle, the other angles of the parallelogram are right angles. This makes  $ACEB$  a rectangle.

Finally, the diagonals of a rectangle bisect each other and are congruent so that  $AD = \frac{1}{2} AE = \frac{1}{2} BC$ .

Coordinate Proof:



Draw right triangle BAC with right angle A at the origin and the legs along the axes.

Assign to vertex C the coordinates  $(2a, 0)$  and to vertex B the coordinates  $(0, 2b)$ .

We know that  $\overline{AD}$  is the median to the hypotenuse, which means point D is the midpoint of  $\overline{BC}$ . We apply the Midpoint Formula to get the coordinates of point D.

$$\left( \frac{2a+0}{2}, \frac{0+2b}{2} \right) = (a, b)$$

### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

### Connections to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Investigating Quadrilateral Properties
- III. Triangles: Pythagorean's Theorem



Now we use the distance formula to find the lengths of the median and the hypotenuse.

$$AD = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

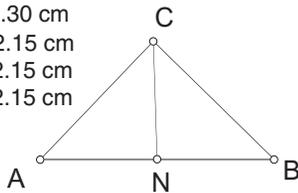
and

$$\begin{aligned} BC &= \sqrt{(2a-0)^2 + (0-2b)^2} \\ &= \sqrt{4a^2 + 4b^2} \\ &= \sqrt{4(a^2 + b^2)} \\ &= 2\sqrt{a^2 + b^2} \end{aligned}$$

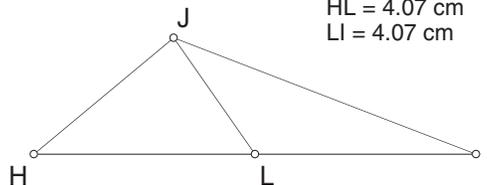
This shows that  $AD = \frac{1}{2}BC$ ; that is, on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse.

3. Other triangles to consider are an acute triangle and an obtuse triangle. Using geometry technology or paper and pencil constructions, we can easily investigate this situation. Possible values are given below.

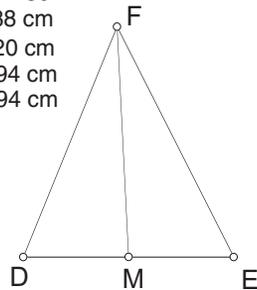
$m\angle ACB = 90^\circ$   
 $AB = 4.30$  cm  
 $CN = 2.15$  cm  
 $AN = 2.15$  cm  
 $NB = 2.15$  cm



$m\angle HJI = 120^\circ$   
 $HI = 8.15$  cm  
 $JL = 2.59$  cm  
 $HL = 4.07$  cm  
 $LI = 4.07$  cm



$m\angle DFE = 50^\circ$   
 $DE = 3.88$  cm  
 $FM = 4.20$  cm  
 $DM = 1.94$  cm  
 $ME = 1.94$  cm



$\triangle ABC$  is a right triangle and demonstrates the conjecture we made and proved.

$\triangle DFE$  shows the median drawn from an acute angle to the opposite side and shows that the median is longer than half of the side to which it is drawn.

$\triangle HJI$  shows the median drawn from an obtuse angle to the opposite side. It shows that the median is shorter than half of the side to which it is drawn, but it is not half the length of that side.



This suggests the following conjecture:

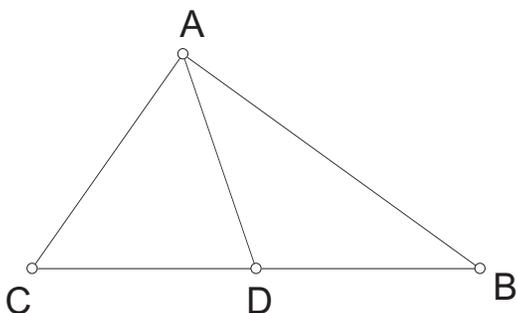
If a median on a triangle is drawn from an acute angle, the median will be longer than half of the side to which it is drawn. If a median on a triangle is drawn from an obtuse angle, the length of the median will be less than half the length of the side to which it is drawn.

4. The converse of our conjecture is the following:

If a median drawn on a triangle is half as long as the side to which it is drawn, then the median is drawn to the hypotenuse of a right triangle.

This is true, as the following argument shows:

Let  $\overline{AD}$  be a median on  $\triangle CAB$  and  $AD = \frac{1}{2}CB$ .



Then, by the definition of a median, D is the midpoint of  $\overline{CB}$ , and we have  $CD = DB = AD$ . This makes  $\triangle CDA$  and  $\triangle BDA$  isosceles triangles with bases  $\overline{CA}$  and  $\overline{BA}$ . Since base angles of an isosceles triangle are congruent,

$$m\angle DCA = m\angle DAC = x, \text{ and}$$

$$m\angle DBA = m\angle DAB = y.$$

Now the angles of  $\triangle CAB$  sum to  $180^\circ$  so that

$$m\angle DCA + m\angle CAB + m\angle DBA = x + (x + y) + y = 180^\circ.$$

This simplifies to

$$2x + 2y = 180$$

$$x + y = 90.$$

This shows that  $\angle CAB$  is a right angle, and  $\triangle CAB$  is a right triangle with hypotenuse  $\overline{CB}$ .



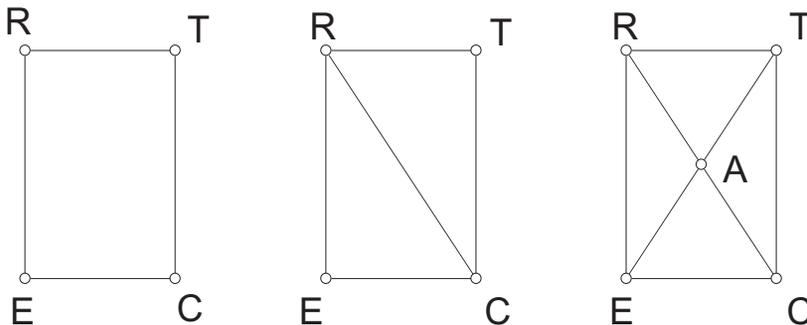
### Extension Questions:

- How does this problem relate to your work with parallelograms?

*Desired Response: A right triangle is formed when a diagonal is drawn on a rectangle. The diagonals of a rectangle are congruent and bisect each other. Therefore, on a right triangle, the median of a right triangle drawn to the hypotenuse is half the diagonal drawn on a rectangle.*

*Consider rectangle  $RECT$ . Draw diagonal  $\overline{RC}$  to form a right triangle,  $\triangle REC$  with hypotenuse  $\overline{RC}$ . Draw diagonal  $\overline{ET}$ , and label the point of intersection  $A$ . Because of the properties of rectangles,  $RA = AC = EA = AT$ .*

*Therefore,  $\overline{EA}$  becomes the median drawn to the hypotenuse  $\overline{RC}$  of  $\triangle REC$  and is half as long as  $\overline{RC}$ .*



- You have shown that on a right triangle the median drawn to the hypotenuse is half as long as the hypotenuse. You were given the choice of three types of proofs. What might be an advantage or disadvantage for each type?

*We can write the proof using congruence transformations, a Euclidean argument, or a coordinate proof.*

*The transformation approach is nice because we can model it with patty paper, and that makes it more visual.*

*The Euclidean approach is a little harder. We have to start with the right triangle and the median drawn on it and realize that we must draw auxiliary segments on it so that we form a rectangle with its diagonals.*

*A coordinate proof is nice because we see how we can use algebra to prove geometric concepts.*

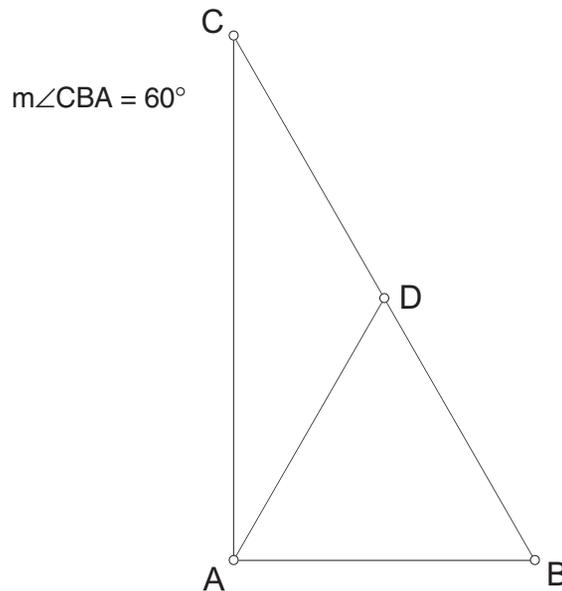
- Suppose that  $\triangle BAC$  is a right triangle with  $\overline{BA} \perp \overline{AC}$  and  $\overline{AD}$  as the median to the hypotenuse. What do we know about  $BD$ ,  $AD$ , and  $CD$ ? How does this relate to your knowledge of circles?



Since  $BD = AD = CD$  and these 4 points are coplanar, we know that points  $B$ ,  $A$ , and  $C$  lie on a circle with center  $D$ . Since points  $B$ ,  $D$ , and  $C$  are collinear, we know that  $\overline{BC}$  is a diameter of the circle.

- What additional patterns emerge if we draw the median to the hypotenuse of the two special right triangles (the triangle with angles that measure  $30^\circ$ ,  $60^\circ$ , and  $90^\circ$  or  $45^\circ$ ,  $45^\circ$ , and  $90^\circ$ )?

Consider a 30-60-90 triangle:



Triangle  $BAC$  is a 30-60-90 triangle with  $m\angle BAC = 90^\circ$  and  $m\angle B = 60^\circ$ .  $\overline{AD}$  is the median to the hypotenuse.

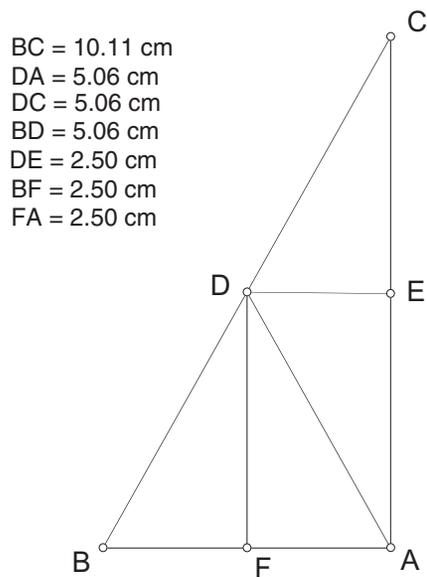
Suppose  $BC = 10$ . Then  $AD = DB = 5$ . Also,  $AB = 5$ . This makes  $\triangle ADB$  equilateral.

The other triangle formed,  $\triangle ADC$ , is an isosceles triangle. The legs are  $AD = DC = 5$  and base angles are 30 degrees.

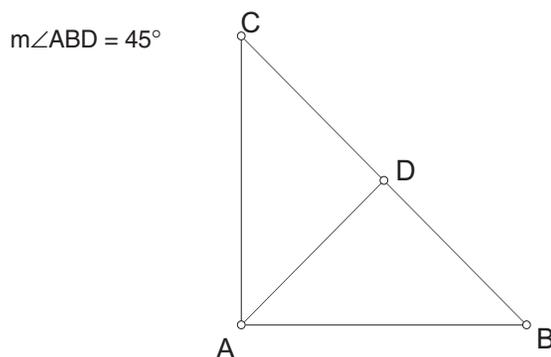
If we draw the median from point  $D$  to segment  $\overline{AB}$ , this will form two 30-60-90 triangles. Also, if we draw the median from  $D$  to  $\overline{AC}$ , we will have 30-60-90 triangles.

The second set of medians separates the original triangle into 4 congruent 30-60-90 triangles.





Now consider the isosceles right triangle:



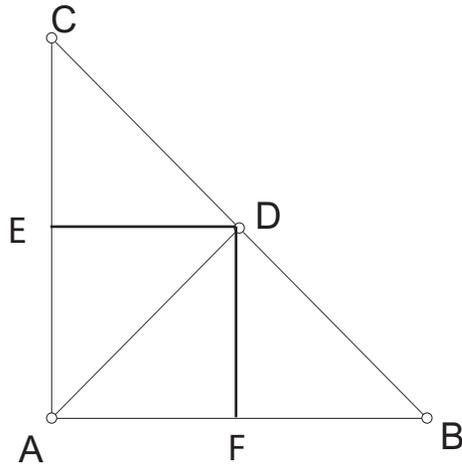
The median to the hypotenuse separates the triangle into two congruent isosceles right triangles:

1. They are isosceles because  $AD = DC = DB$ .
2. They are right triangles because the median from the vertex angle of an isosceles triangle is also an altitude so that  $\overline{AD} \perp \overline{BC}$ .
3.  $\triangle ADC \cong \triangle ADB$  by SSS because of (1) and because  $\overline{AC}$  and  $\overline{AB}$  are congruent legs of the original isosceles right triangle.

If we draw the medians from point  $D$  to segments  $\overline{AB}$  and  $\overline{AC}$ , it will separate the original isosceles right triangle into four congruent isosceles right triangles.



$$m\angle ABD = 45^\circ$$

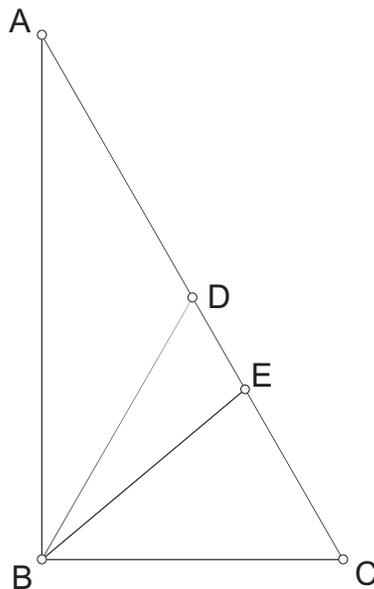


$$\triangle CED \cong \triangle AED \cong \triangle AFD \cong \triangle BFD \text{ by SSS}$$

The isosceles right triangle pattern is different from the 30-60-90 triangle pattern. Drawing the first median to a hypotenuse gives two isosceles right triangles. Drawing the medians on these repeats this pattern.

In the 30-60-90 triangle pattern, the first median drawn to the hypotenuse gives an isosceles triangle (30-30-120) and an equilateral triangle. Drawing medians to the base of isosceles triangle and the equilateral triangle gives four 30-60-90 triangles.

- The median divides the given right triangle into two triangles. Make and prove a conjecture about the areas of the triangles.



*The two triangles have equal area. To demonstrate this, drop a perpendicular from  $B$  to  $\overline{AC}$  to form a segment  $\overline{BE}$ . This segment is an altitude to both triangles, with bases  $\overline{AD}$  and  $\overline{DC}$ . These two segments are equal in length. The triangles have equal bases and the same height, thus they have the same area.*



## Tell Me Everything You Can About...

1. Given a line segment,  $\overline{RO}$ , that is a diagonal of a rhombus RHOM, construct the rhombus. Describe your construction procedure, and explain why your figure is a rhombus.
2. List as many conclusions that you can make about the triangles that result from the construction as possible.

Justify your conclusions.

3. Explain why and how the diagonals may be used to determine the area of the rhombus.



## Teacher Notes

### Materials:

One compass and straightedge per student

Unlined paper, patty paper, and construction tools (including geometry software)

### Connections to Geometry

#### TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(B) constructs and justifies statements about geometric figures and their properties; and

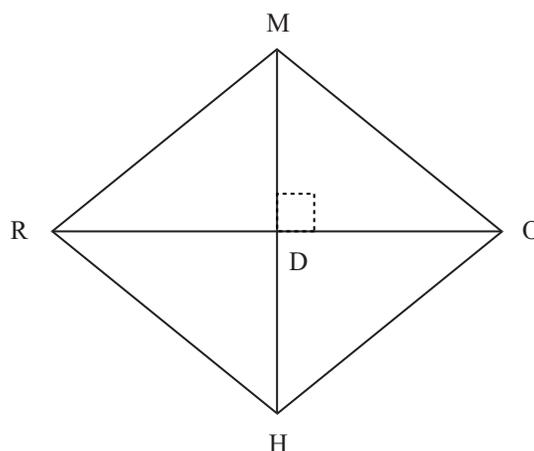
(C) demonstrates what it means to prove mathematically that statements are true;

### Scaffolding Questions:

- What is the definition of a rhombus?
- How is a square related to a rhombus?
- How are the diagonals of a square related?
- How are the diagonals of a rhombus related?
- What special triangles do you see on your constructed rhombus?
- What kind of triangle congruencies do you see?

### Sample Solutions:

1.



Construction process using patty paper:

**Step 1:** To construct the perpendicular bisector of  $\overline{RO}$ , fold the patty paper so that R reflects onto O. The fold line is the perpendicular bisector.

**Step 2:** Mark a point, M, on this perpendicular bisector. A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment, so  $RM = MO$ .

