1.2 Transformations

Overview: Participants investigate the effects of changing the parameters of quadratic function of the form $y = ax^2 + c$. They apply this understanding by fitting a quadratic to real data. Participants extend their understanding and investigate the effects of changing the parameter h in quadratic functions of the form $y = (x - h)^2$.

Objective: **Algebra I TEKS**
(d.1.B) The student investigates, describes, and predicts the effects of changes in a on the graph of $y = ax^2$.
(d.1.C) The student investigates, describes, and predicts the effects of changes in c on the graph of $y = x^2 + c$.

Terms: parameter, transformation, scale factor, translation

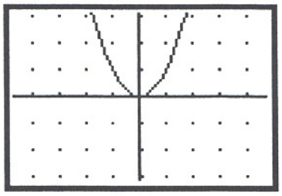
Materials: graphing calculators, patty paper

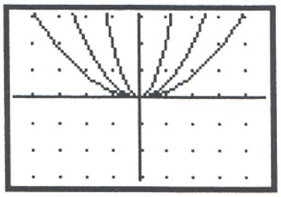
Procedures: Participants should be seated at tables in groups of 3 – 4.

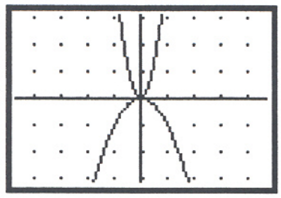
Transformations of functions is an important concept to aid students in graphing various functions and understanding the behavior of various functions. In these activities, participants explore the effects of changing parameters of quadratic functions. They use the power of graphing calculators to find many examples quickly, make and check conjectures, and apply what they have learned. Exploring transformations with parabolas is a natural starting place as participants can watch the vertex "travel" around the coordinate system. In later courses, students will apply the lessons learned to other parent functions, and they will add other transformations to their graphing toolkit.

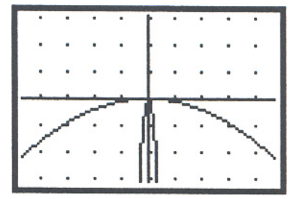
Have participants work together in groups, comparing observations on **Activities 1 – 3**. Discuss their answers to Exercise 5 for each activity. Also, look at table values. See the notes for each activity for an example.

Activity 1: Investigating the Role of *a*

1. 

2. 

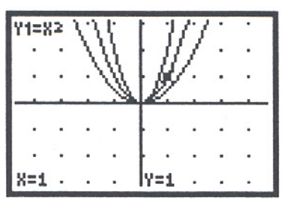
3. 

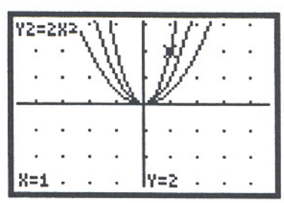
4. 

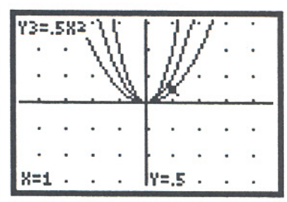
5. The value a is a vertical scale factor. For $a > 1$, the parabola is **vertically stretched**. As $|x|$ increases, the y-values increase faster than for $y = x^2$. For $0 < |a| < 1$, the parabola is **vertically compressed**. As $|x|$ increases, the y-values decrease faster than for $y = x^2$. For $a < 0$, the graph is a **reflection** over the x-axis.

   - Does it change the shape of the graph? [For $a = -1$, the shape of the graph does not change. It is a reflection over the x-axis. For $a > 1$, the shape does change because the parabola is vertically stretched. For $0 < |a| < 1$, the shape also changes because the parabola is vertically compressed.

Choose an Exercise and look at table values, both on the graphs and in the table as shown. Use the questions below to discuss.







