1.1 Quadratic Relationships

Overview: Participants use lists to develop a quadratic function representing the volume of a sandbox with a fixed depth. Using the quadratic function, participants solve quadratic equations numerically and graphically.

Objective: **Algebra I TEKS**
(d.2) The student understands there is more than one way to solve a quadratic equation and solves them using appropriate methods.
(d.2.A) The student solves quadratic equations using concrete models, tables, graphs, and algebraic methods.
(d.2.B) The student relates the solutions of quadratic equations to the roots of their functions.

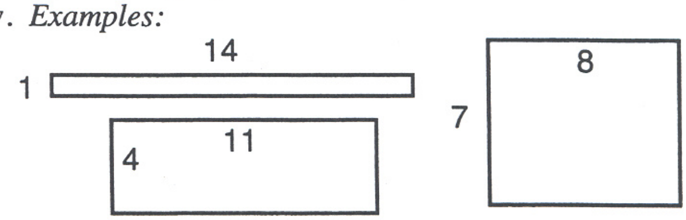
Terms: quadratic function, zero of a function, root of a function, solution of an equation

Materials: graphing calculators, pieces of lumber or cardboard to simulate lumber

Procedures: Participants should be seated at tables in groups of 3 – 4.

Activity 1: Building a Sandbox
Work through the activity with participants, using the overhead graphing calculator to demonstrate. Begin by discussing the situation of building a rectangular sandbox. Use the 1 foot wide lumber or cardboard to simulate a sandbox.

1. Have participants roughly sketch some possible sandboxes from a bird's eye view. *Examples:*



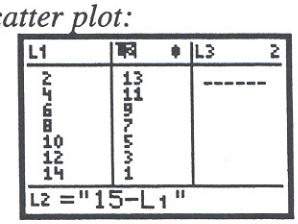
- What is fixed in this situation? [Two things are fixed. The depth of the sandbox is 1 foot deep and the perimeter of the sandbox is 30.]
- How will you fill in the depth column? [The depth is fixed. It will always be 1 foot.}
- If the perimeter is 30, what kind of widths make sense for the situation? [Widths ranging from more than 0 feet to less than 15 feet.]
- How does the length relate to the width? [The length is always $15 - width$.]
- How does the volume relate to the width and length? [The volume is the product of the width, length, and depth, $v = w * l * d$. In this case since the depth is always 1, $v = w * l * 1 = w * l$]

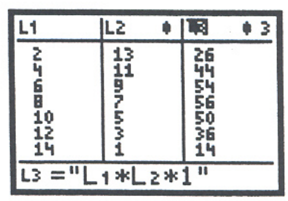
2. *Sample dimensions*:

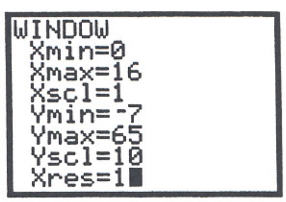
Width	Length	Depth	Volume
2	13	1	26
4	11	1	44
6	9	1	54
8	7	1	56
10	5	1	50
12	3	1	36
14	1	1	14

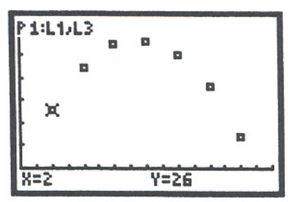
3. Encourage participants to predict the general shape of the graph.
4. Use the following to help participants enter the data into lists.
   Ask participants to put the table values for width into a list in the calculator.
   - How is the length related to the width? [The sum of the width and the length is always 15.]
   - How can you use an expression for the length in terms of the width to fill in the lengths into the list in your calculator. [15 – *list* 1.]
   - What is the depth of each sandbox? [Depth is always fixed at 1 foot.]
   - What expression can you use for volume? [(*list* 1)(*list* 2) • 1.]

   Sample scatter plot:





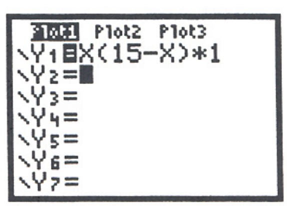


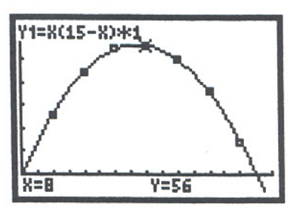


Note: Some calculators allow you to name lists. For this situation, you could name lists WIDTH, LENTH, and VOLUM.