**Step 3:** Locate point H on the perpendicular bisector of  $\overline{RO}$ , on the opposite side from point M, so that  $RH = RM$ . Then, since H is equidistant from R and O,  $HO = RH$ .

Because  $HD = DM$ , and  $\overline{RO}$  is perpendicular to  $\overline{HM}$ ,  $\overline{RO}$



is the perpendicular bisector of  $\overline{HM}$ . Any point on  $\overline{RO}$  is equidistant from H and M. Thus,  $HO = MO$ . Hence,  $HO = MO = RM = RH$ . The figure is a rhombus because it is a quadrilateral with four congruent sides.

2. Possible answers:

Triangles RMO, RHO, HRM, and HOM are isosceles triangles because  $HO = MO = RM = RH$ .

Triangles RMO and RHO are congruent by SAS.

Triangles HRM and HOM are congruent by SAS because  $HO = MO = RM = RH$ .

Since a rhombus is a parallelogram, its opposite angles are congruent.

Therefore,  $\angle RMO \cong \angle RHO$ , and  $\angle HRM \cong \angle HOM$ .

$\triangle RMD \cong \triangle OMD \cong \triangle HOD \cong \triangle HRD$

Since the diagonals of the rhombus are perpendicular, they divide the rhombus into 4 right triangles with  $\angle RMD \cong \angle OMD$  and  $\angle HOD \cong \angle HRD$ .  $\overline{HM}$  and  $\overline{RO}$  are also perpendicular bisectors of each other.  $HD = MD$  and  $RD = DO$ . Thus the 4 right triangles are congruent by SAS.

3. The diagonals of the rhombus may be used to find the area of the rhombus. The diagonals of the rhombus divide it into 4 congruent right triangles. Therefore, the area of the rhombus is 4 times the area of any one of the right triangles. The area of a right triangle can be found by taking one-half the product of its legs. One leg is one-half the length of one diagonal of the rhombus (diagonal 1). The other leg is one-half the length of the other diagonal of the rhombus (diagonal 2).

$$\text{Area of Rhombus} = 4\left(\frac{1}{2}\right)\left(\frac{\text{diagonal 1}}{2}\right)\left(\frac{\text{diagonal 2}}{2}\right)$$

so

$$\text{Area of Rhombus} = \frac{1}{2}(\text{diagonal 1})(\text{diagonal 2})$$

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

#### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

#### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Investigating Quadrilateral Properties



### Extension Questions:

- What characteristics of a rhombus helped you decide how to construct RHOM?

*A rhombus is a parallelogram with 4 congruent sides.*

*Responses would vary from this point on. The following response represents the construction approach used in the sample solution.*

*This leads to the diagonals of the rhombus being perpendicular bisectors of each other. Therefore, start by constructing a segment and its perpendicular bisector. Then decide the length you want the rhombus' sides to be and locate points on the perpendicular bisector of your segment to complete the rhombus.*

- How does a square compare with a rhombus?

*A square is a rhombus with congruent diagonals. You could also say it is a rhombus with adjacent angles being right angles.*

- How does a kite compare with a rhombus?

*A kite is a quadrilateral with perpendicular diagonals. It has to have two pairs of adjacent sides that are congruent, but it does not necessarily have congruent opposite sides.*

*A kite does not have to be a parallelogram.*



## Tiling with Four Congruent Triangles

Shannon is helping her mom decide on triangular tile patterns for the kitchen they are going to remodel. Using colored construction paper cutouts, they saw they could easily use right triangles (isosceles or scalene), since two congruent right triangles form a rectangle. Shannon began to wonder about using congruent acute or congruent obtuse triangles, which would give a very different look. This led to her question:

1. Can 4 congruent triangles always be arranged to form a triangle? Describe how you would explore this question.
2. If your response to # 1 is no, give a counter-example.
3. If your response to # 1 is yes, state a conjecture about how the original triangle and its 3 copies are related to the newly formed triangle?
4. If your response to # 1 is yes, write a proof that uses transformations to justify your conjecture.



## Teacher Notes

### Materials:

Construction paper, patty paper, unlined paper, and construction tools (including geometry software)

### Connections to Geometry

#### TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

(B) constructs and justifies statements about geometric figures and their properties;

### Scaffolding Questions:

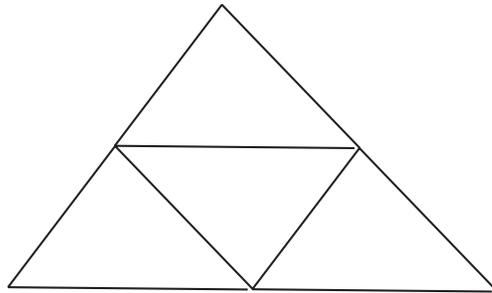
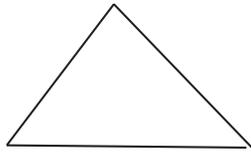
- How can you arrange the triangles so that sides are collinear?
- How do you know that the sides of the smaller triangles are collinear?
- After you have arranged the 4 triangles, where are their vertices located?
- One triangle could be described as an inner triangle. What can you say about its vertices?
- As you arrange the 4 triangles, think about transformations. What transformations are you using?
- How can you label information on the triangles to make it easier for you to describe these transformations?
- Recall the definition of a triangle midsegment. Recall the triangle midsegment properties we investigated in class. How does this activity relate to these investigations?

### Sample Solutions:

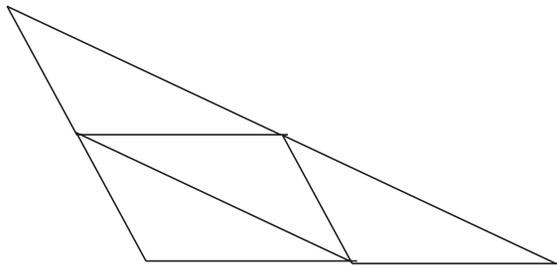
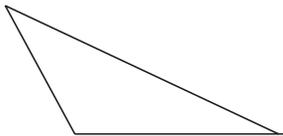
1. To explore this question cut out 4 copies of an acute triangle using different colors for each triangle. Arrange the triangles so that they do not overlap and they form a triangle. Repeat this using 4 copies of an obtuse triangle. You could also draw a triangle on patty paper and make 3 copies. Then arrange the 4 triangles to form the larger triangle.



Acute triangle example:



Obtuse triangle example:



2. Since a triangle may be formed with the four triangles, there is no apparent counterexample.

3. In both cases the 4 congruent triangles form a larger triangle that is similar to the original triangle and its copies.

The sides of the larger triangle are twice as long as the corresponding sides of the original triangle. The area of the large triangle is four times the area of the original triangle because it is made up of the 4 non-overlapping congruent triangles.

(C) demonstrates what it means to prove mathematically that statements are true;

(D) uses inductive reasoning to formulate a conjecture; and

(E) uses deductive reasoning to prove a statement.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;



(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures.

If the large triangle is similar to the original triangle, the ratio of the area of the large triangle to the area of the original triangle is the square of the ratio of the corresponding sides.

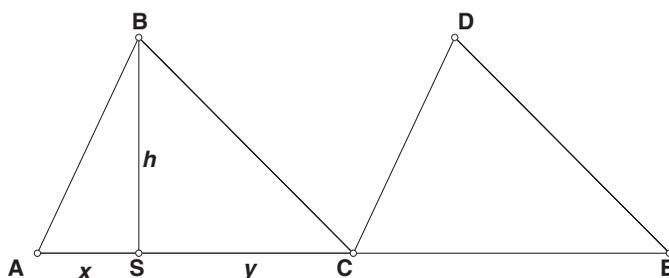
$$\frac{\text{Area of large triangle}}{\text{Area of original triangle}} = \left(\frac{2}{1}\right)^2 = \frac{4}{1}$$

#### 4. Transformational Proof:

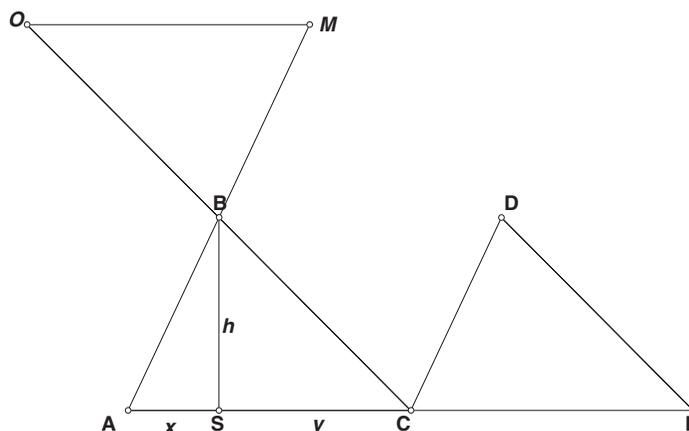
**Step 1:** Draw  $\triangle ABC$  so that segment  $\overline{AC}$  is horizontal. Draw altitude  $\overline{BS}$ .

Let  $BS = h$ ,  $AS = x$ , and  $SC = y$ .

**Step 2:** Translate  $\triangle ABC$   $x + y$  units to the right to form  $\triangle CDE \cong \triangle ABC$ .  $A$ ,  $C$ , and  $E$  are collinear.

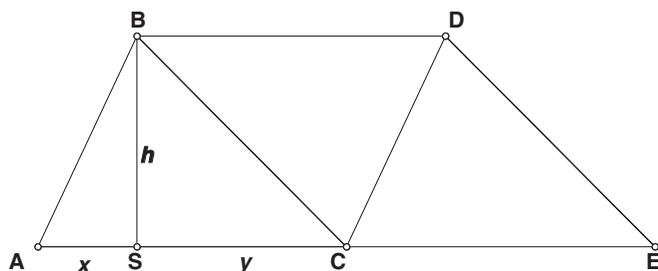


**Step 3:** Rotate  $\triangle ABC$   $180^\circ$  counterclockwise about point  $B$  to form  $\triangle BMO \cong \triangle ABC$ ,  $\overline{BO} \cong \overline{CB}$ , and  $\overline{BM} \cong \overline{CD}$ .

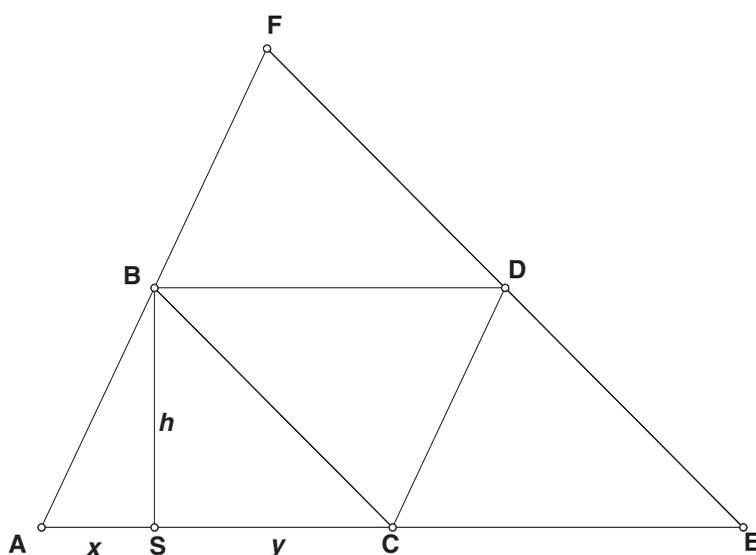


**Step 4:** Translate  $\triangle BMO$   $y$  units to the right and  $h$  units down to form  $\triangle DCB \cong \triangle ABC$ .





**Step 5:** Translate  $\triangle ABC$   $x$  units to the right and  $h$  units up to form  $\triangle BFD \cong \triangle ABC$ .  $A$ ,  $B$ , and  $F$  are collinear.  $F$ ,  $D$ , and  $E$  are collinear.



The four triangles are congruent. The three angles of the smaller triangle are congruent to the three corresponding angles of the larger triangle.

$$\angle A \cong \angle A, \angle ABC \cong \angle F, \angle ACB \cong \angle E$$

The large triangle is similar to the original triangle by AAA similarity.

$$\triangle AFE \approx \triangle ABC$$

### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

- I. Structure: Triangles Tell It All



### Extension Questions:

- In this activity what have you shown to be true?

*Triangles of any shape can be used to tile a plane because the four congruent triangles can be arranged in a non-overlapping way to form a larger triangle.*

- How are the vertices of the 4 original triangles related to the vertices of the larger triangle that is formed?

*Three of the triangles may be considered to be outer triangles, and the fourth triangle is inside the larger triangle. One vertex from each of the outer triangles is a vertex of the larger triangle.*

*The vertices of the inner triangle are on the sides of the larger triangle and will become the midpoints of the sides of the larger triangle. Each side of the large triangle is formed by two corresponding sides of the congruent triangles. Therefore, each side of the large triangle is twice as long as a side of the original triangle.*

- How does this result relate to the Triangle Midsegment Theorems?

*It is the converse of the following theorem:*

*If the midsegments of a triangle are drawn, then four triangles are formed, and the four triangles are congruent.*

- How can you prove the result of this activity using algebraic thinking?

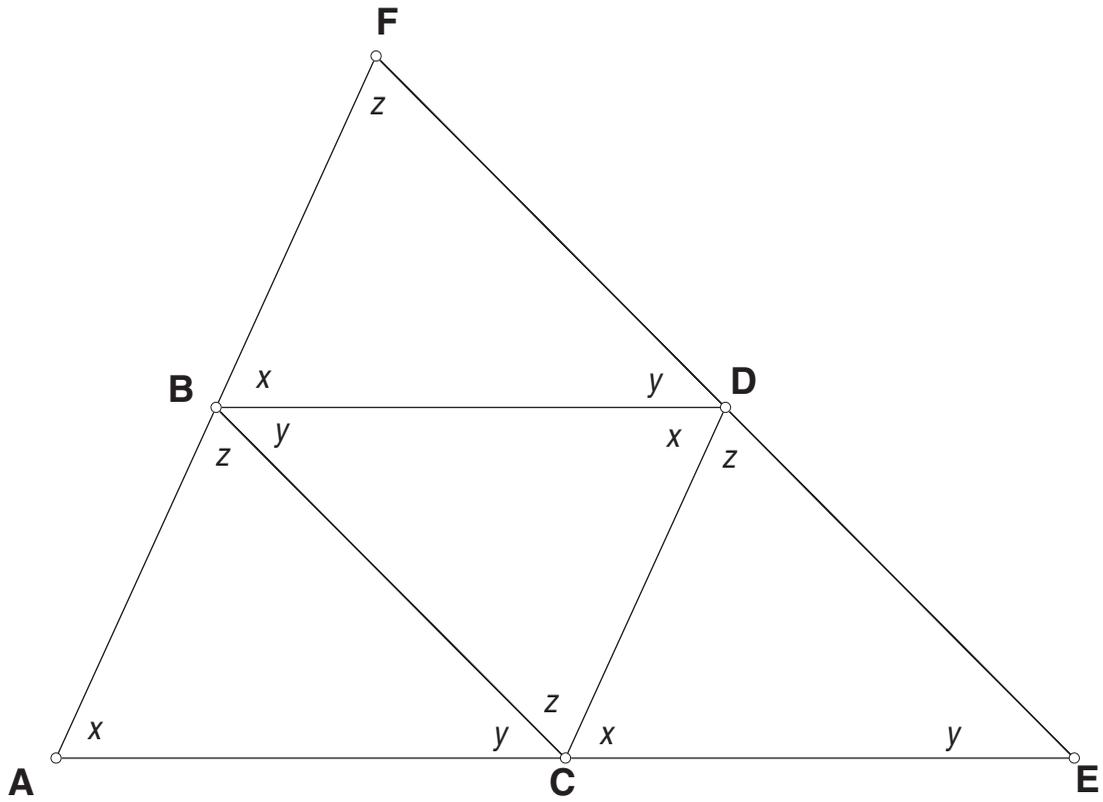
*To be sure the 4 congruent triangles are forming a triangle, we need to show that points A, B, and F are collinear; points F, D, and E are collinear; and points C, E, and A are collinear.*

*Since  $\triangle ABC \cong \triangle DCB \cong \triangle CDE \cong \triangle BFD$ , let the measures of the congruent corresponding angles be  $x$ ,  $y$ , and  $z$ , as shown on the diagram.*

*We know that the sum of the angles of a triangle equal  $180^\circ$ . Therefore, we know that  $x + y + z = 180$ .*

*This shows that B is collinear with A and F, D is collinear with F and E, and C is collinear with E and A. Therefore, the 4 congruent triangles form the large triangle with no overlap.*





## Student Work Sample

As an extension to the problem, a student was asked to justify his conjecture using a coordinate proof. The student started with and labeled the coordinates. A more complete proof would have included an explanation of why he chose the coordinates of the other points on his diagram. The student used E and G interchangeably from time to time.

Some of the solution guide criteria exemplified by this work are the following:

- Demonstrates geometric concepts, processes, and skills.