X	Y1	Y2
-3	9	18
-2	4	8
-1	1	2
0	0	0
1	1	2
2	4	8
3	9	18

Y2=2X2

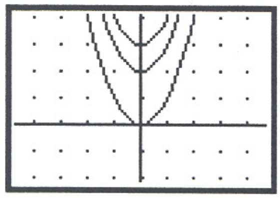
X	Y1	Y3
-3	9	4.5
-2	4	2
-1	1	.5
0	0	0
1	1	.5
2	4	2
3	9	4.5

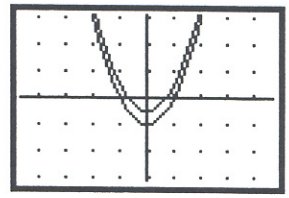
Y3=.5X2

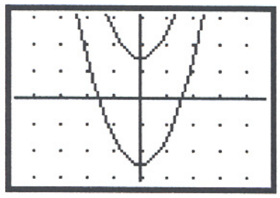
- How do the y-values (function values) of $y = 2x^2$ compare with those of the parent function $y = x^2$? [The y-values are twice as much.]
- How do the y-values (function values) of $y = 0.5x^2$ compare with those of the parent function $y = x^2$? [The y-values are half as much.]
- Why did the vertex remain the same? [Any number times zero is still zero, $x \cdot 0 = 0$.]

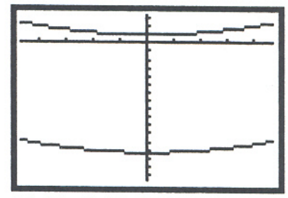
This process of looking at y-values to compare functions may seem unnecessary because it is so obvious, but it lays important groundwork for students. In later courses, students will be expected to discern when a question is referring to function values (y-values) and when a question is referring to x-values.

Activity 2: Investigating the Role of *c*

1. 

2. 

3. 

4. 

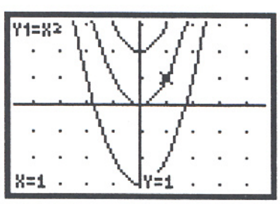
[4.7, 4.7] [-250, 50] 15
WINDOW

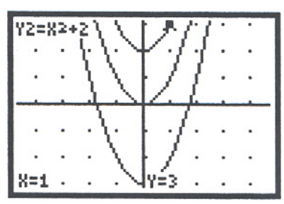
5. For $c > 0$, the graph is vertically translated (shifted) up *c* units. For $c < 0$, the graph is vertically translated (shifted) down *c* units.
   - Does the shape change? [No, vertical translations are shape preserving transformations.]

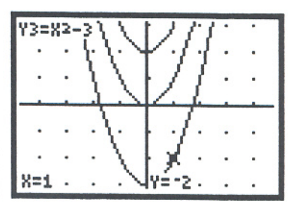
Put a piece of patty paper over the graph for Exercise 1 and trace the parent function $y = x^2$. Slide or shift the patty paper up and down until the parent function is directly over the translated functions to show that indeed the shape does not change.

- Why did we not use patty paper to look at the transformed functions $y = ax^2$? [Dilations are shape changing transformations. If you take the parent function traced on the patty paper and try to make it "fit" one of the stretched or compressed functions from Activity 1, it will not work. The shapes are different.]

Choose an Exercise and look at table values, both on the graphs and in the table as shown. Use the questions below to discuss.







X	Y1	Y2
-3	9	11
-2	4	6
-1	1	3
0	0	2
1	1	3
2	4	6
3	9	11

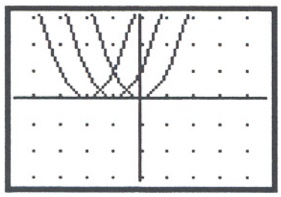
$Y_2 = X^2+2$

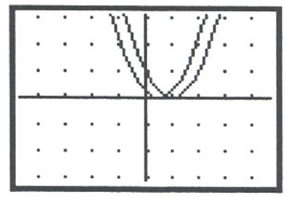
X	Y1	Y3
-3	9	6
-2	4	1
-1	1	-2
0	0	-3
1	1	-2
2	4	1
3	9	6

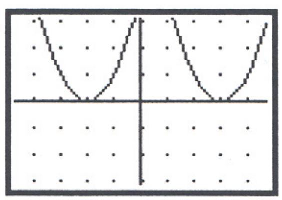
$Y_3 = X^2-3$

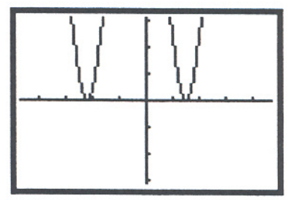
- How do the y-values (function values) of $y = x^2 + 2$ compare with those of the parent function $y = x^2$? [The y-values are 2 more.]
- How do the y-values (function values) of $y = x^2 - 3$ compare with those of the parent function $y = x^2$? [The y-values are 3 less.]
- Why did the vertex change? $[0 + 2 = 2,\ 0 - 3 = -3.]$

Activity 3: Investigating the Role of *h*
Horizontal translations are not listed in the Algebra I TEKS. This activity is intended to enhance teachers understanding of transformations and is not intended for an average algebra I class.