5. (*list* 1)(*list* 2) • 1

6. *Sample:*





Extension: Ask participants to predict how the situation and the graph would change if the depth of the sandbox is 1.5 feet instead of 1 foot. Change the volume function to be $V = x(15 - x) * 1.5$ and graph. How would the situation and the graph change if the depth of the sandbox is 0.75 feet. Again change the volume function to be $V = x(15 - x) * 0.75$ and graph. This is an introduction to transformations, which will be explored in depth in 1.2 Transformations.

Note: The purpose of the following questions, solving quadratic equations, is to familiarize participants with the different types of equations that arise from quadratic functions. One of the common struggles that students have is differentiating between questions that ask for an input value and questions that ask for an output value. We use some non-algebraic solution methods here to introduce quadratic equations. We want to build confidence in reading questions and solving equations with power of technology and students will then be able to solve symbolically with more understanding.

Briefly discuss each of the solution methods shown below, emphasizing the power of multiple representations in promoting understanding.

7. Solution:
$V = 38.1875 ft^3$

Graph:

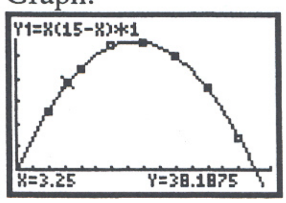
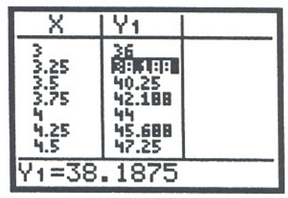
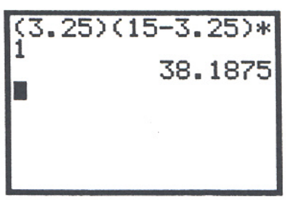


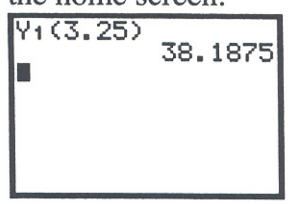
Table:



Home screen:

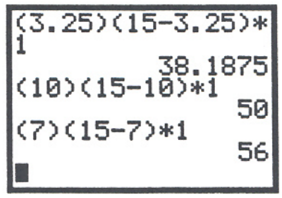


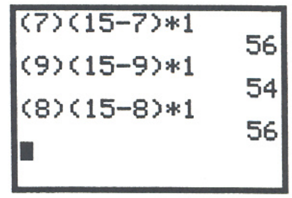
Using function notation on the home screen:



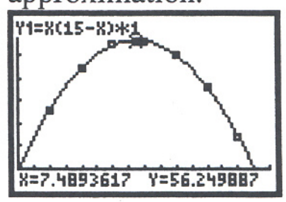
- Does this question give an input value and ask for an output value or does the question give an output value and ask for an input value? [The question gives an input value, 3.25 feet, and asks for an output value, the volume. You could also use the terms *domain* and *range* in this question and answer.]

8. Solution: $width = 7.5 ft$
   You can do some work on the home screen to get a feel for the answer:

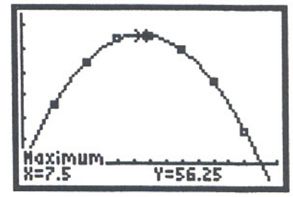




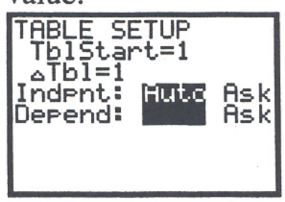
Graph, using trace to get an approximation:

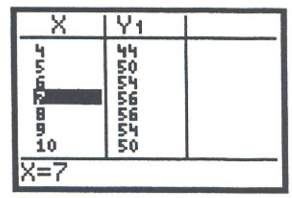


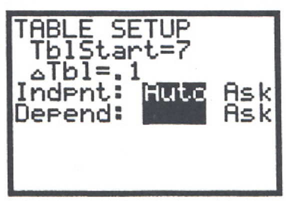
Graph, using the calculator to find the maximum value:

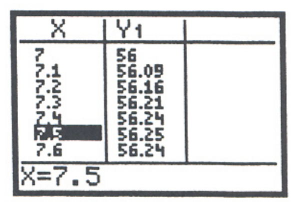


Table, looking for the *x*-value that yields the highest *y*-value:





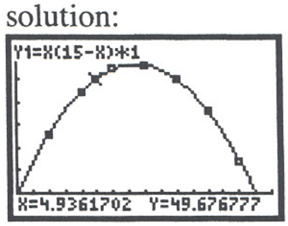




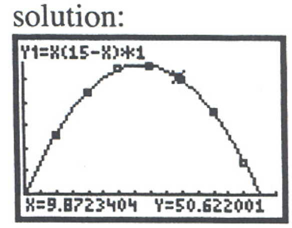
- Does this question give an input value and ask for an output value or does the question give an output value and ask for an input value? [The question gives an output value, the maximum volume, and asks for an input value, the width.]
- Where do we usually see a maximum question like this? [Traditionally, maximum and minimum problems have usually been reserved for calculus, but can readily be examined using technology in earlier courses. This kind of question naturally arises when studying quadratic functions.]