*The student shows the correct use of the distance formula. He correctly simplifies the radicals and determines the ratio of corresponding sides in the smaller and larger triangles.*

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem.

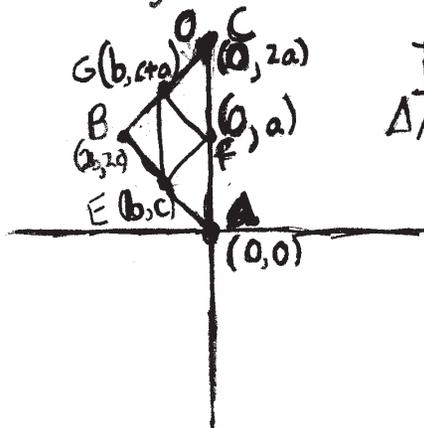
*The student correctly used the distance formula to get the length of each side of the triangles. Also, he compared the length of the sides to get the ratio to prove similarity of the triangles.*

- Communicates a clear, detailed, and organized solution strategy.

*The student states his conjecture that the smaller triangles are similar to the larger triangle. He indicates that he will use the distance formula. He shows all the necessary steps in using the distance formula and in simplifying the ratios of the lengths of the corresponding sides. He states his conclusion that the triangles are similar and gives a reason for this conclusion.*



# Tiling with 4 Congruent Triangles



I must prove that  $\triangle CEF$ ,  $\triangle AEF$ ,  $\triangle BEG$ , and  $\triangle EFG$  are similar to  $\triangle ABC$ . Using the distance formula to do this,

$$\overline{AC} = \sqrt{0^2 + (2a)^2} \text{ or } \sqrt{4a^2} \text{ or } 2a$$

$$\overline{CE} = \sqrt{b^2 + a^2} \text{ or } \sqrt{a^2} \text{ or } a$$

$$\overline{AF} = \sqrt{0^2 + a^2} \text{ or } a$$

$$\overline{EG} = \sqrt{(b-b)^2 + (c-a)^2} \text{ or } \sqrt{0^2 + a^2} \text{ or } a$$

$$\frac{\overline{CE}}{\overline{AC}} = \frac{\overline{AF}}{\overline{AC}} = \frac{\overline{EG}}{\overline{AC}} = \frac{a}{2a} = \frac{1}{2}$$

$$\overline{AB} = \sqrt{(2b)^2 + c^2} \text{ or } \sqrt{4b^2 + 4c^2} \text{ or } 2\sqrt{b^2 + c^2}$$

$$\overline{AG} = \sqrt{b^2 + c^2}$$

$$\overline{BG} = \sqrt{(2b-b)^2 + (c-c)^2} \text{ or } \sqrt{b^2 + c^2}$$

$$\overline{EF} = \sqrt{b^2 + (c-a-a)^2} \text{ or } \sqrt{b^2 + c^2}$$

$$\frac{\overline{AG}}{\overline{AB}} = \frac{\overline{BG}}{\overline{AB}} = \frac{\overline{EF}}{\overline{AB}} = \frac{\sqrt{b^2 + c^2}}{2\sqrt{b^2 + c^2}} = \frac{1}{2}$$

$$\overline{BC} = \sqrt{(2b-0)^2 + (2c-2a)^2} \text{ or } \sqrt{4b^2 + 4c^2 - 8ac + 4a^2} \text{ or } 2\sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{BE} = \sqrt{(2b-b)^2 + (2c-c+a)^2} \text{ or } \sqrt{b^2 + (c-a)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{CE} = \sqrt{b^2 + (2a-a+c)^2} \text{ or } \sqrt{b^2 + (a-c)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\overline{FG} = \sqrt{b^2 + (c-a)^2} \text{ or } \sqrt{b^2 + c^2 - 2ac + a^2}$$

$$\frac{\overline{BE}}{\overline{BC}} = \frac{\overline{CE}}{\overline{BC}} = \frac{\overline{FG}}{\overline{BC}} = \frac{\sqrt{b^2 + c^2 - 2ac + a^2}}{2\sqrt{b^2 + c^2 - 2ac + a^2}} = \frac{1}{2}$$

This means  $\triangle CEF$ ,  $\triangle AEF$ ,  $\triangle BEG$ ,  $\triangle EFG$ , are congruent and similar to  $\triangle ABC$  because the corresponding sides are 1:2.





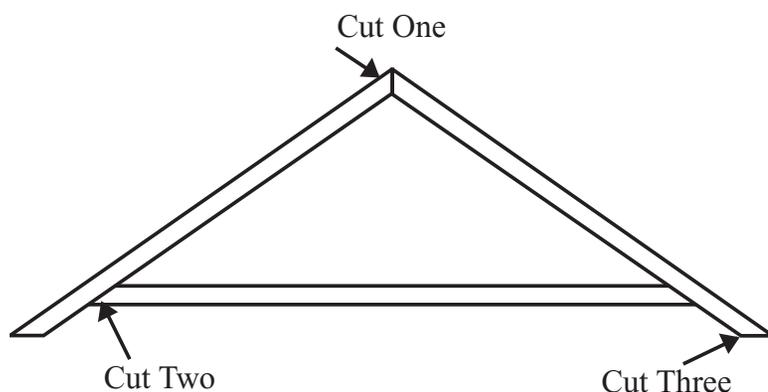
## Shadow's Doghouse

You are planning to build a new doghouse for your German Shepherd, Shadow. The doghouse will be a rectangular solid: 24 inches across the front, 36 inches deep, and 30 inches high. The cross section of the pitched roof is an isosceles triangle. The distance from the floor of the doghouse to the peak of the roof is 39 inches.

There will be 4 roof trusses (shown below), which are isosceles triangles, to support the pitched roof.

To determine the materials you need to purchase and how you will construct the frame, you make careful plans on paper before you begin construction.

1. Make a sketch of the doghouse showing the front, side, and roof.
2. How could you ensure the 4 roof trusses are precisely the same shape and size using the least number of measurements? Describe at least two ways. What geometry concepts are being used?
3. The roof trusses are to be cut from 1-inch by 2-inch boards, as diagrammed below:



At what angles should cuts one, two, and three be made? What geometry concepts are being used?

4. The roof is to have a 3-inch overhang. That means that there will be a 3-inch extension of the roof at the front, back, and sides. The roof is to be made from two pieces of plywood and covered with shingles. The shingles



are laid so that they overlap, with the exposed (visible) area of a shingle measuring 6 inches by 6 inches. How many shingles are needed? Explain how you determine this, citing the geometry concepts you use.

5. The roof, sides, front, back, and floor of the doghouse will be made of plywood, which is available in 4-foot by 8-foot sheets and half-sheets (2 feet by 8 feet or 4 feet by 4 feet). The opening on the front of the doghouse will be 12 inches wide by 18 inches high. The front and back pieces are to be cut as pentagons, not as rectangles with an added triangle. How many sheets and/or half-sheets of plywood are needed for the entire doghouse, and how would you lay out the pieces to be cut? What geometry concepts are being used to ensure opposite walls are precisely the same size?
6. After estimating your total costs, you decide to consider making Shadow's doghouse smaller, with a floor 20 inches wide and 30 inches deep. The shape of the new doghouse would be similar to the original doghouse. A dog that is the size of Shadow requires a doghouse with a floor area of at least 4 square feet and a capacity of at least 9 cubic feet. Would the smaller doghouse be large enough for Shadow? Explain.





## Teacher Notes

### Materials:

One ruler, calculator, and protractor per student

Unlined paper for drawing

### Connections to Geometry

#### TEKS:

(b.1) **Geometric structure.** The student understands the structure of, and relationships within, an axiomatic system.

The student:

(A) develops an awareness of the structure of a mathematical system, connecting definitions, postulates, logical reasoning, and theorems;

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

The student:

(A) determines if the converse of a conditional statement is true or false;

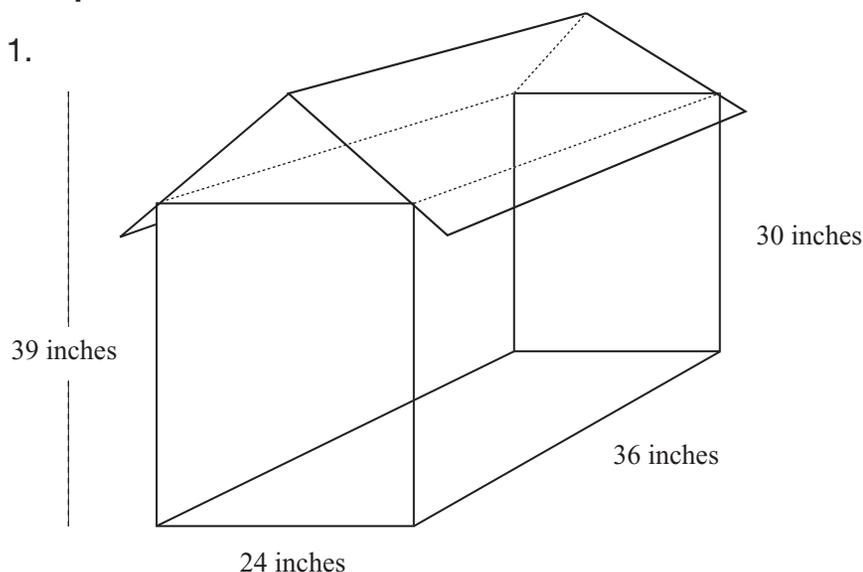
(d.1) **Dimensionality and the geometry of location.** The student analyzes the relationship between three-dimensional objects and related two-dimensional representations and uses these representations to solve problems.

### Scaffolding Questions:

- What does it mean for geometric figures to be congruent?
- What are the possible ways to prove triangles are congruent?
- What would be reasonable ways to build the roof trusses so that they are congruent triangles?
- How can Pythagorean Triples help you determine measurements in this problem?
- What are the angle characteristics for various triangles?
- What special triangles are formed by the roof trusses?
- What do you know about the altitude to the base of an isosceles triangle?
- What are the shape and dimensions of the plywood pieces that will form the roof?
- How can you plan the plywood pieces you will need to cut for the doghouse?
- If figures are similar, what do you know about the ratio of the corresponding linear measurements, the ratio of areas, and the ratio of volumes?

### Sample Solutions:

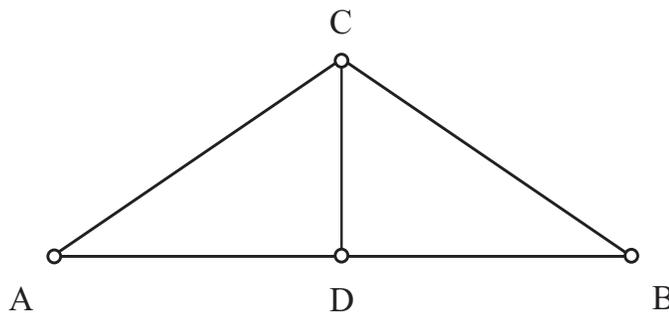
1.



2. For the triangular roof trusses to be the same shape and size, we must design and build them so that we have congruent triangles. Triangles are congruent by *SSS*, *SAS*, *ASA*, or *SAA*. The most reasonable congruence relationship to use in building the roof trusses is probably *SSS*. We would determine how long the boards need to be and where they will be joined. Then we would cut the boards.

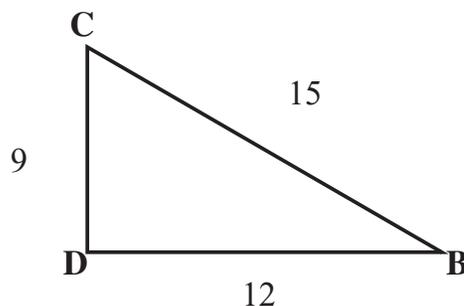
We know the beams will be 24 inches across and 9 inches high.

$AB = 24$ ,  $CD = 9$ ,  $\overline{CD}$  is perpendicular to  $\overline{AB}$ ,  $AC = CB$



Since the beams form an isosceles triangle, we know that the sides that will support the roof are congruent, and that the altitude segment  $\overline{CD}$  to the base  $\overline{AB}$  bisects segment  $\overline{AB}$ .

Using right triangles and Pythagorean Triples, we find the lengths of the congruent sides. Segments  $AC$  and  $BC$  are 15 inches.



Measuring the angles with a protractor, we find that  $m\angle A = m\angle B \approx 37^\circ$  and  $m\angle ACB \approx 106^\circ$ .

If we build the triangular trusses to be congruent by *SSS*, we would cut four 24-inch boards and eight 18-inch boards. Three inches of each board is for the roof overhang.



The student:

(C) uses top, front, side, and corner views of three-dimensional objects to create accurate and complete representations and solve problems.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations.

(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(D) analyzes the characteristics of three-dimensional figures and their component parts.

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

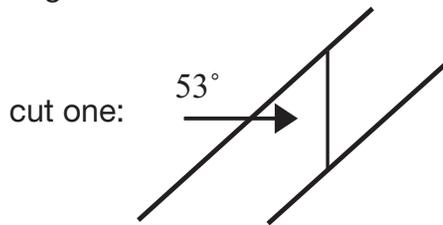
(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(4) describes and applies the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

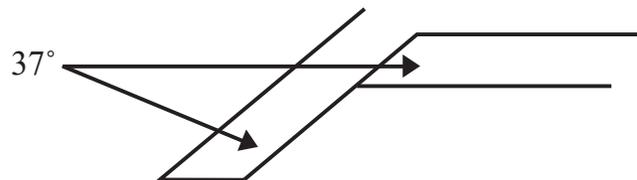
If we build the triangular trusses to be congruent by ASA, we would cut the four 24-inch boards and then measure the base angles on each end to be 37 degrees.

3. Since the vertex angle of the isosceles triangle formed by the outer edges of the boards measures 106 degrees, and the angled edge lies along the altitude to the base of the isosceles triangle, cut one needs to be made at an angle of 53 degrees. This is because the altitude from the vertex angle of an isosceles triangle is also the bisector of that angle.



The top and bottom edges of the baseboard of the truss (the base of the isosceles triangle) and the angled edge of the bottom of the boards that are the sides of the beam are parallel. Since corresponding angles formed by transversals on parallel lines are congruent, cuts two and three need to be made at angles of 37 degrees.

cuts two and three:



4. To determine the number of shingles needed, we compute the surface area to be covered. The surface area is  $42(18) = 756$  square inches. The number of shingles needed for one side is

$$\frac{756 \text{ square inches}}{36 \text{ square inches for one shingle}} = 21 \text{ shingles.}$$

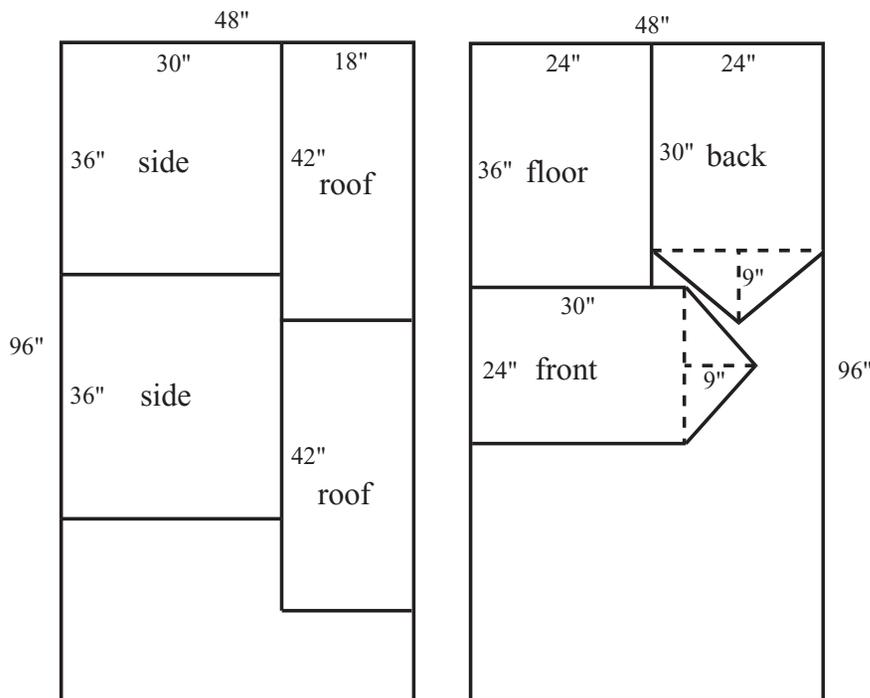
5. The total square area of plywood needed for the doghouse is



$$2[30(36) + 18(42) + 24(30) + 0.5(9)(24)] + 24(36) = 6192 \text{ square inches.}$$

One full sheet of plywood has an area of  $48(96) = 4608$  square inches. One and one-half sheets would provide an area of 6912 square inches. However, the layout of the pieces (side, front, back, roof, and floor) must be considered.

The following possible layout shows that two full pieces of plywood are needed to efficiently cut the pieces as planned.



The sides are congruent rectangles, and the front and back are congruent pentagons since corresponding sides and corresponding angles are congruent.