1. 

2. 

3. 

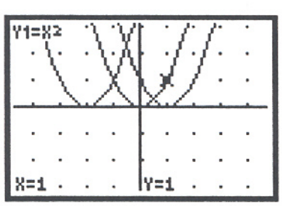
4. 

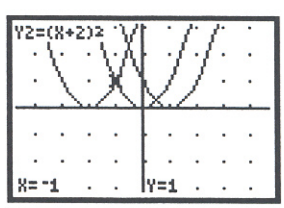
[-47, 47] [-31, 31]
WINDOW

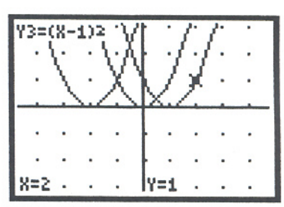
5. For $h > 0$, the graph is horizontally translated (shifted) left h units. For $h < 0$, the graph is horizontally translated (shifted) right h units.
   - Does the shape change? [No, horizontal translations are shape preserving transformations.]

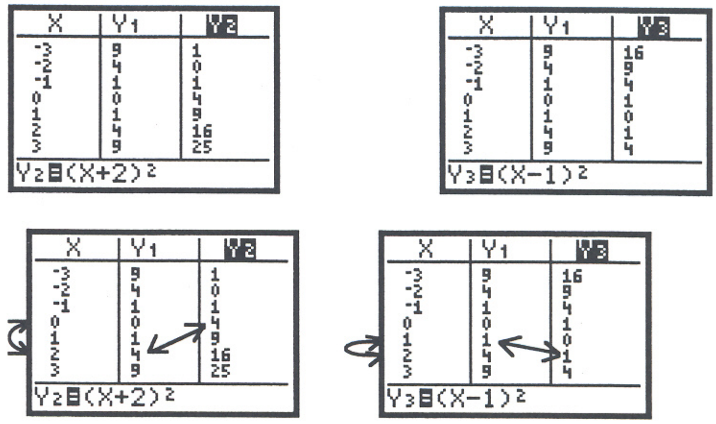
Put a piece of patty paper over the graph for Exercise 1 and trace the parent function $y = x^2$. Slide or shift the patty paper left and right until the parent function is directly over the translated functions to show that indeed the shape does not change.

Choose an Exercise and look at table values, both on the graphs and in the table as shown. Use the questions below to discuss.









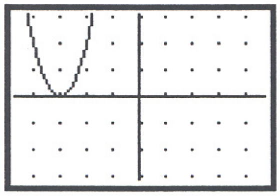
- How do the y-values (function values) of $y = x^2 + 2$ compare with those of the parent function $y = x^2$? [The y-values have all been shifted up two in the table, left two on the graph.]
- How do the y-values (function values) of $y = x^2 - 1$ compare with those of the parent function $y = x^2$? [The y-values have all been shifted down one in the table, right one on the graph.]
- Why did the vertex change? [The y-value of zero has been shifted too.]

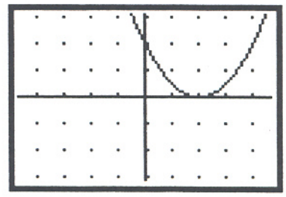
Activity 4: Transformations

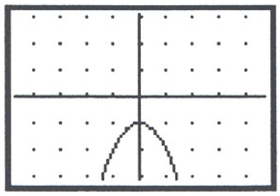
Change	New Equation	Change in Graph
Add 3 to the function	$y = x^2 + 3$	Vertical translation up 3
Multiply by 1/3	$y = \frac{1}{3}x^2$	Scale change of 1/3
Multiply by 3	$y = 3x^2$	*Scale change of 3*
Replace x with $(x-2)$	$y = (x-2)^2$	Horizontal translation right 2
Multiply by – 1	$y = -x^2$	*Reflection over the x-axis*
Subtract 2 from the function	$y = x^2 - 2$	Vertical translation down 2
Replace x with $(x+1)$	$y = (x+1)^2$	Horizontal translation left 1
Multiply by 2	$y = 2x^2$	*Scale change of 2*
Replace x with $(x-3)$	$y = (x-3)^2$	Horizontal translation right 3