9. Solution:
$width = 5\,ft, 10\,ft$

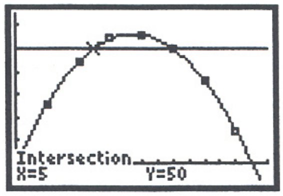
Graph, using trace to get an approximation for one solution:



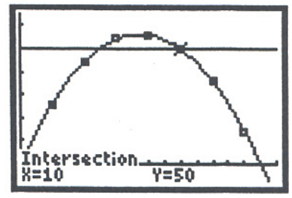
Graph, using trace to get an approximation for the other solution:



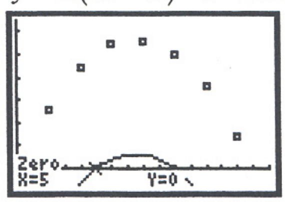
Graph, using the calculator to get an exact answer for one solution, by finding the intersection of $y = x(15 - x) \bullet 1$ and $y = 50$:



Graph, using the calculator to find the other solution, by finding the intersection of $y = x(15 - x) \bullet 1$ and $y = 50$:



Graph, using the calculator to get an exact answer for one solution, by finding the zero (root) of $y = x(15 - x) \bullet 1 - 50$:



Graph, using the calculator to find one solution, by finding the (root) zero of $y = x(15 - x) \bullet 1 - 50$:

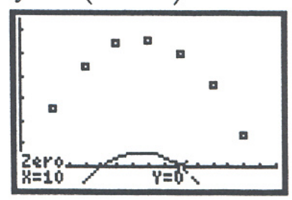


Table: "Trace" to the x-value where the volume is 50. Two solutions.

X	Y1
4	44
5	50
6	54
7	56
8	56
9	54
10	50

X=5

X	Y1
4	44
5	50
6	54
7	56
8	56
9	54
10	50

X=10

Find the intersection where $y_1 = y_2$. Two solutions.

X	Y1	Y2
4	44	50
5	50	50
6	54	50
7	56	50
8	56	50
9	54	50
10	50	50

X=5

X	Y1	Y2
4	44	50
5	50	50
6	54	50
7	56	50
8	56	50
9	54	50
10	50	50

X=10

Find the zero (root) of $y_1 - y_2 = 0$. Two solutions.

X	Y1
4	-6
5	0
6	4
7	6
8	6
9	4
10	0

X=5

X	Y1
4	-6
5	0
6	4
7	6
8	6
9	4
10	0

X=10

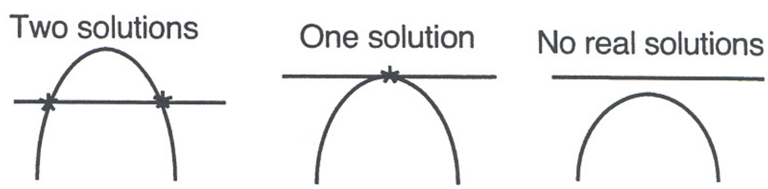
- Does this question give an input value and ask for an output value or does the question give an output value and ask for an input value? [The question gives an output value, 50 ft^3, and asks for an input value, the width.]

An important discussion to have with participants is to compare the 3 different table methods and 3 different graph methods – trace, intersection, zero.

- How does solving quadratic equations in many ways add to your understanding?
- Why might one be more inclined to use the zero method when solving a quadratic equation and not a linear equation. [Linear equations are solved algebraically by getting all of the variables on one side and the numbers on the other side and solving for the variable. Quadratic equations are often solved algebraically by getting everything on one side equal to zero and then solving using factoring, completing the square, or with the quadratic equation.]

Note: When solving linear equations, there is one solution. Often students mistakenly find only one solution to a quadratic equation when solving symbolically. Now with a picture in their heads of a quadratic equation being a parabola intersecting a line, they will be more apt to consider how many solutions they are looking for.

- If you think of the solution to a quadratic equation as the intersection between a parabola and a line, how many solutions are possible? Make a sketch of each to justify your answer. [Two, one, or no real solutions.]