6. The floor space of the new doghouse would be 20 inches by 30 inches, or 600 square inches. Convert to square feet.

$$600 \text{ in}^2 \cdot \frac{1 \text{ ft}^2}{144 \text{ in}^2} = 4.167 \text{ ft}^2$$

The floor space is more than 4 ft<sup>2</sup>.

Since the two doghouses are similar in shape, the other dimension of the doghouse may be determined by using ratios.

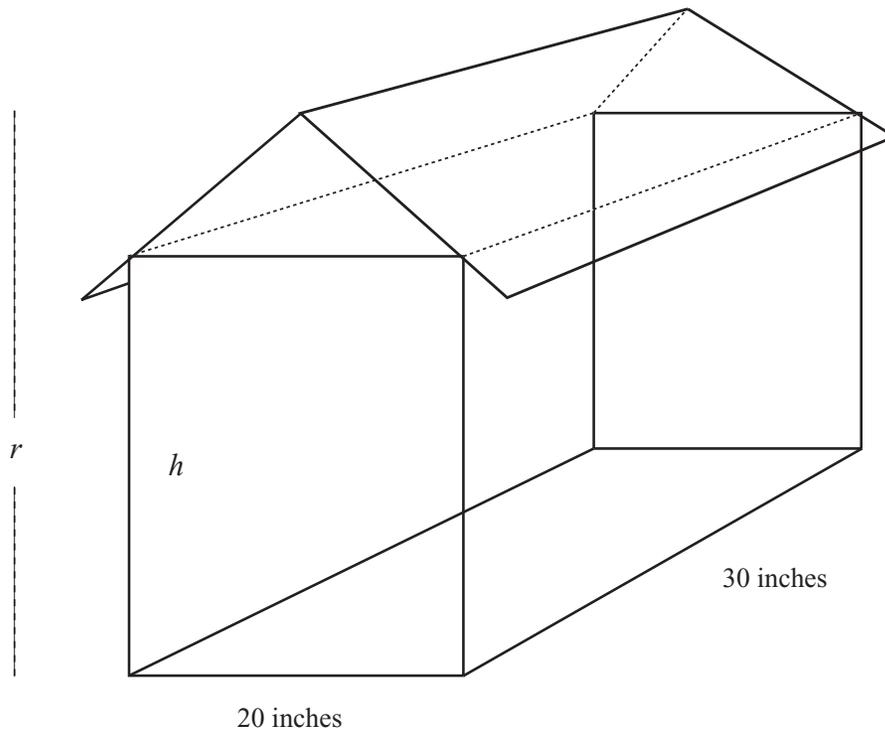
### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

### Connections to High School Geometry: Supporting TEKS and TAKS Institute:

- III. Triangles: Foundations and Trussess
- V. Solid Figures: Bayou City Dome





If the shapes are similar, then

$$\frac{h}{20} = \frac{30}{24} \quad \text{and} \quad \frac{r}{20} = \frac{39}{24}$$

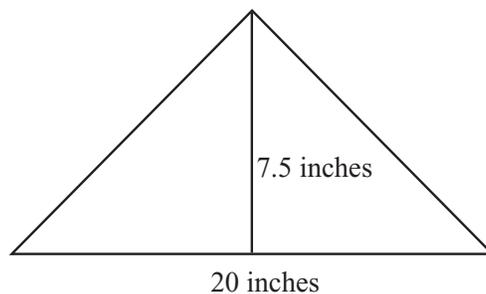
$$h = 20 \left( \frac{30}{24} \right) = 25 \text{ in} \quad r = 20 \left( \frac{39}{24} \right) = 32.5 \text{ in}$$

The capacity of the original doghouse is the volume of the rectangular prism portion plus the volume of the triangular prism of the roof.

The volume of the rectangular prism is 20 inches by 30 inches by 25 inches, or 15000 in<sup>3</sup>.

$$15000 \text{ in}^3 \cdot \frac{1 \text{ ft}^3}{12 \text{ in} \cdot 12 \text{ in} \cdot 12 \text{ in}} = 8.68 \text{ ft}^3$$

The volume of the triangular prism is the area of the triangle times the length. The height of the triangle is 32.5 – 25 = 7.5 inches.



$$\text{Area of the triangle} = \frac{1}{2} 7.5(20) = 75 \text{ in}^2$$

The volume of the prism is  $75(30) = 2250 \text{ in}^3$ .

$$2250 \text{ in}^3 \cdot \frac{1 \text{ ft}^3}{12 \text{ in} \cdot 12 \text{ in} \cdot 12 \text{ in}} = 1.30 \text{ ft}^3$$

The volume of the new doghouse would be  $8.68 + 1.30 \text{ ft}^3$ , or  $9.98 \text{ ft}^3$ .

This doghouse would be large enough for Shadow.

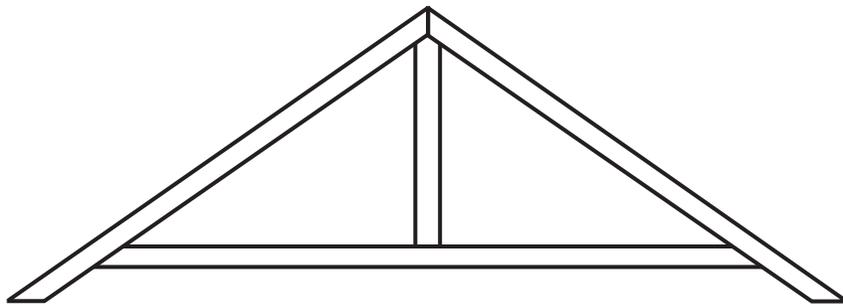
### Extension Questions:

- Build a scale model of Shadow's doghouse.

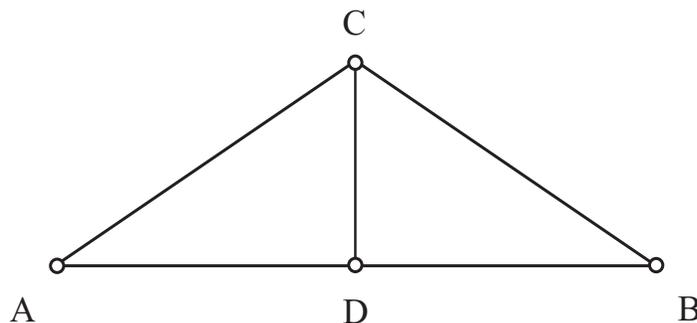
*Answers will vary.*

- Describe how you could build the roof trusses using congruent scalene triangles.

*Assuming the peak of the roof will be centered on the doghouse, each truss would consist of two congruent scalene triangles as shown below:*



*The truss would consist of the 24-inch base, a 9-inch board that is perpendicular to the base, and the 15-inch roof supports. This is using two scalene triangles that are congruent by SSS:*



$$AD = BD = 12, CD = 9 \text{ and } AC = BC = 15.$$

For the remaining questions, suppose that you decide to build the doghouse so that the floor is a regular hexagon.

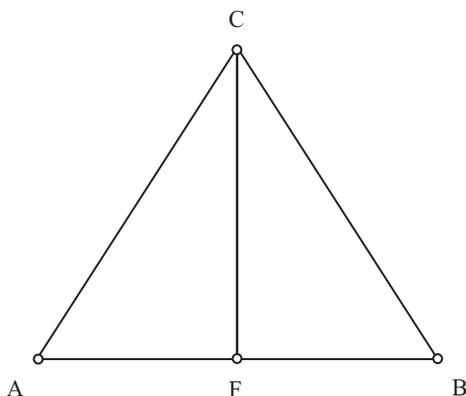
- Determine the dimensions, to the nearest inch of the hexagon, and describe how you will draw the hexagon on the plywood in order to cut it. Remember that you need at least 4 square feet of floor space.

The hexagon will consist of 6 congruent equilateral triangles. To determine the side of the equilateral triangle, solve the following area problem:

Area of polygon:

One-half the apothem of the polygon times its perimeter = one-half times the altitude of the triangle space times six times the base of the triangle.

Since the apothem is the altitude of the equilateral triangle, and the altitude divides the equilateral triangle into two 30-60-90 triangles, use the Pythagorean Formula to express the altitude,  $a$ , in terms of a side,  $s$ , of the equilateral triangle.



$\triangle ABC$  is an equilateral triangle, and  $\overline{CF}$  is the altitude to  $\overline{AB}$ .

Let  $CF = a$ , and  $BC = s$ . Then  $FB = \frac{s}{2}$ , and

$$a^2 + \left(\frac{s}{2}\right)^2 = s^2$$

$$a^2 = s^2 - \frac{s^2}{4}$$

$$= \frac{3s^2}{4}$$

$$\text{so that } a = \frac{\sqrt{3}}{2}s.$$



Next, get an expression for the area of the hexagon:

$$\begin{aligned}A &= \frac{1}{2}ap \\ &= \frac{1}{2}a(6s) \\ &= 3\left(\frac{\sqrt{3}}{2}s\right)s \\ &= \frac{3\sqrt{3}}{2}s^2.\end{aligned}$$

Now

$$\begin{aligned}\text{let } \frac{3\sqrt{3}}{2}s^2 &= 4 \\ 3\sqrt{3}s^2 &= 8 \\ s^2 &= \frac{8}{3\sqrt{3}} \\ &\approx 1.54 \text{ sq. ft.} \\ \text{so that } s &\approx 1.24 \text{ ft} \\ &= 14.88 \text{ inches.}\end{aligned}$$

Rounding to the nearest inch, we need to cut a regular hexagon with a side that is 15 inches.

One way to draw the hexagonal floor is to draw a circle of radius 15 inches. Then mark off and draw on the circle chords that are 15 inches long.

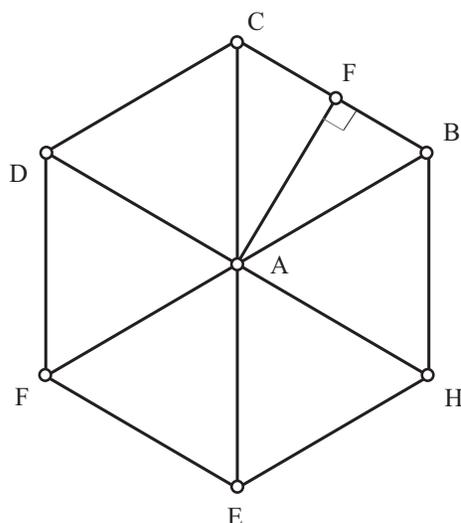
- Assume the floor-to-roof edge must still be 30 inches, and the door opening must be at least 12 inches wide by 18 inches high. How much plywood will be needed for the sides?

You would need to cut six congruent rectangles that are 15 inches wide by 30 inches high and then cut the opening in one of them. One 4-foot by 8-foot piece of plywood would be enough, measuring 15 inches across the 4 feet width twice and 30 inches along the 8 feet length three times.

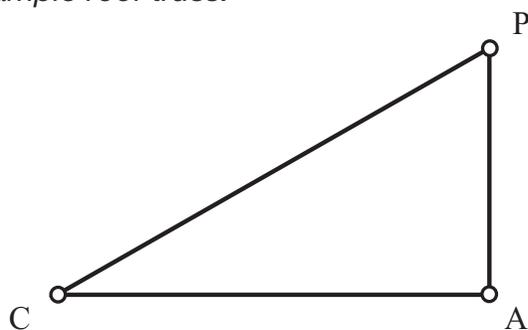
- The roof for the redesigned doghouse will be a hexagonal pyramid with a height (altitude) of 9 inches. Describe how you would design the beams to support the roof and the plywood sections that will form the roof.



The base of the hexagonal pyramid that will be the roof will be congruent to the floor of the doghouse. The base is shown below.

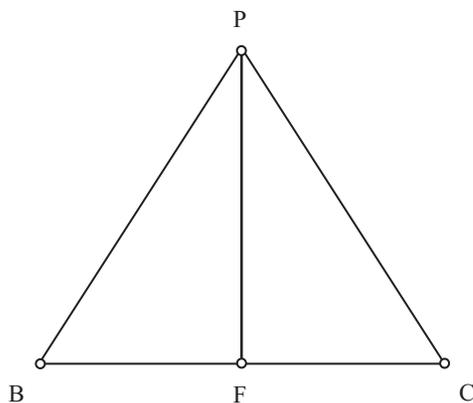


Six congruent triangular roof trusses can be made. Let  $P$  be the point at the top of the roof. The triangle  $CAP$  is a sample roof truss.



$CA = 15$  inches,  $PA = 9$  inches, and  $CP = 17.49$  inches (approximately) by the Pythagorean Formula. (The board corresponding to segment  $PA$  needs to be a hexagonal post, and it is a common leg of all six right triangles forming the roof trusses.)

The 6 congruent, plywood triangles that would form the roof would look like the following triangle:



$BC = 15$  inches.  $BP = PC = 17.49$  inches, and the segments are the lateral edges of one of the six congruent triangle faces of the pyramid that forms the roof.

- How will the capacity (volume) of the new doghouse compare with that of the original doghouse?

The capacity of the new doghouse will be the volume of the regular hexagonal prism that is Shadow's room plus the volume of the hexagonal pyramid that is the roof:

Prism Volume = height times base area

Pyramid Volume = one-third height times base area.

Base Area:

$$\text{Find the apothem of the hexagon: } a = \frac{\sqrt{3}}{2} s = \frac{\sqrt{3}}{2} \cdot 15 \approx 12.99$$

$$A = \frac{1}{2} ap = 0.5(12.99)(6 \cdot 15) = 584.55 \text{ square inches}$$

Prism Volume:

$$V = 30 \cdot 584.55 = 17536.5 \text{ cubic inches}$$

Pyramid Volume:

$$V = \frac{1}{3} \cdot 9 \cdot 584.55 = 1753.65 \text{ cubic inches}$$

Total Volume:

$$V = 17536.5 + 1753.65 = 19290.15 \text{ cubic inches}$$

Convert to cubic feet.

$$19290.15 \text{ cubic inches} \cdot \frac{1 \text{ cubic foot}}{12 \text{ cubic inches}} = 11.16 \text{ cubic feet}$$

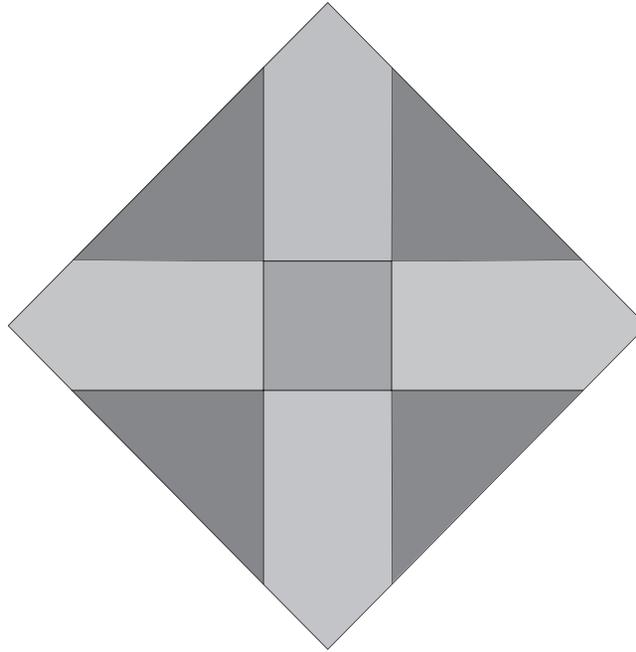
The volume is 11.16 cubic feet.





## The School Flag

A flag for a high school displays a cross on a square as shown below:



The square in the center of the cross will be silver, the rest of the cross will be red, and the remainder of the flag is to be black.

The total area of the flag will be 400 square inches, and the cross should fill 36% of the square flag.

The total cost of a flag is based on the parts: silver (most expensive), red (next in cost), and black (least expensive). These cost issues prompt the question:

What percent of the area of the flag is each colored section?



**Materials:**

One calculator per student

Construction paper in different colors, unlined paper, and construction tools (including geometry software)

**Connections to Geometry****TEKS:**

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(A) uses constructions to explore attributes of geometric figures and to make conjectures about geometric relationships; and

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.3) **Geometric structure.** The student understands the importance of logical reasoning, justification, and proof in mathematics.

## Teacher Notes

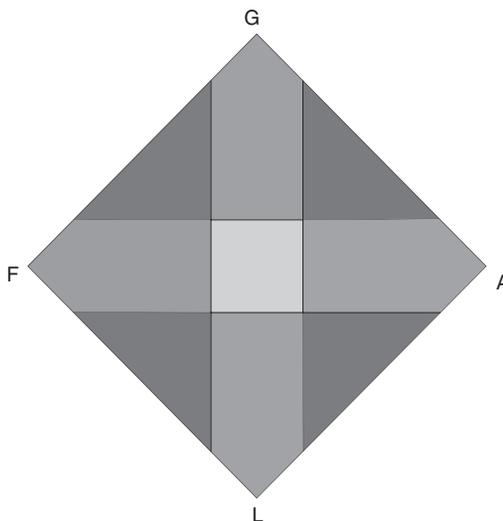
**Scaffolding Questions:**

- Make a model using different colors of construction paper.
- If you break the flag apart into pieces, what kinds of polygons do you see?
- What are the dimensions of the shapes that could vary?
- What information does the problem give you that restricts the dimensions?
- How could you dissect your model and rearrange the pieces so that you have the same colored pieces grouped together to form rectangles or squares?
- What properties of a square help you find congruent triangles?
- As you rearrange pieces think about transformations. What transformations are you using?