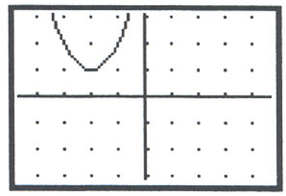
2. a. Horizontal translation left 5, vertical translation down 1
   b. Scale change of 3, vertical translation up 2
   c. Reflection across the x-axis, scale change of 1/3, horizontal translation left 1.
3. a. $y = 2(x-1)^2$
   b. $y = -(x+2)^2$
   c. $y = (x-3)^2 - 2$

d. $y = (x+3)^2 + 1$

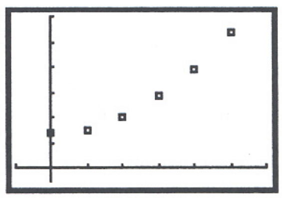
4. 

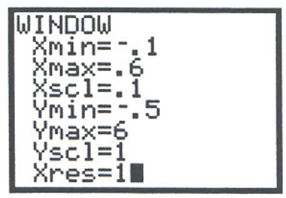
5. 

6. 

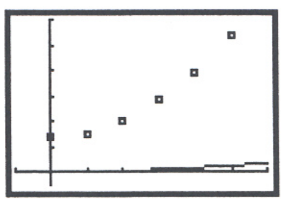
7. 

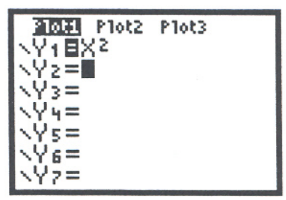
8.

a. 

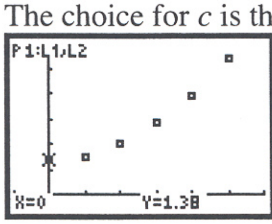


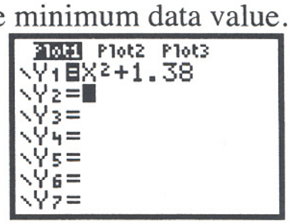
b. As the elapsed time increases, the distance from the motion detector increases.

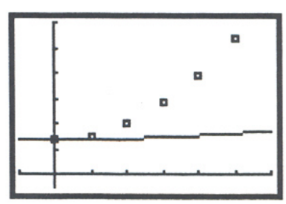
c. 

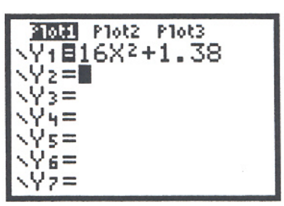


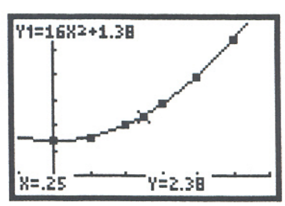
d. The choice for c is the minimum data value.







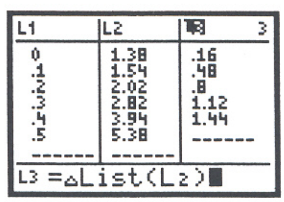
e. 

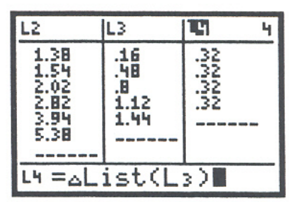


- What is the significance of $a = 16$ in this problem? [The formula for the distance of an object dropped from an initial height of d_0 at an initial velocity of zero is $d = \frac{1}{2}at^2 + d_0$. The acceleration due to the force of gravity, a, is $-32\frac{ft}{sec^2}$, so $d = \frac{1}{2}(-32)t^2 + d_0 = -16t^2 + d_0$. (The proof of this statement is dependent on an understanding of derivatives that are studied in calculus.) The 16 in the problem situation is positive because the motion detector is measuring the distance from the motion detector to the book instead of the distance from the book to the ground.