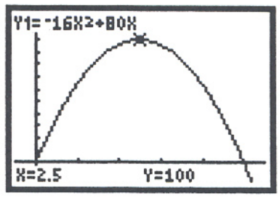


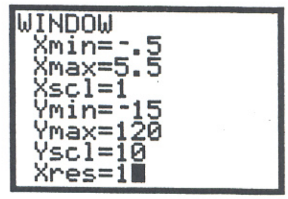
- If you use the "zero" method, that is setting everything equal to zero and solving, how will your sketch of the possible solutions change? [The sketch is essentially the same except the line is now the x-axis, $y = 0$.]

Note: In this problem, we used a quadratic function to represent the volume of a sandbox as the width and length varied with a fixed perimeter and a fixed depth. The units of measure of volume are cubic, in this case, cubic feet. Some discussion may arise that area can be modeled with a quadratic function and the unit of measure is square units, while volume can be modeled with a cubic function, the unit of measure is cubic units. In the particular scenario of the sandbox, the volume can be modeled by a quadratic function because one of the three dimensions, depth, is a fixed quantity. Hence, $V = l \cdot w \cdot d = x(15 - x) \cdot 1$, which is a quadratic function.

Activity 2: Projectile Motion
Have participants work through the activity, using the following to discuss.

1.





- What are a reasonable domain and range for the situation? [See the above window.]

2. Solution:
$h = 96 ft$

Graph:

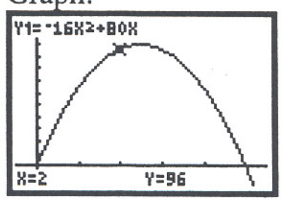
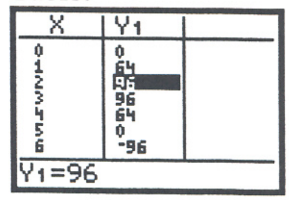
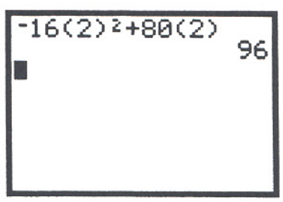


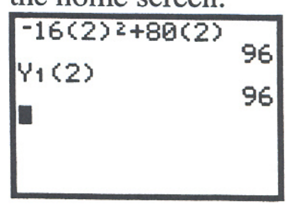
Table:



Home screen:



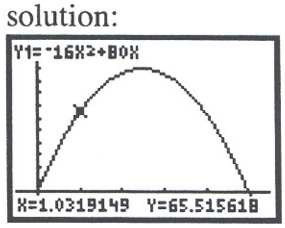
Using function notation on the home screen:



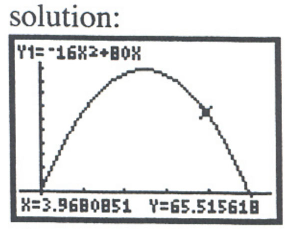
- Does this question give an input value and ask for an output value or does the question give an output value and ask for an input value? [The question gives an input value, 2 feet. and asks for an output value, the height.]

3. Solution:
$width = 1 ft, 4 ft$

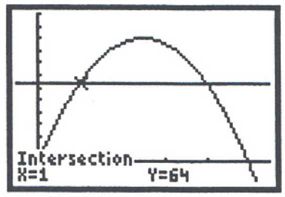
Graph, using trace to get an approximation for one solution:



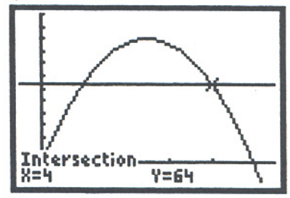
Graph, using trace to get an approximation for the other solution:



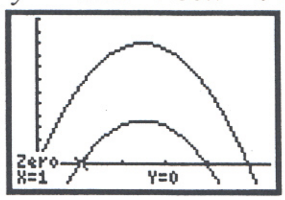
Graph, using the calculator to find one solution, by finding the intersection of $y = -16x^2 + 80x$ and $y = 64$:



Graph, using the calculator to find the other solution, by finding the intersection of $y = -16x^2 + 80x$ and $y = 64$:



Graph, using the calculator to find one solution, by finding the zero (root) of $y = -16x^2 + 80x - 64$:



Graph, using the calculator to find the other solution, by finding the zero (root) of $y = -16x^2 + 80x - 64$:

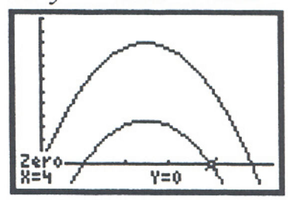


Table: "Trace" to the x-value where the volume is 64. Two solutions.