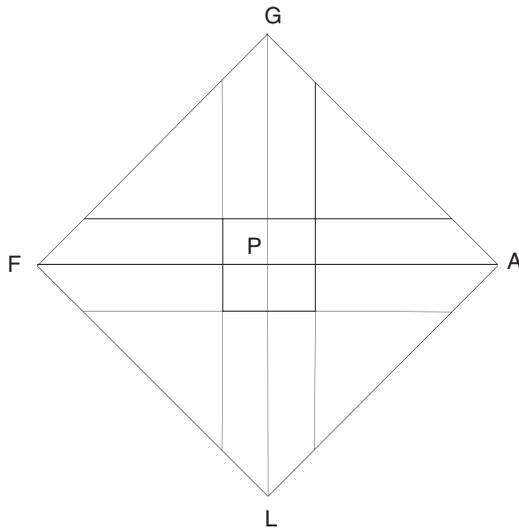
**Sample Solution:**

The flag can be cut into pieces, and the pieces can be rearranged so that each color forms a rectangular region.

Label the original square FLAG.

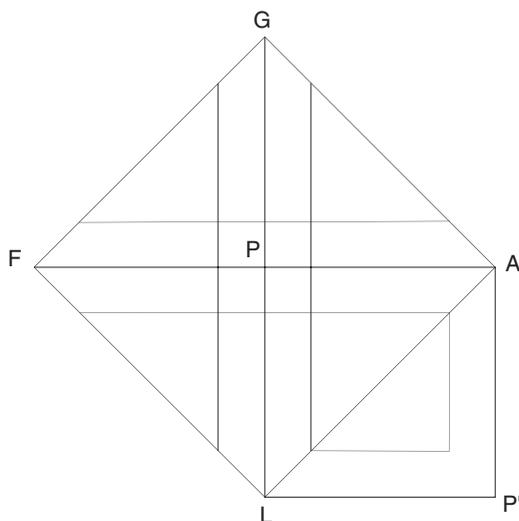


Draw diagonals  $\overline{FA}$  and  $\overline{LG}$ . Label their point of intersection P.



Since the diagonals of the square are congruent, bisect each other, and are perpendicular, triangles FPL, LPA, APG, and GPF are congruent by SAS.

Translate  $\triangle FPG$  so that segment  $\overline{FG}$  coincides with segment  $\overline{LA}$  to form the following:



Triangles LPA and LP'A form a square that is one-half of the flag. Therefore, the area of square PLP'A is 200 square inches.

The student:

(B) constructs and justifies statements about geometric figures and their properties;

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals.



(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(A) finds areas of regular polygons and composite figures.

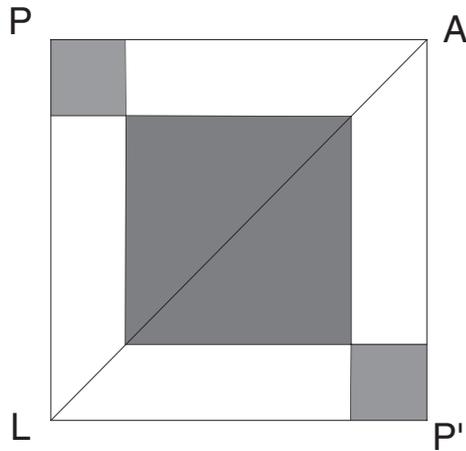
(e.2) **Congruence and the geometry of size.** The student analyzes properties and describes relationships in geometric figures.

The student:

(B) based on explorations and using concrete models, formulates and tests conjectures about the properties and attributes of polygons and their component parts;

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

Now focus on the colors of the regions of square  $PLP'A$ :



The original square flag has an area of 400 square inches. Square  $PLP'A$  has an area of 200 square inches. If we let  $s$  be the side of square  $PLP'A$ , then

$$\begin{aligned} s^2 &= 200 \\ s &= 10\sqrt{2} \\ &\approx 14.14 \text{ inches} \end{aligned}$$

Since the cross fills 36% of the flag, the black portion is 64% of the flag. Therefore, in the half-flag the black portion is 64% of 200 square inches, or 128 square inches.

In the figure above, the black portion is square. If we let  $b$  be the side of the black square, then

$$\begin{aligned} b^2 &= 128 \\ b &= 8\sqrt{2} \\ &\approx 11.31 \text{ inches} \end{aligned}$$

Now  $y$ , the side of the two small silver squares, must be

$$y = \frac{14.14 - 11.31}{2} \approx 1.42 \text{ inches.}$$

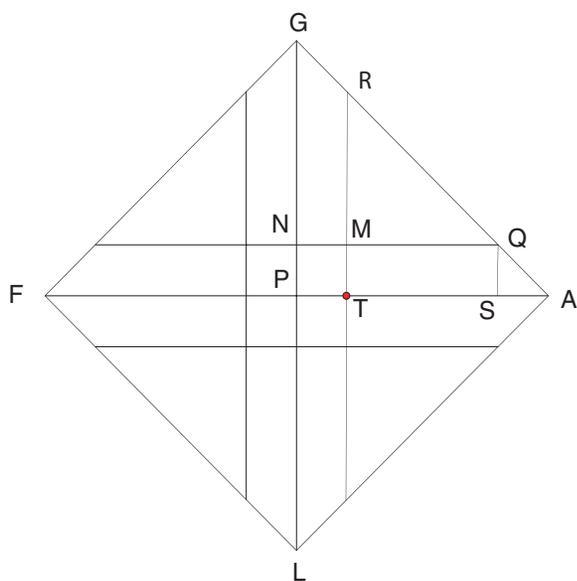
The area of the two small silver squares in the half-flag is  $A = 2(1.42)^2 \approx 4$  square inches.



The area of the red portion of the cross in the half-flag must be  $200 - 128 - 4 = 68$  square inches.

The color percents of the half-flag will be the same for the original flag. Therefore, 64% of the flag is to be black,  $\frac{68}{200} = 34\%$  is to be red, and  $\frac{4}{200} = 2\%$  is to be silver.

Another approach to solving this problem would be to recognize that the variable in this situation is the side of the silver square. The lengths may be expressed in terms of this variable.



$NMTQ$  is a square. Let the sides of this square be  $x$  units. Let  $\overline{QS}$  be perpendicular to  $\overline{AP}$ . Because parallel lines are equidistant,  $QS$  is also  $x$  units.  $SA$  would also be  $x$  units because  $QAS$  is an isosceles right triangle.  $MTSQ$  is a rectangle, so  $MQ = TS$ . As shown in the previous solution,  $AP$  is  $10\sqrt{2}$  units.  $MQ$  is  $10\sqrt{2} - 2x$ .

$RMQ$  is a right isosceles triangle with area

$$\frac{1}{2}(10\sqrt{2} - 2x)(10\sqrt{2} - 2x).$$

The area of the four black triangles is

$$A = 4 \cdot \frac{1}{2}(10\sqrt{2} - 2x)(10\sqrt{2} - 2x)$$

$$A = 2(10\sqrt{2} - 2x)^2.$$

As indicated earlier, since the cross fills 36% of the flag, the black portion is 64% of the flag. Therefore, the black

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

**Texas Assessment of Knowledge and Skills:**

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

IV. Planar Figures: Bayou City Lake

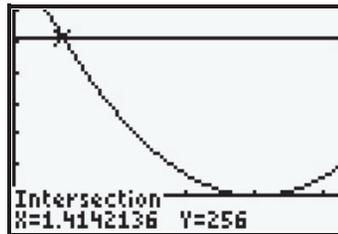
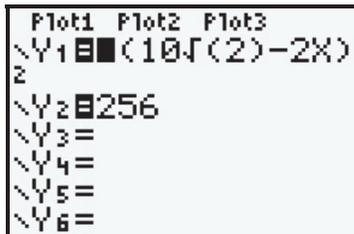


portion is 64% of 400 square inches, or 256 square inches.

$$A = 2(10\sqrt{2} - 2x)^2$$

$$256 = 2(10\sqrt{2} - 2x)^2$$

This equation may be solved using a graphing calculator. The area function may be graphed or a table of values may be found.



X	Y <sub>1</sub>	Y <sub>2</sub>
1.4	257.29	256
1.41	256.38	256
1.42	255.48	256
1.43	254.57	256
1.44	253.67	256
1.45	252.77	256
1.46	251.87	256

$X = 1.41$

$x$  is 1.41.

The area of the silver square is  $(2x)^2$ , or 7.9524 square inches.

The percent of the total area of 400 is  $\frac{7.9524}{400} \approx 0.02$  or 2% of the total area. The red area must be 36% minus 2%, or 34%.



### Extension Questions:

- What transformations could you use to rearrange the pieces of the flag to place same colored pieces together?

*Only horizontal and vertical translations of the triangular piece FPL were needed.*

*We translated a distance equal to half the diagonal of the square flag to the right and down.*

- What shapes were formed by the colored areas?

*The black region was a square formed by the 4 congruent isosceles right triangles. The silver pieces were two congruent squares. The red pieces were 4 congruent trapezoids with one leg being an altitude of the trapezoid. The red pieces could have been cut and rearranged into a rectangle, but that did not make the problem easier to solve.*

- What key ideas help you solve the problem?

*We worked with half of the flag. We were able to arrange it into a square, so all we needed was the formula for the area of a square to help us get the side length of that square and the black square. The rest was basically computation.*

- How could you extend this problem?

*Change the shape of the flag to be a non-square rectangle. Change the design on the flag. Look for patterns in the different flags.*



## Student Work

A copy of a student's work on this problem appears on the next page.

This work exemplifies many of the solution guide criteria. For example:

- Shows an understanding of the relationships among elements.

*The student recognized the right triangles and used the Pythagorean Theorem. He recognized that one of the black right triangles is one-fourth of the black area of the square. He recognized that the segment labeled  $x + y + z$  was the same as the side of the square—20 cm in length .*

- Demonstrates geometric concepts, process, and skills.

*The student used the Pythagorean Theorem correctly. He showed the algebraic processes necessary to solve for the lengths of the unknown segments.*

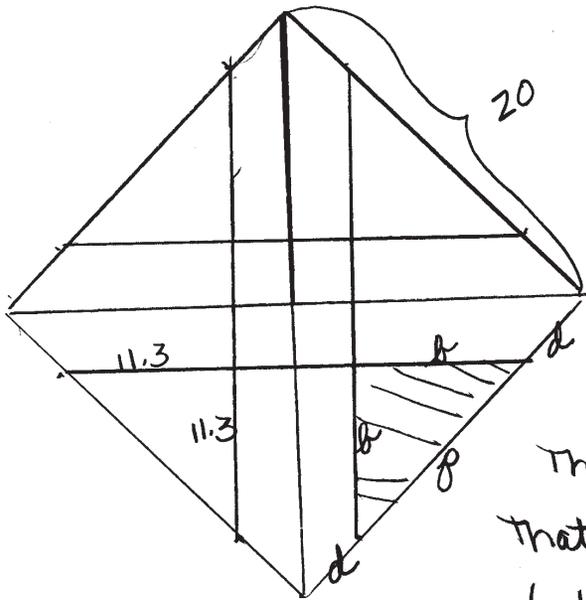
- Communicates clear, detailed, and organized solution strategy

*Overall, the solution is organized clearly. However, when setting up equations, there should have been more detailed explanation of how and why the equations were set up the way they were. For example, the student set up  $2x^2 = b^2$  but there is no explanation of why.*

This student was advised that his solution would have been more complete if he had justified his steps and indicated the reasons for the steps in his solution. A teacher might ask the student some of the following questions to clarify his thinking:

- How did you know that the triangle with legs labeled  $b$  had two congruent legs?
- Why does  $2b^2 = p^2$ ?
- How do you know that  $20 - p = 2d$ ?
- What justifies your statement that  $2x + y = 20$ ?





red 34%  
 silver 2%  
 black 64%

$$\frac{1}{4} \text{ of the square} = \frac{1}{4}(20)^2 = 100 \text{ m}^2$$

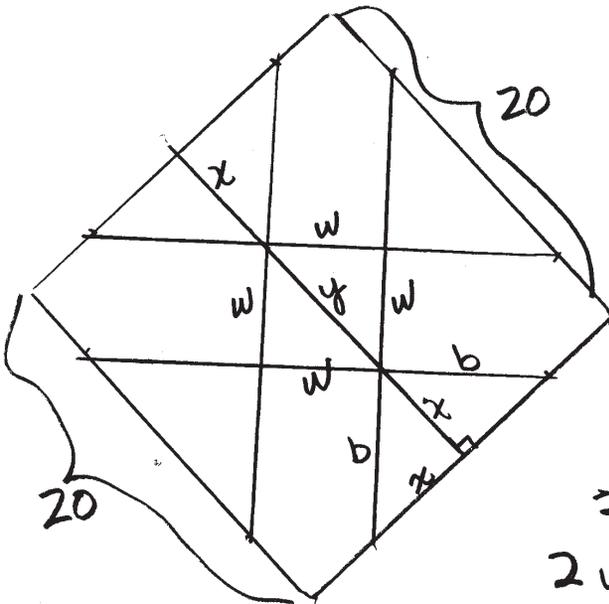
The black part is 64% of it or  $64 \text{ m}^2$ .  
 That is one of the black triangles

$$\frac{1}{2}bh = 64 \quad b = h$$

$$\frac{1}{2}b^2 = 64 \quad b^2 = 128 \quad b \approx 11.3$$

$$2b^2 = p^2 \quad 2(128) = p^2 \quad p^2 = 256 \quad p = 16$$

$$20 - p = 2d \quad 20 - 16 = 2d \quad 4 = 2d \quad 2 = d$$



$$2x^2 = b^2 \quad 2x^2 = 128$$

$$x^2 = 64 \quad x = 8$$

$$2x + y = 20$$

$$2(8) + y = 20$$

$$16 + y = 20$$

$$y = 4$$

$$2w^2 = 4^2$$

$$w^2 = 8$$

$$2w^2 = y^2$$

$$2w^2 = 16$$

$$w = \sqrt{8}$$

$$w^2 = \text{area of silver} = 8 \quad \frac{8}{400} = 2\%$$

$$100\% - (2\% + 64\%) = \text{area of red} = 100\% - 66\% = 34\%$$





## The Shortest Cable Line

Two houses are to be connected to cable TV by running cable lines from the houses to a common point of connection with the main cable. The main cable runs underground along the edge of the street the houses face. Without measuring, determine where that connection point should be placed to minimize the amount of cable run from the houses to the street. Explain your reasoning.



# Teacher Notes

**Materials:**

One ruler per student

Unlined paper and patty paper, and geometry software

**Connections to Geometry TEKS:**

(b.2) **Geometric structure.** The student analyzes geometric relationships in order to make and verify conjectures.

The student:

(B) makes and verifies conjectures about angles, lines, polygons, circles, and three-dimensional figures, choosing from a variety of approaches such as coordinate, transformational, or axiomatic.

(b.4) **Geometric structure.** The student uses a variety of representations to describe geometric relationships and solve problems.

The student:

selects an appropriate representation (concrete, pictorial, graphical, verbal, or symbolic) in order to solve problems.

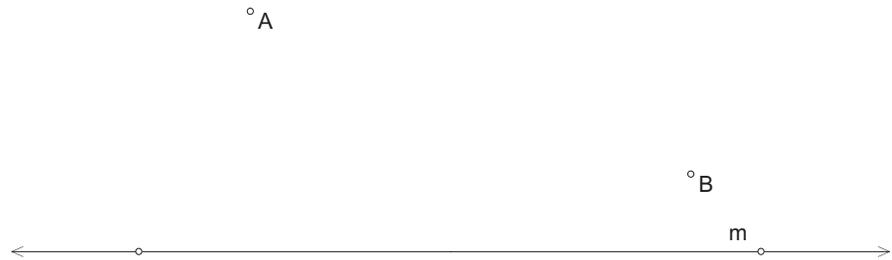
**Scaffolding Questions:**

- What path gives the shortest distance between two points?
- How can you position 3 points to minimize the distance between them?
- What transformations can you use to create a straight line path between point B and an image of point A?