Extension:
Ask participants to find first and second differences for the data.

- What do the second differences imply about the choice of a quadratic function for a model for the data? [Since second differences are constant, the data can be modeled with a quadratic function.]
- How are the second differences and your value for a related? [The second differences as shown below are 0.32 feet per 0.1 sec per 0.1 sec, $0.32\frac{ft}{\frac{0.1sec}{0.1sec}} = 32\frac{ft}{sec^2}$. The acceleration due to the force of gravity, the force pulling the book down to the ground, is $-32\frac{ft}{sec^2}$. The second difference, 32, is positive because the motion detector is measuring the distance from the motion detector to the book instead of the distance from the book to the ground.



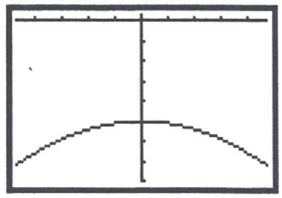


Answers to Reflect and Apply
Discuss with participants the term "appropriate viewing windows," especially with respect to the graph of quadratic functions.

- What should an appropriate viewing window show about a quadratic function? [A complete graph]
- What is a complete graph of a quadratic function? [The window would include x-intercepts, if any, the direction the parabola opens, and the vertex. Some participants may want to show the y-intercept.]

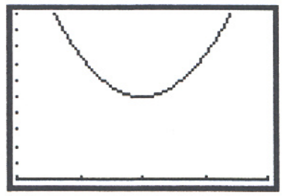
Sample answers. Window may vary, but should show similar graphs.

1.



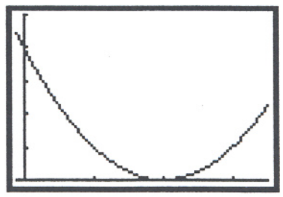
```
WINDOW
 Xmin=-4.7
 Xmax=4.7
 Xscl=1
 Ymin=-80
 Ymax=3.1
 Yscl=10
 Xres=1
```

2. A window without the *y*-intercept.



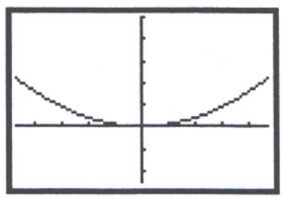
```
WINDOW
 Xmin=1900
 Xmax=2100
 Xscl=50
 Ymin=0
 Ymax=10000
 Yscl=1000
 Xres=1
```

2. A window that shows the *y*-intercept.



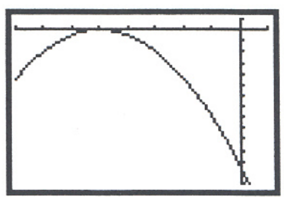
```
WINDOW
 Xmin=-100
 Xmax=3500
 Xscl=1000
 Ymin=0
 Ymax=5000000
 Yscl=1000000
 Xres=1
```

3.



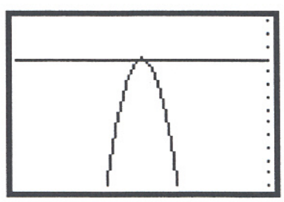
```
WINDOW
 Xmin=-4.7
 Xmax=4.7
 Xscl=1
 Ymin=-.025
 Ymax=.05
 Yscl=.01
 Xres=1
```

4. A window that shows the *y*-intercept.



```
WINDOW
 Xmin=-8
 Xmax=1
 Xscl=1
 Ymin=-14000
 Ymax=1000
 Yscl=1000
 Xres=1
```

4. A window without the *y*-intercept.



```
WINDOW
 Xmin=-5.5
 Xmax=-4.5
 Xscl=1
 Ymin=-10
 Ymax=3.1
 Yscl=1
 Xres=1
```

5. y4, A
6. y3, D
7. y2, C
8. y1, B
9. y2, D
10. y1, A
11. y4, B
12. y3, C

Summary: Using technology to see many examples quickly, participants connect transformations of quadratic functions with the vertex form of the equation of a parabola $y = a(x - h)^2 + b$. Participants use transformations to fit a quadratic function to data, i.e. the distance of a dropped object from a motion detector.