X	Y1
0	0
1	64
2	96
3	96
4	64
5	0
6	-96

X=1

X	Y1
0	0
1	64
2	96
3	96
4	64
5	0
6	-96

X=4

Find the intersection where $y_1 = y_2$. Two solutions.

X	Y1	Y2
0	0	64
1	64	64
2	96	64
3	96	64
4	64	64
5	0	64
6	-96	64

X=1

X	Y1	Y2
0	0	64
1	64	64
2	96	64
3	96	64
4	64	64
5	0	64
6	-96	64

X=4

Find the zero (root) of $y_1 - y_2 = 0$. Two solutions.

X	Y1
0	-64
1	0
2	32
3	32
4	0
5	-64
6	-160

X=1

X	Y1
0	-64
1	0
2	32
3	32
4	0
5	-64
6	-160

X=4

- Does this question give an input value and ask for an output value or does the question give an output value and ask for an input value? [The question gives an output value, 64 ft, and asks for an input value, the time.]

- How does solving quadratic equations in many ways add to your understanding?

4. Solution:
$t = 5$ sec

Graph:

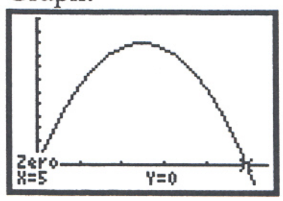
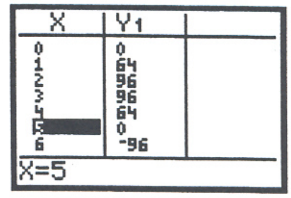


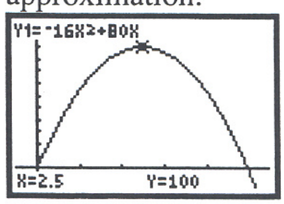
Table:



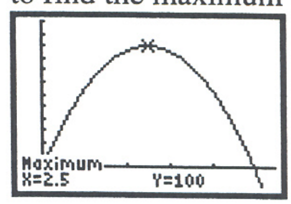
- Does this question give an input value and ask for an output value or does the question give an output value and ask for an input value? [The question gives an output value, 0 feet. and asks for an input value, the time.]

5. Solution:
height = 100 *ft*

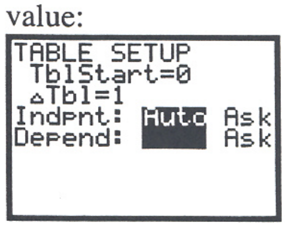
Graph, using trace to get an approximation:

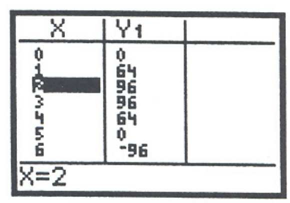


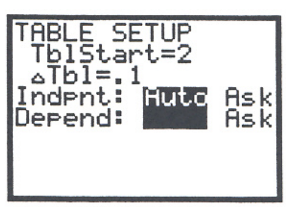
Graph, using the calculator to find the maximum value:

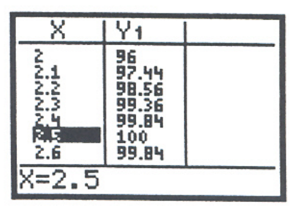


Table, looking for the *x*-value that yields the highest *y*-value:









6. The big idea of this question is to use the symmetry of a parabola to find the vertex. Once you know the roots of a parabola, you can find the *x*-coordinate of the vertex by finding the average of the roots. Then you can

find the y-coordinate of the vertex by substituting the x-coordinate into the equation.

Answers to Reflect and Apply

1. a. $h(2.5) = -16(2.5)^2 + 64(2.5)$, $h = 60$. Participants may also answer, $-16x^2 + 64x = 60$, $x = 2.5$. If they do, discuss that there is another solution to the equation in addition to the one that is shown in the graph.
   b. $-16x^2 + 64x = 48$, $x = 1, 3$
   c. $-16x^2 + 64x = 60$, $x = 1.5, 2.5$
2. $v_0 = 96 \frac{ft}{\text{sec}}$

Summary: Using the natural quadratic relationship of volume where the depth is fixed, participants build a quadratic function. They solve arising quadratic equations in several non-algebraic ways, making connections and building generalizations. Participants further their study by solving equations that arise from a projectile motion situation.

Activity 1: Building a Sandbox

The Cano family is building a rectangular sandbox one foot deep. Diana has decided to use lumber that is one foot wide. She collected 30 feet of lumber to enclose the sandbox.

1. Sketch a few possible sandboxes.

2. Fill in the table with some possible dimensions:

Width	Length	Depth	Volume

3. Predict: what do you think a scatter plot of (width, volume) will look like?