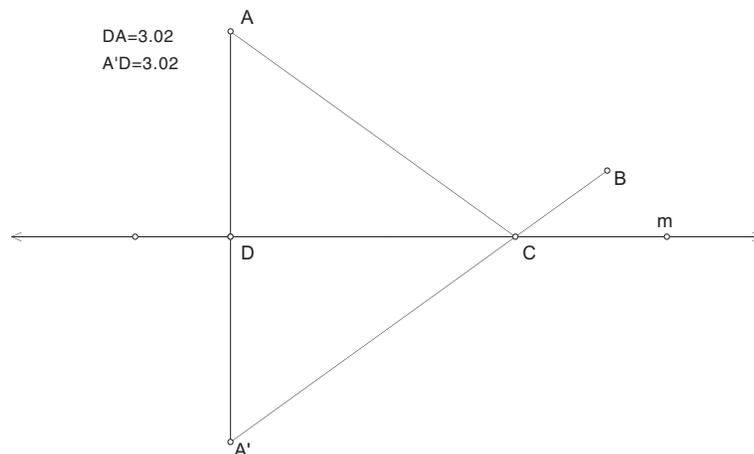
**Sample Solution:**

Let  $C$  be the connection point along the street to which the cable lines from points  $A$  and  $B$  will run. To minimize the amount of cable run from the houses to this point, we need to minimize  $AC + CB$ . To do this we use congruence transformations.

Let line  $m$  represent the main cable.



Reflect point  $A$  over line  $m$  to get its image,  $A'$ . Draw  $\overline{A'B}$  and label the point of intersection with line  $m$  as point  $C$ .



DA=3.02  
A'D=3.02



Since  $A'$ ,  $C$ , and  $B$  are collinear points,  $A'C + CB = A'B$  is the shortest distance between points  $A'$  and  $B$ .

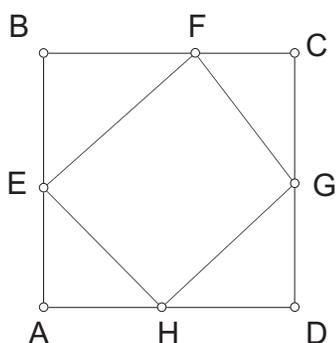
Reflecting point  $A$  over line  $m$  to get its image  $A'$  means that line  $m$  is the perpendicular bisector of  $\overline{AA'}$ , and any point on line  $m$  is equidistant from points  $A'$  and  $A$ . Therefore,

$$AC = A'C, \text{ and } A'B = AC + CB.$$

This shows that point  $C$  minimizes  $AC + CB$ .

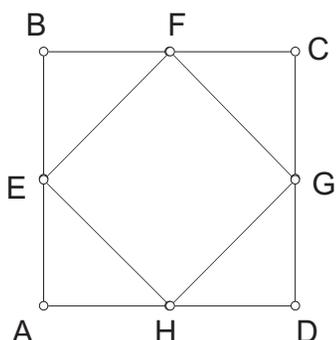
### Extension Questions:

- A square garden is to be divided up into 4 right triangular regions surrounding a quadrilateral region with a pathway (EFGH) surrounding the quadrilateral region. One vertex of the quadrilateral is the midpoint of a side of the square garden, as shown below:



$ABCD$  is a square, and  $E$  is the midpoint of  $\overline{AB}$ . Where should the other vertices of quadrilateral  $EFGH$  be to minimize its perimeter? Use congruence transformations (isometries) to investigate and explain your conclusion.

*First, consider making quadrilateral  $EFGH$  a square by placing  $F$ ,  $G$ , and  $H$  at the midpoints of the other three sides.*



(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem; and

(D) finds surface areas and volumes of prisms, pyramids, spheres, cones, and cylinders in problem situations

(e.3) **Congruence and the geometry of size.** The student applies the concept of congruence to justify properties of figures and solve problems.

The student:

(A) uses congruence transformations to make conjectures and justify properties of geometric figures; and

(B) justifies and applies triangle congruence relationships.

**Texas Assessment of Knowledge and Skills:**

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

II. Transformation: Reflections

**Student Response:**

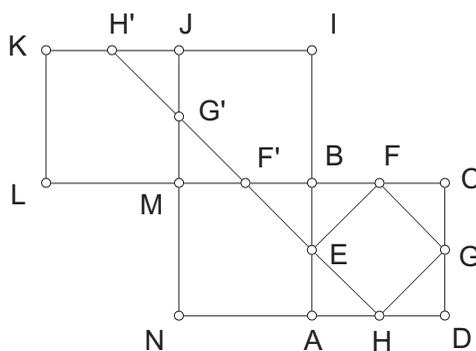
In order to solve this problem, you should use the minimal path conjecture. First, reflect one of the points across the cable line. Then connect the image that results to the other point's pre-image. Once you've found the intersecting point, you connect each pre-image to it, and the result is the minimal path.

Since  $EFGH$  is formed by connecting the midpoints of square  $ABCD$ , triangles  $EBF$ ,  $FCG$ ,  $GDH$ , and  $HAE$  are congruent isosceles right triangles.

This means that  $\overline{EF} \cong \overline{FG} \cong \overline{GH} \cong \overline{HE}$ , so that  $EFGH$  is a rhombus.

Since  $m\angle EFB = m\angle CFG = 45^\circ$ ,  $m\angle EFG = 180 - (45 + 45) = 90^\circ$ , so that  $EFGH$  is a square.

We use reflections to show that the path,  $EF + FG + GH + EH$ , is equal in length to a straight line path, which is the shortest distance between two points.



Reflect Square  $EFGH$  over line  $\overleftrightarrow{AB}$  to form  $\overline{F'E} \cong \overline{FE}$ . Reflect the newly positioned square,  $EFGH$ , over line  $\overleftrightarrow{BC}$  to form  $\overline{F'G'} \cong \overline{FG}$ . Finally, reflect this newly positioned square,  $EFGH$ , over line  $\overleftrightarrow{MJ}$  to form  $\overline{G'H'} \cong \overline{GH}$ .

Now  $H, E, F', G',$  and  $H'$  are collinear, and  $HE + EF' + F'G' + G'H'$ , which is the shortest distance between  $H$  and  $H'$ ,

is equal to  $HE + EF + FG + GH$ , the perimeter of the inner square.



# Chapter 7:

## *Similarity*





# Introduction

In this chapter the problems require the students to apply the properties of similarity to justify properties of figures and to solve problems using these properties.

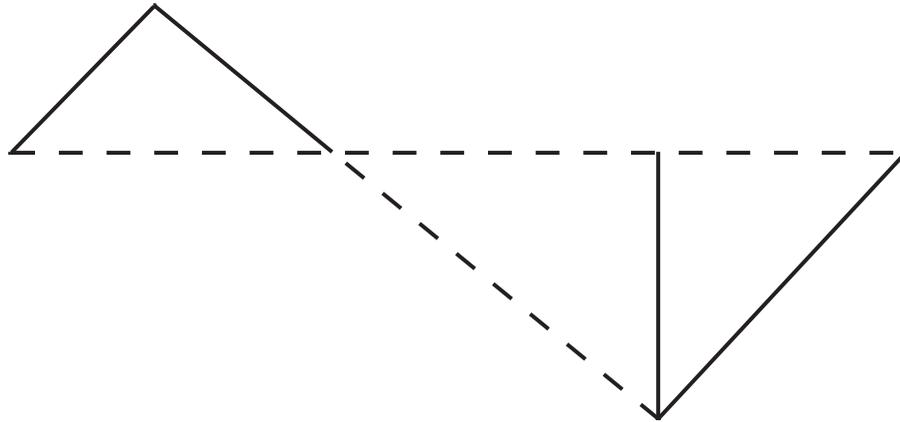
Geometry consists of the study of geometric figures of zero, one, two, and three dimensions and the relationships among them. Students study properties and relationships having to do with size, shape, location, direction, and orientation of these figures. (*Geometry, Basic Understandings, Texas Essential Knowledge and Skills*, Texas Education Agency, 1999.)



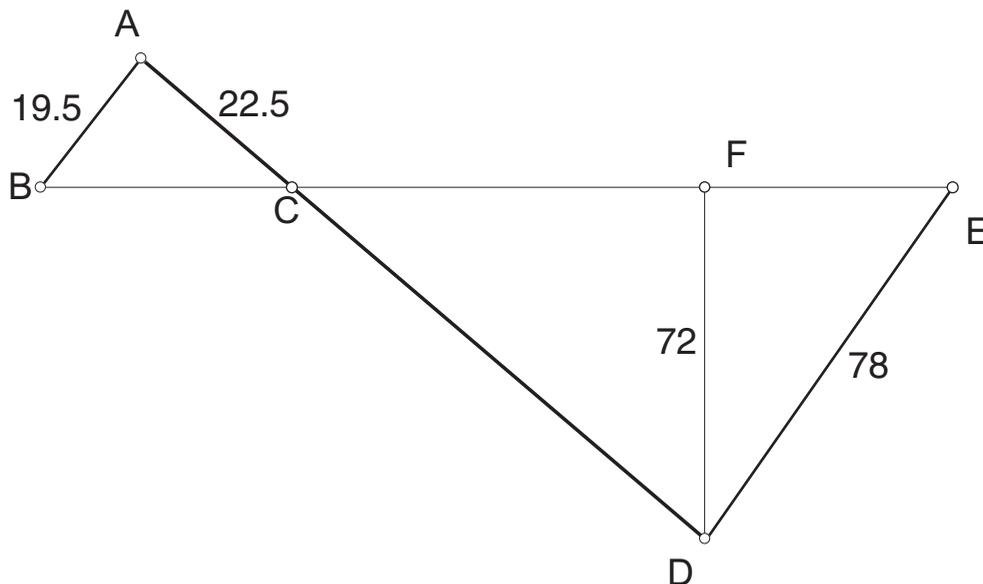


## Ancient Ruins

Archaeologists flying over a remote area in the interior of Mexico saw what appeared to be the ruins of an ancient ceremonial temple complex below. This diagram shows what they saw:



Based on their photographs they believed that some of the walls were still intact. These are drawn as solid segments, and the walls that had crumpled or had obviously been there are drawn as dashed segments. Before they can excavate the site they need to construct an accurate scale drawing or model. Based on their altitude flying over the ruins and the measurements made on the photograph, they generated the following drawing of the ruins:



The lengths, in feet, of the walls  $\overline{AB}$ ,  $\overline{AC}$ ,  $\overline{DE}$ , and  $\overline{DF}$  were 19.5, 22.5, 78 and 72, respectively.  $\overline{AB} \parallel \overline{DE}$  and  $\overline{DF} \perp \overline{CE}$ .

To plan for the excavation they need to know a number of things about the site.

1. The archaeologists estimate it will require 45 minutes to an hour to excavate each foot of the exterior walls of the temple site. Approximately how long will it take to complete this task? Explain in detail how you determined this.
2. The entrance into the ceremony preparation room appears to be along wall  $\overline{BC}$  and directly opposite vertex A. What would this mean geometrically? What is the distance from point A to the entrance? Were any of the walls of this room perpendicular to each other?
3. The archaeologists believe that about 9 square feet of space was needed in the main temple for each person. How many people could occupy the temple (triangle CDE)? How many priests and assistants could occupy what appears to be the ceremony preparation room (triangle ABC)?





**Materials:**

One calculator per student

Unlined paper and construction tools

**Connections to Geometry****TEKS:**

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

## Teacher Notes

**Scaffolding Questions:**

- What information do you need to determine in order to compute the perimeters?
- What appears to be true about triangles ABC and CDE?
- What special segment on a triangle is question 3 referring to?
- Can the Pythagorean Theorem or Pythagorean Triples help you find answers to the questions?
- How are the ratios of corresponding linear dimensions on similar triangles related to the ratios of their areas? The ratio of their volumes?

**Sample Solution:**

1. To find the perimeters the missing sides, segments  $\overline{BC}$ ,  $\overline{DC}$ , and  $\overline{EC}$ , need to be found. Triangles ABC and DEC appear to be similar triangles.

Segments AB and DE are parallel, so  $\angle ABC \cong \angle DEC$  because they are alternate interior angles on parallel lines. Because they are vertical angles,  $\angle ACB \cong \angle DCE$ .

Thus,  $\triangle ABC \approx \triangle DEC$  because two angles of one triangle are congruent to two angles of the other triangle. Similar triangles have proportional corresponding sides:

$$\frac{AB}{DE} = \frac{BC}{EC} = \frac{AC}{DC} = \frac{1}{4} \text{ since } AB = 19.5 \text{ and } DE = 78.$$

Use the proportion to get DC:  $\frac{AC}{DC} = \frac{22.5}{DC} = \frac{1}{4}$  so DC = 90.

To get EC, find FE and FC. Consider the right triangles DFE and DFC. Pythagorean Triples may be used.

Triangle DFE has DE = 78 = 6 times 13, and DF = 72 = 6 times 12, so FE = 6 times 5 = 30.

Triangle DFC has DF = 72 = 18 times 4, and DC = 90 = 18 times 5, so FC = 18 times 3 = 54.



Thus,  $EC = 30 + 54 = 84$ .

To get BC, use the proportion:  $\frac{BC}{EC} = \frac{BC}{84} = \frac{1}{4}$ , so  $BC = 21$ .

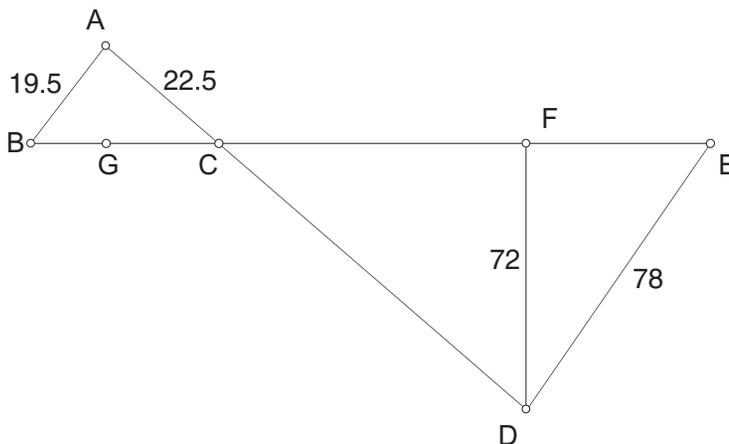
Perimeter of Triangle ABC =  $19.5 + 22.5 + 21 = 63$  feet.

Perimeter of Triangle DEC =  $78 + 84 + 90 = 252$  feet.

The time it will take to excavate the perimeters is  
 $(63 \text{ ft.} + 252 \text{ ft.}) \cdot (45 \text{ min. per foot}) = 14175 \text{ min.} = 236.25 \text{ hours}$   
 $(63 \text{ ft.} + 252 \text{ ft.}) \cdot (1 \text{ hour per foot}) = 315 \text{ hours}$

The excavation time will be between 236.25 hours and 315 hours.

2. Since the entrance along wall  $\overline{BC}$  is directly opposite vertex A, the distance from point A to the entrance should be along the perpendicular from A to the line containing  $\overline{BC}$ . Let point G be the entrance. Then segment  $\overline{AG}$  is the altitude to segment  $\overline{BC}$  (since triangle ABC is acute).



To find the length of this altitude, use the fact that the ratio of corresponding altitudes on similar triangles is equal to the ratio of the lengths of the corresponding sides, so

$$\frac{AG}{DF} = \frac{AC}{DC} = \frac{1}{4}, \text{ and } AG = 18 \text{ feet.}$$

No pair of walls in this room is perpendicular to each other since  $(19.5)^2 + (21)^2 \neq (22.5)^2$

3. Since  $\overline{DF} \perp \overline{EC}$ ,  $\overline{DF}$  is the altitude to  $\overline{EC}$ .

$$\text{Thus, the area of } \triangle DEC = \frac{1}{2}(72)(84) = 3024 \text{ square feet}$$

(2) uses ratios to solve problems involving similar figures;

(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples; and

(4) describes and applies the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

**Texas Assessment of Knowledge and Skills:**

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

**Connection to High School Geometry: Supporting TEKS and TAKS Institute:**

II. Transformations: Dilations



Dividing the area by 9 square feet of space per person shows that the temple could have housed 336 people.

To get the area of  $\triangle ABC$ , either use the triangle area formula or the property that the ratio of the areas of similar triangles is the square of the ratio of the corresponding sides:

$$\frac{\text{area of } \triangle ABC}{\text{area of } \triangle DEC} = \frac{A}{3024} = \left(\frac{1}{4}\right)^2.$$

Solving this for A shows the area to be 189 square feet. Dividing this by 9 square feet reveals that the ceremony preparation room could have housed 21 priests and assistants.

### Extension Questions:

- In the proposal that the archaeologists must write to excavate the site, they must compare the perimeters and the areas of the ceremony preparation room to those of the main temple. How could this be done?

*The ratio of the areas is the square of the ratio of the corresponding sides of the similar figures. Here is why:*

*On the first triangle let  $S_1$  and  $h_1$  be a side and the altitude to that side, respectively.*

*On the second triangle let  $S_2$  and  $h_2$  be the corresponding side and the altitude to those in the first triangle, respectively. Then  $\frac{S_2}{S_1} = \frac{h_2}{h_1} = r$  so that  $S_2 = rS_1$  and  $h_2 = rh_1$ .*

$$\text{Now } \frac{\text{area of triangle 2}}{\text{area of triangle 1}} = \frac{\frac{1}{2}h_2S_2}{\frac{1}{2}h_1S_1} = \frac{(rh_1)(rS_1)}{h_1S_1} = r^2,$$

*which is the square of the ratio of the sides.*

- It is believed that each of the two rooms were built in the shape of a triangular pyramid. What can be said about the capacity of the rooms? Is it possible to determine their capacity?

*If it is assumed that the triangular pyramids were similar figures, then the ratio of their volumes is the cube of the ratio of corresponding sides.*

*In this case, the ratio of the volumes would be  $\left(\frac{1}{4}\right)^3 = \frac{1}{64}$ .*

*To compute the volumes, compute the following:*

$$\frac{1}{3}(\text{altitude to pyramid base})(\text{base area}).$$



*Even though the ratio of the volumes is known, there is not enough information to find the volumes. Both base areas can be computed. The altitude of one of the triangular pyramids must be known to find the altitude of the other. If one of the altitudes were known, the volumes could be computed.*



## Student Work Sample

A student working individually did the work sample on the next page.

- Demonstrates geometric concepts, process and skills

*The student applied a ratio for similarity and the Pythagorean theorem to find the length of the missing sides in the problem. He also found correctly corresponding sides to set up equations for the ratio.*

Although the way the solution is organized is clear in both symbolic and verbal forms, it is still missing an explanation of why the student could apply certain mathematical principles in the situation. For example, the student used a ratio to find the length of CD without explaining why he could use a ratio. He should have explained that the triangles ABC and CDE are similar so that he can use a ratio to find the missing length.



## Ancient Ruins

$$1) \frac{19.5}{225} = \frac{78}{x}$$

$$19.5x = 1755$$

$$x = 90 = CD$$

$$90^2 - 72^2 = CF^2$$

$$8100 - 5184 = CF^2$$

$$\sqrt{2916} = 54 = CF$$

$$78^2 - 72^2 = FE^2$$

$$6084 - 5184 = FE^2$$

$$\sqrt{900} = 30 = FE$$

$$\frac{19.5}{x} = \frac{78}{84}$$

$$78x = 1638 \quad x = 21 = BC$$

$$\text{time} = 236 \text{ hrs } 15 \text{ min}$$

first we setup a ratio of  $\frac{AB}{AC} = \frac{ED}{CD}$   
 to find CD. Then we used pythagorean theorem  
 to find CF + FE to get CE. Next we set up the  
 proportion  $\frac{AB}{BC} = \frac{ED}{CE}$  to get 21. Finally we found the sum  
 of all sides that multiplied it by 215 min then  
 divided it by 60 and converted it into time.

2) That  $AX \perp BC$

$$\frac{19.5}{x} = \frac{78}{72}$$

$$78x = 1404$$

$$x = 18 = AX = BC$$

$$3) A = \frac{B \times H}{2} = \Delta CDE$$

$$A = 3024 = \Delta CDE$$

336 ft<sup>2</sup> per person in  $\Delta CDE$

$$A = \frac{b \times h}{2} = \Delta ABC$$

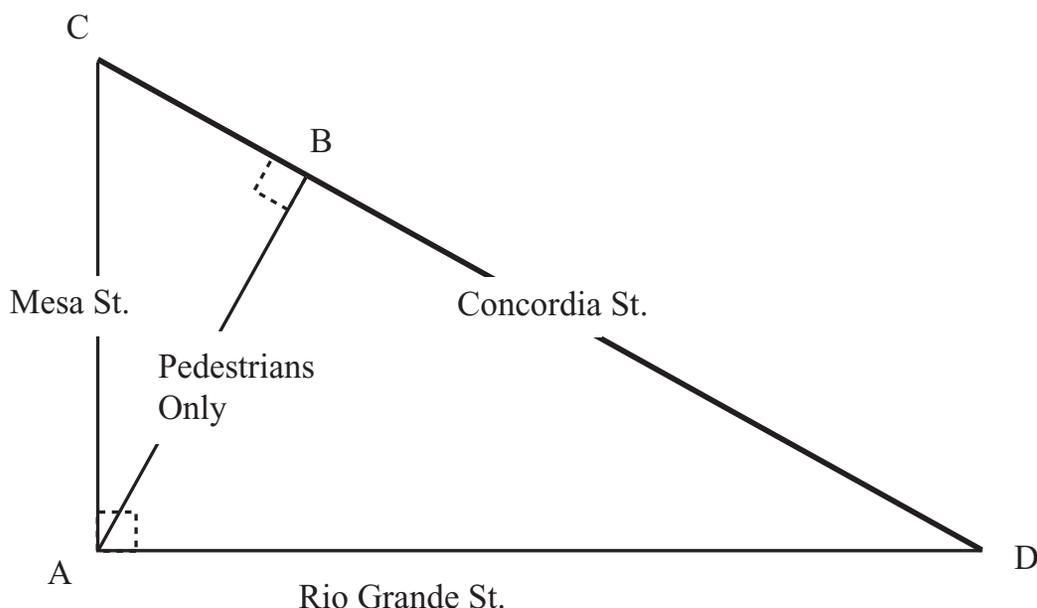
$$A = \frac{18 \times 21}{2} = 189 \text{ ft}^2$$

21 ft<sup>2</sup> per person in  $\Delta ABC$





## Sightseeing Walk



Yuma City has an historic region downtown between Mesa St., Rio Grande St., and Concordia St. Mesa and Rio Grande Streets intersect to form a right angle. There is a Pedestrians Only path from the intersection of Mesa and Rio Grande to Concordia that intersects Concordia at a right angle. A sightseer started at the intersection of Mesa and Rio Grande and walked the 6 blocks long path to Concordia. She then walked 4 blocks along Concordia to Mesa and back to the intersection of Mesa and Rio Grande. After that she walked to the intersection of Rio Grande and Concordia.

Answer the following questions, completely justifying your answers with geometric explanations.

1. How far did the sightseer walk?
2. If another sightseer had started at the intersection of Mesa and Concordia and walked along Concordia Street to the intersection of Concordia and Rio Grande, how far would he have walked?



**Materials:**

One ruler and compass per student

Unlined paper, patty paper, and geometry software

**Connections to Geometry****TEKS:**

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(1) uses numeric and geometric patterns to make generalizations about geometric properties, including properties of polygons, ratios in similar figures and solids, and angle relationships in polygons and circles;

(2) uses properties of transformations and their compositions to make connections between mathematics and the real world in applications such as tessellations or fractals; and

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

(2) uses ratios to solve problems involving similar figures;

## Teacher Notes

**Scaffolding Questions:**

- What segments represent the paths taken by the sightseer?
- What are the known distances?
- What are the unknown distances?
- What types of triangles do you see in this problem?
- What relationships about this/these triangle types can help you find unknown quantities?

**Sample Solutions:**

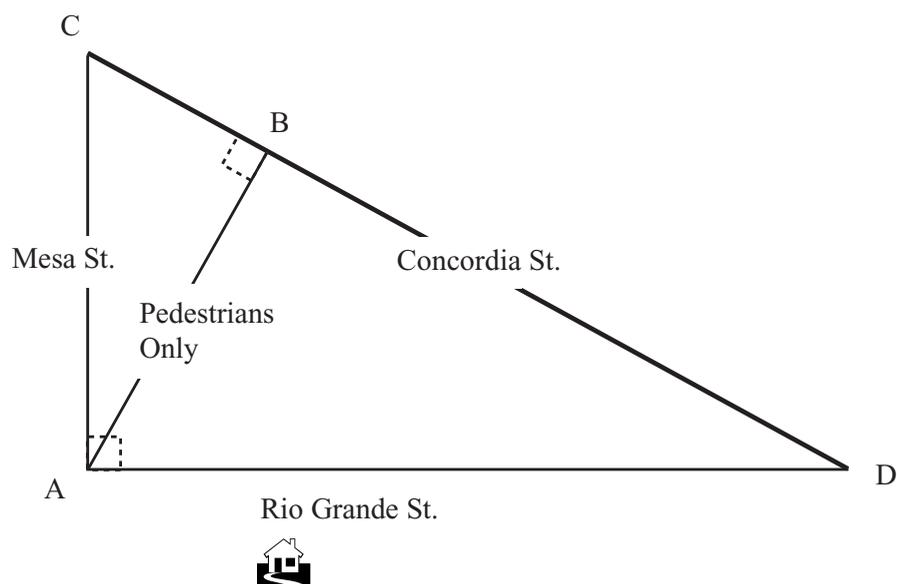
1. The distance the sightseer walks is given by  $AB + BC + CA + AD$ .

We know that  $AB = 6$  and  $BC = 4$ . To find  $CA$  we apply the Pythagorean Formula to right triangle  $ABC$ .

$$CA = \sqrt{AB^2 + BC^2} = \sqrt{36 + 16} = \sqrt{52} \approx 7.21 \text{ blocks}$$

There are three right triangles:  $\triangle CAD$ ,  $\triangle CBA$ , and  $\triangle ABD$ , with right angles  $\angle CAD$ ,  $\angle CBA$ , and  $\angle ABD$ .

In the large right triangle  $CAD$ , let the measure of acute angle  $C$  be  $x$  degrees. Then, since the acute angles of a right triangle are complementary, the measure of angle  $D$  is  $90 - x$  degrees.



In triangle CBA, the measure of angle C is  $x$  degrees, so the measure of angle CAB is  $90 - x$  degrees.

In triangle ABD, the measure of angle D is  $90 - x$  degrees, so the measure of angle BAD is  $x$  degrees

Now we have  $\triangle CAD \sim \triangle CBA \sim \triangle ABD$ , because if three angles of one triangle are congruent to the corresponding three angles of another triangle, the triangles are similar. (Angle-Angle-Angle similarity)

Using triangles CBA and ABD, we have

$$\frac{CB}{AB} = \frac{CA}{AD}$$

$$\frac{4}{6} = \frac{\sqrt{52}}{AD}$$

$$AD = \frac{6\sqrt{52}}{4}$$

$$AD \approx 10.82 \text{ blocks}$$

The sightseer walked  $AB + BC + CA + AD = 6 + 4 + 7.21 + 10.82 = 28.03 \approx 28$  blocks.

2. The distance from the intersection of Mesa and Concordia to the intersection of Concordia and Rio Grande is given by

$$CD = CB + BD = 4 + BD.$$

To find BD we can use the same similar triangles as in Problem 1:

$$\frac{CB}{AB} = \frac{AB}{BD}$$

$$\frac{4}{6} = \frac{6}{BD}$$

$$BD = 9 \text{ blocks}$$

The distance from Mesa and Concordia to Concordia and Rio Grande is 4 plus 9 or 13 blocks.

(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples; and

(4) describes and applies the effect on perimeter, area, and volume when length, width, or height of a three-dimensional solid is changed and applies this idea in solving problems.

#### Texas Assessment of Knowledge and Skills:

Objective 6: The student will demonstrate an understanding of geometric relationships and spatial reasoning.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

#### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

III. Triangles: Pythagorean Theorem

#### Teacher's Comment:

"It surprised me how hard they worked on this problem, because they are usually difficult to keep on task. I was pleased with how quickly they recognized similar triangles and how well they set up the proportions because the concept was still new to them. I was disappointed in their written explanations because they were usually incomplete although they understood the problem. The instructional strategies I will modify to improve student success is that I would insist on better written explanations for all problems."



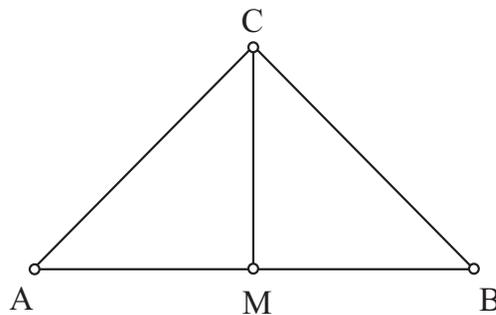
### Extension Questions:

- What important geometric concepts did you consider in this problem?

*To find the missing distances along the sightseer's route, we showed we had 3 similar right triangles. Using the resulting proportions we were able to find all missing dimensions along the streets.*

- On the other side of Concordia St. is an old cemetery in which a famous sheriff, Mo Robbins, is believed to be buried. The cemetery forms an isosceles right triangle with legs 10 blocks long. The hypotenuse of the triangular cemetery runs along Concordia St. Sheriff Robbins' gravesite is believed to be located at the centroid of the triangular cemetery, but there is no tombstone to mark his grave. How would you locate the gravesite of this famous person?

Cemetery:



*The centroid is the intersection of the medians. Its distance from each vertex is two-thirds the length of that median. Since triangle ACB is an isosceles right triangle with right angle C, the median from vertex C is the median to the hypotenuse. Triangle CMB is also a right isosceles triangle.  $CM = MB$ . Therefore, the median CM is half the length of the hypotenuse, AB.*

*Because the legs of isosceles right triangle ACB are 10 blocks long,*

$$AB = 10\sqrt{2} \text{ and } CM = \frac{1}{2}AB = 5\sqrt{2}.$$

*The centroid is located  $\frac{2}{3}(5\sqrt{2}) = 4.71$  blocks along segment CM from point C.*

- Suppose the triangular region formed by the streets were an equilateral triangle and the distance between any two intersections were 10 blocks. How will your answers to Problem 1 change?

*Since the triangular region formed by the streets is equilateral, the pathway, segment  $\overline{AB}$ , is an altitude and divides the region into two 30-60-90 triangles. This is because an altitude on an equilateral triangle is also an angle bisector and a median.*

*In 30-60-90 triangle ACB, the hypotenuse  $AC = 10$ , so the shorter leg  $CB = 5$  and the longer leg  $AB = 5\sqrt{3} \approx 8.66$  blocks. Also  $AC = DA$ .*

*The sightseer walked  $AB + BC + AC + AD = 8.66 + 5 + 10 + 10 = 33.66$  blocks.*





## Student Work Sample

An individual student produced the work on the next page. The work exemplifies the following criteria:

- Uses geometric and other mathematical principles to justify the reasoning used to analyze the problem

*The student applied theorems and corollaries correctly to get answers in that she substituted the correct numbers in the equations provided by corollaries. The student referenced specific corollaries from the book, but she did not write what they meant. The student also provided a good combination of verbal explanation and symbolic process.*

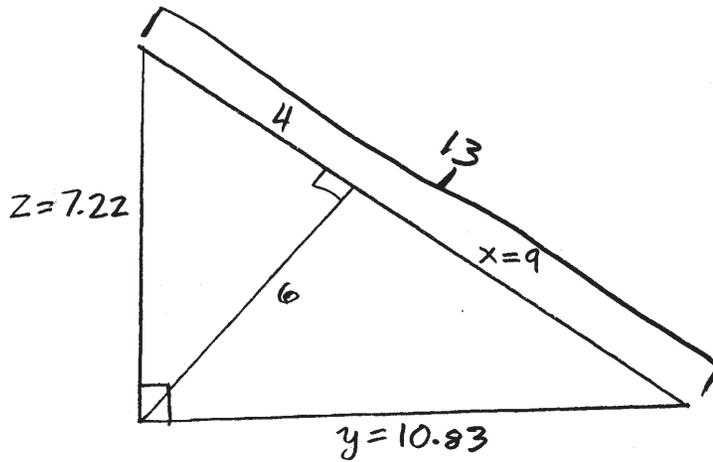
The following criteria was not met:

- Uses appropriate terminology and notation

*The student inappropriately used equality signs and used  $x$  to represent two different things (one for the altitude and the other for the missing part of hypotenuse).*



# Sight Seeing Walk



① I used corollary theorem 10.2, which states, if the altitude is drawn to the hypotenuse of the right triangle, then its (altitude) length is the geometric mean of the length of two segments of the hypotenuse.  $\frac{\text{one side}}{x} = \frac{x}{\text{2nd side}}$ , this allowed me to swap it and find  $x$  (one of the segments of the hypotenuse).  $\frac{4}{6} = \frac{6}{x} = 36 = 4x = \boxed{9=x}$

I also used the pythagorean theorem, which states if a triangle is a right triangle with legs of length  $A+B$  and hypotenuse of length  $C$ , then  $a^2 + b^2 = c^2$

$$6^2 + 4^2 = c^2$$

$$36 + 16 = c^2$$

$$\boxed{z = 7.22}$$

$$\sqrt{52} = c$$

$$7.22 = c$$

I used corollary 10.3, which states, the leg of a right triangle is the geometric mean of the total and adjacent part of the hypotenuse

$$\frac{\text{tot hyp}}{\text{leg}} = \frac{\text{leg}}{\text{adjacent segment of hypotenuse}}$$

$$\frac{13}{y} = \frac{4}{9} = \frac{y^2}{117}$$

$$\sqrt{y^2} = \sqrt{117}$$

$$y = 10.83$$

I added the lengths of Mesa St. + Concord Street + Pedestrian + Rio Grande to get the distance she walked = 29.05 blocks

② I added the lengths of the two segments of Concordia St. that are divided by Pedestrian to get the total length of Concordia St.

$$\begin{array}{r} 9 \\ + 4 \\ \hline 13 \end{array}$$

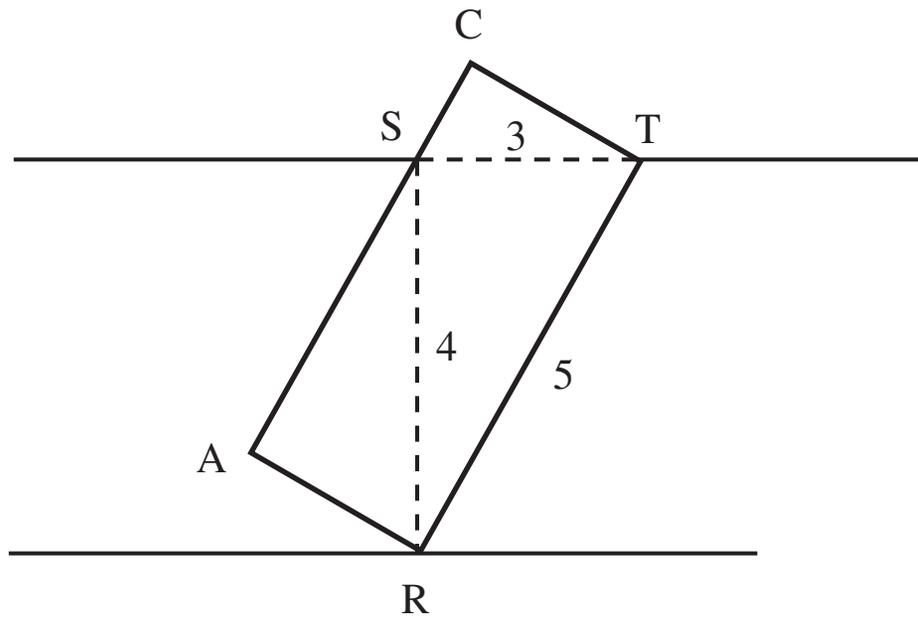
13 blocks





## Will It Fit?

Rectangular cartons that are 5 feet long need to be placed in a storeroom that is located at the end of a hallway. The walls of the hallway are parallel. The door into the hallway is 3 feet wide and the width of the hallway is 4 feet. The cartons must be carried face up. They may not be tilted. Investigate the width and carton top area that will fit through the doorway.



## Teacher Notes

### Materials:

One calculator per student

### Connections to Geometry

#### TEKS :

(c) **Geometric patterns.** The student identifies, analyzes, and describes patterns that emerge from two- and three-dimensional geometric figures.

The student:

(3) identifies and applies patterns from right triangles to solve problems, including special right triangles (45-45-90 and 30-60-90) and triangles whose sides are Pythagorean triples.

(e.1) **Congruence and the geometry of size.** The student extends measurement concepts to find area, perimeter, and volume in problem situations.

The student:

(C) develops, extends, and uses the Pythagorean Theorem; and

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

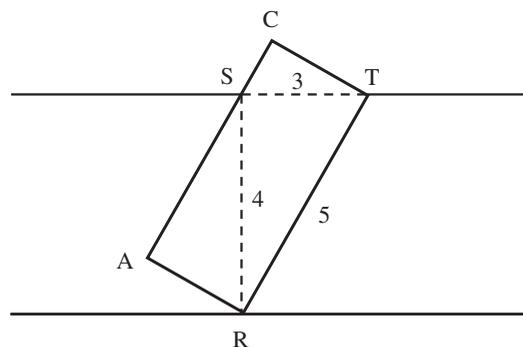
(2) uses ratios to solve problems involving similar figures;

(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples.

### Scaffolding Questions:

- What segments represent the width of the trunk?
- How does knowing the dimensions of triangle TSR help you?
- What triangles involve the width of the trunk?
- What triangles involve the width of the hallway?
- Explain any angle relationships in the triangles.
- How are the triangles you see related? How does this help you?
- How can you use your solution to Problem 1 to help you with Problem 2?

### Sample Solution:



Triangle SRT is a right triangle because it has three sides that are Pythagorean Triples. That is, the sum of the squares of two sides is the square of the third side.  $3^2 + 4^2 = 5^2$

Triangle SCT is a right triangle because  $\overline{AC} \perp \overline{CT}$ .

$\overline{AC} \parallel \overline{RT}$  so the alternate interior angles are congruent.

$\angle RSA \cong \angle TRS$ .

$\angle TRS$  and  $\angle STR$  are complementary angles because they are the two acute angles of a right triangle SRT.

$\angle CTS$  and  $\angle STR$  are complementary angles because the figure CTRA is a rectangle with right angle CTR.

$\angle CTS \cong \angle TRS$  because they are two angles that are



complementary to the same angle.

Thus, the three right triangles,  $\triangle CTS$ ,  $\triangle SRT$ , and  $\triangle ASR$ , have congruent acute angles:  $\angle CTS \cong \angle TRS \cong \angle RSA$ .

Therefore,  $\triangle CTS \approx \triangle SRT \approx \triangle ASR$  because they are right triangles with acute angles that are congruent.

Now write the proportion between the corresponding sides:

$$\frac{CT}{SR} = \frac{ST}{TR}$$

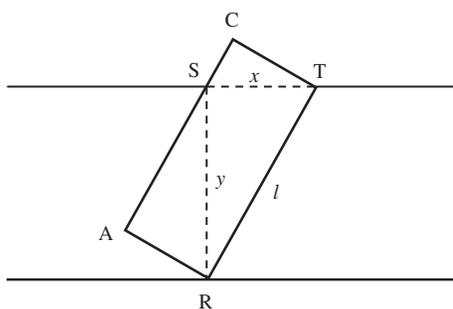
$$\frac{CT}{4} = \frac{3}{5}$$

$$CT = 2.4$$

The widest carton that will fit through the opening has a width of 2.4 feet, and the top surface area is 2.4 feet times 5 feet = 12 square feet.

### Extension Questions:

- Generalize your results for Problem 1 for a hallway opening of  $x$  feet and a hallway width of  $y$  feet if the maximum carton dimensions are  $l$  feet length and  $x^2 + y^2 = l^2$ .



Because  $x^2 + y^2 = l^2$ , the triangle STR is a right triangle.

$$\frac{CT}{SR} = \frac{ST}{TR}$$

Let  $CT = w$ .

$$\frac{w}{y} = \frac{x}{l}$$

$$w = \frac{xy}{l}$$

### Texas Assessment of Knowledge and Skills:

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

III. Triangles: Pythagorean Theorem



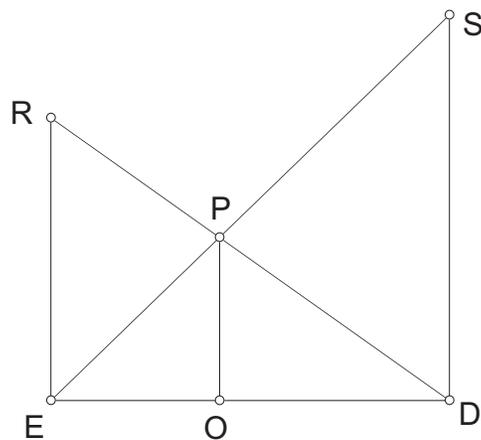
*This shows that the maximum carton width is the product of the door width and the hallway width divided by the carton length. The maximum carton surface is the width,  $w$ , times the length,  $l$ , or  $w \cdot l = \frac{xy}{l} \cdot l = xy$ . The maximum carton area is the product of the door width and the hallway width.*



## Spotlights

1. The spotlight display for an outdoor rock concert is being planned. At the sides of the stage a red spotlight is mounted on a pole 18 feet high, and a green spotlight is mounted on a pole 27 feet high. The light from each spotlight must hit the base of the other pole as shown in the diagram. How high above the ground should the stage be so that the spotlights meet and highlight the upper body of a performer who is about 6 feet tall?

The red spotlight is at point  $R$ . The green spotlight is at point  $S$ . Ground level is segment  $\overline{ED}$ .



2. If the poles are 30 feet apart and the stage is 20 feet long, how should the stage be positioned so that the spotlights meet on the upper body of the performer when he is center stage?
3. Generalize your results in Problem 1 if the poles for the spotlights are  $a$  feet and  $b$  feet long.



## Teacher Notes

### Materials:

One ruler per student

Unlined paper and geometry software

### Connections to Geometry

#### TEKS:

(d.2) **Dimensionality and the geometry of location.** The student understands that coordinate systems provide convenient and efficient ways of representing geometric figures and uses them accordingly.

The student:

(A) uses one- and two-dimensional coordinate systems to represent points, lines, line segments, and figures;

(B) uses slopes and equations of lines to investigate geometric relationships, including parallel lines, perpendicular lines, and special segments of triangles and other polygons;

(f) **Similarity and the geometry of shape.** The student applies the concept of similarity to justify properties of figures and solve problems.

The student:

(1) uses similarity properties and transformations to explore and justify conjectures about geometric figures;

(2) uses ratios to solve problems involving similar figures;

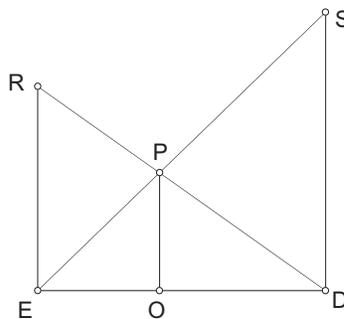
(3) in a variety of ways, the student, develops, applies, and justifies triangle similarity relationships, such as right triangle ratios, trigonometric ratios, and Pythagorean triples; and

### Scaffolding Questions:

- In what figures do you see segment  $\overline{PO}$  as a side or special segment?
- Since  $\overline{PO}$  is perpendicular to segment  $\overline{ED}$ , what does this tell you about  $\overline{PO}$  and triangle EPD? Does this help? Why or why not?
- Can you find similar triangles involving segment  $\overline{PO}$ ?
- What proportions can you write involving  $\overline{PO}$ ?
- How can you use algebraic notation to make your explanation easier to follow?
- Experiment with equivalent ways of writing a proportion to discover a way to solve for the length of  $\overline{PO}$ .
- Does the distance between the light poles seem to matter?
- How can you generalize from 18 feet and 27 feet long poles to poles of arbitrary lengths,  $a$  feet and  $b$  feet?

### Sample Solutions:

1.



Let  $h = PO$ , which is the height at which the spotlight beams meet.

We know that  $RE = 18$  and  $SD = 27$ . We can use two pairs of similar triangles to compare  $h$  to the lengths of the poles.

$\triangle EOP \sim \triangle EDS$  by Angle-Angle-Angle similarity. The triangles have right angles EOP and EDS, and they share the common angle,  $\angle PEO$ .



Similarly,  $\triangle DOP \sim \triangle DER$ .

Let  $EO = x$  and  $OD = y$ . Consider the following proportions that we get from the similar triangles:

$$1. \quad \frac{PO}{SD} = \frac{EO}{ED} \Rightarrow \frac{h}{27} = \frac{x}{x+y}$$

$$2. \quad \frac{PO}{RE} = \frac{OD}{ED} \Rightarrow \frac{h}{18} = \frac{y}{x+y}$$

Experimenting with properties of proportions gives an equation with  $h$  and the pole heights:

$$1. \quad \frac{x}{h} = \frac{x+y}{27}$$

$$2. \quad \frac{y}{h} = \frac{x+y}{18}$$

Adding these equations results in the equation

$$\begin{aligned} \frac{x}{h} + \frac{y}{h} &= \frac{x+y}{27} + \frac{x+y}{18} \\ \frac{x+y}{h} &= \frac{x+y}{27} + \frac{x+y}{18} \\ (x+y)\frac{1}{h} &= (x+y)\left(\frac{1}{27} + \frac{1}{18}\right) \\ \frac{1}{h} &= \frac{1}{27} + \frac{1}{18} = \frac{18+27}{(27)(18)} = \frac{45}{(27)(18)} \end{aligned}$$

Take the reciprocal of both sides.

$$h = \frac{(27)(18)}{45} = \frac{54}{5} = 10.8 \text{ feet}$$

This shows that the spotlight beams will meet at a point 10.8 feet above the ground.

Therefore, if the performer is about 6 feet tall, the stage should be 4.8 feet off the ground.

### Texas Assessment of Knowledge and Skills:

Objective 7: The student will demonstrate an understanding of two- and three-dimensional representations of geometric relationships and shapes.

Objective 8: The student will demonstrate an understanding of the concepts and uses of measurement and similarity.

### Connection to High School Geometry: Supporting TEKS and TAKS Institute:

II. Transformations: Dilations



It also shows that the distance between the poles,  $x + y$ , does not matter. This distance was not needed to determine  $h = 10.8$  feet.

2. The spotlight poles are to be 30 feet apart, so  $x + y = 30$ . The point at which the spotlights meet must be determined. In other words, find the value of  $x$  or  $y$ .

$$\begin{aligned}\frac{x}{x+y} &= \frac{h}{27} \\ \frac{x}{30} &= \frac{10.8}{27} \\ x &= 10.8 \cdot \frac{30}{27} \\ x &= 12 \text{ feet}\end{aligned}$$

The spotlights meet 12 feet in from the red spotlight and, therefore, 18 feet in from the green spotlight. We want this point to correspond to the center of the 20-foot long stage.

We need 10 feet of the stage to the left of this point and 10 feet of this stage to the right of this point. This means the pole for the red spotlight should be 2 feet from the left edge of the stage, and the pole for the green spotlight should be 8 feet from the right edge of the stage.

3. To generalize our results for spotlight poles of lengths  $a$  feet and  $b$  feet, we would have the same pairs of similar triangles and the same proportions. We simply need to replace 18 feet with  $a$  feet and 27 feet with  $b$  feet.

$$\begin{aligned}\frac{1}{h} &= \frac{1}{b} + \frac{1}{a} \\ \frac{1}{h} &= \frac{a+b}{ab} \\ h &= \frac{ab}{a+b}\end{aligned}$$



### Extension Questions:

- What are the key concepts needed to solve Problem 1?

We needed to relate the height at which the spotlight beams meet to the lengths of the poles. Segment  $\overline{PO}$  is the altitude from point  $P$  to  $\overline{ED}$  in triangle  $EPD$ . This does not help since we know nothing about this triangle. By the same reasoning, it does not help to consider  $\overline{PO}$  as a leg of right triangles  $EOP$  or  $POD$ .

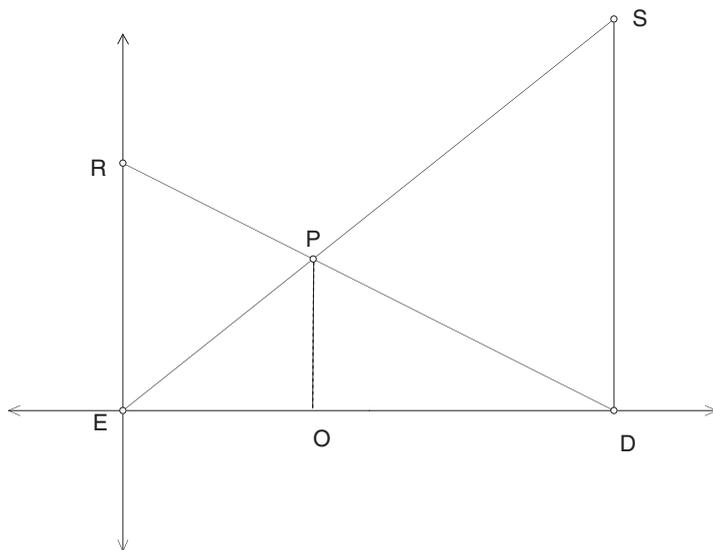
The big idea to use is similar triangles so we can write proportions involving  $\overline{PO}$  and the lengths of the light poles.

- Why doesn't the distance between the poles matter?

Segment  $\overline{PO}$  divides segment  $\overline{ED}$  into two pieces: segments  $\overline{EO}$  and  $\overline{OD}$ . When we solve the proportions for  $PO$ , the sum of these unknown lengths is a factor on both sides of the equation.

- How could you use algebra and a coordinate representation to solve this problem?

The coordinates of the base of the stage can be  $(0,0)$  and  $(30,0)$ . The red spotlight will be located at  $(0,18)$ , and the green spotlight will be located at  $(30,27)$ . The diagram below shows this.



The slope of segment  $\overline{RD}$  is  $\frac{-18}{30} = \frac{-3}{5}$ , and its  $y$ -intercept is 18.

The slope of segment  $\overline{ES}$  is  $\frac{27}{30} = \frac{9}{10}$ , and its  $y$ -intercept is 0.

The equations of these two segments are  $y = \frac{-3}{5}x + 18$  and  $y = \frac{9}{10}x$ .

Solve this system to get  $x = 12$ . Substitute for  $x$  in either equation and solve to get  $y = 10.8$ .





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